Problem sheet 6
Feb 15th 2005

MT290 COMPLEX VARIABLE

Ex. 1
Evaluate the following integrals around C, the circle $|z| = 1$ taken anticlockwise, using Cauchy's integral formula (or the formula for derivatives if appropriate).

$$\text{(a)} \int_C \frac{\cos z}{z}\, dz; \qquad \text{(b)} \int_C \frac{e^{\pi z}}{(z-\frac{1}{4})^2(z-4)}\, dz.$$

Ex. 2
Throughout this question $|z| < 2$. Let $f(z) = \frac{1}{z} + g(z)$, where $|g(z)| < A$ (a fixed constant). Let γ_r be the circle: re^{it}, $0 \le t \le 2\pi$. Use the estimation lemma to show that for $r < 2$

$$\left| \int_{\gamma_r} f(z)\, dz - 2\pi i \right| \le 2\pi A r.$$

Deduce that

$$\int_{\gamma_r} f(z)\, dz \longrightarrow 2\pi i$$

as $r \to 0$. [*Note:* there is nothing special about the $|z| < 2$ here, it's there just to restrict our attention to a finite region.

Ex. 3
Show that $|e^{iz}| \le 1$ when $\mathrm{Im} z \ge 0$. Let

$$I_R = \int_{\gamma_R} \frac{e^{iz}}{1+z^2}\, dz,$$

where γ_R is the semicircle given by $\phi(t) = Re^{it}$, $0 \le t \le \pi$. Show that, for all large R,

$$|I_R| \le \frac{2\pi}{R}.$$

Hence deduce that $I_R \to 0$ as $R \to \infty$.

Ex. 4
(Compare our discussion of $\int_\gamma \frac{1}{z^2+4}\, dz$.) Let $f(z)$ be a non-constant polynomial of degree at least two, and C(R) a circle centred on 0 with radius R. Use the estimation lemma to show that

$$\lim_{R\to\infty} \int_{C(R)} \frac{1}{f(z)}\, dz \;=\; 0.$$

Hence show, using the deformation of contours theorem, that if all the roots of $f(z) = 0$ lie within the circle $|z| = R$, then

$$\int_{C(R)} \frac{1}{f(z)} dz \;=\; 0.$$

Ex. 5
Solve the quadratic equation $x^2 - 4x + 5 = 0$. Hence write $x^2 - 4x + 5 = (x-\alpha)(x-\beta)$. Use the semicircular contour $\gamma(R)$: $(-R, R) \cup \{Re^{it} : 0 \le t \le \pi\}$ to evaluate the following two integrals:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 - 4x + 5}\, dx, \qquad \int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 4x + 5}\, dx.$$

Note: Workshops this week:
Thursday 3pm: 325
Friday 12pm: Lec B
From Feb. 24th onwards:
Let's see if Thursday 11am attracts more people...
Thursday 11am: 325
Thursday 3pm: 325
But no Friday workshops any longer.