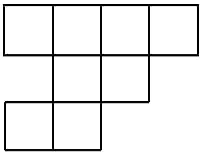
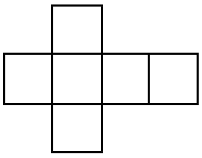
Ex. 1
Find the Rook polynomials of the following boards:

(a)  (b) 

Ex. 2
Alice, Bob, Charles, Diane and Edger are having a drink after work. There are only four drinks on their table: ale, Bacardi, cider, and Drambuie. Bob and Charles hate ale, Charles hates Bacardi, Alice and Edger hate cider and Bob hates Drambuie. In how many ways can four people get a drink they like?

Ex. 3
Count the number of primes in the interval $\{1, 2, \cdots, 100\}$ using the PIE.
State precisely what are the sets A_i etc. in your solution.

Ex. 4
An important function with arithmetic information about integers is the Euler φ-function. (For example, the property $a^p \equiv a \bmod p$ for primes p can be generalized to $a^{\varphi(n)+1} \equiv a \bmod n$, if $\gcd(a, n) = 1$. And this has important consequences in cryptography such as the RSA encryption).
Define $\varphi : \mathbb{N} \to \mathbb{N}, \varphi(n) = |\{a \in \mathbb{N}, 1 \le a \le n : \gcd(a, n) = 1\}|$.
Example: for $n = 10$: $1, 3, 7, 9$ are the only integers $1 \le a \le 10$ that are coprime to 10. Hence $\varphi(10) = 4$. Prove the following properties, or give (with proof!) general formulae for the values of the φ-function.

a) $\varphi(p) = p - 1$, for prime p.

b) Let p, q, r denote distinct primes. Give a formula for $\varphi(pq)$ and one for $\varphi(pqr)$, using the PIE. (Define the sets A_i precisely).

c) Give a formula for prime powers $\varphi(p^e)$.

d) Let $\gcd(n_1, n_2) = 1$. Give a formula for $\varphi(n_1 n_2)$ in terms of $\varphi(n_1)$ and $\varphi(n_2)$.

e) Recall that each integer n can uniquely be written in the form $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$. where the $p_1 < \ldots < p_r$ are primes.

   Give a formula for $\varphi(n)$.

Ex. 5
Work through the following example (not to be handed in).
Imitate the proof for $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$ to prove an identity for $\binom{3n}{n}$.
Verify it in the special case $n = 4$.

Solution: I can think of two natural ways to generalize:

Start with $(1+x)^{2n}(1+x)^n = (1+x)^{3n}$.
Now determine the coefficient of x^n.
The RHS gives $\binom{3n}{n}$.
The LHS gives $\sum_{k=0}^n \binom{2n}{k}\binom{n}{n-k} = \sum_{k=0}^n \binom{2n}{k}\binom{n}{k}$.
So,

$$\binom{3n}{n} = \sum_{k=0}^{n} \binom{2n}{k}\binom{n}{k}.$$

Another possibility is: $(1+x)^n(1+x)^n(1+x)^n = (1+x)^{3n}$. Now determine the coefficient of x^n.
The RHS gives $\binom{3n}{n}$.
The LHS gives $\sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i}\binom{n}{j}\binom{n}{n-i-j} = \sum_{i,j\geq 0 \text{ with } i+j\leq n} \binom{n}{i}\binom{n}{j}\binom{n}{i+j}$.
So,

$$\binom{3n}{n} = \sum_{i,j\geq 0 \text{ with } i+j\leq n} \binom{n}{i}\binom{n}{j}\binom{n}{i+j}.$$

The fact that these results are different is no contradiction. There are simply many different identities...

The details for $n = 4$.
Work out rows 4 and 8 in the Pascal triangle:
1,4,6,4,1
1,8,28,56,70,56,28,8,1
and $\binom{12}{4} = 495$.
The first formula gives

$$1 \times 1 + 4 \times 8 + 6 \times 28 + 4 \times 56 + 1 \times 70 = 1 + 32 + 168 + 224 + 70 = 495.$$

The second formula:
Take all allowed combinations of $(i, j, i + j)$:
000/011/022/033/044/101/112/123/134/202/213/224/303/314/404
This gives:

$$\begin{aligned} & (1+16+36+16+1) + (16+96+96+16) + (36+96+36) + (16+16) + 1 \\ = \; & 70 + 224 + 168 + 32 + 1 = 495. \end{aligned}$$