Exercise Sheet 6

MT454 Combinatorics

1. (a) Show that
$$\frac{x(1+x)}{(1-x)^3}$$
is the generating function for the sequence (u_n) defined by $u_n = n^2$.

   (b) Let $A(x)$ be the generating function for a sequence (a_n). For any $n \geq 0$, let
$$s_n = a_0 + a_1 + \cdots + a_n$$
and let $S(x)$ be the generating function for (s_n). Show that
$$S(x) = \frac{A(x)}{1-x}.$$
Use this formula together with part (a) to find a formula for $\sum_{i=0}^{n} i^2$. [Hint: Use Lemma 4.10 at some point!]

2. Let $U(x) = F_0 + F_1 x + F_2 x^2 + \cdots$ be the generating function for the Fibonacci numbers. Find polynomials $A(x)$ and $B(x)$ such that
$$U(x) = \frac{A(x)}{B(x)}.$$

3. Find an explicit formula for u_n, where $u_0 = 1$, $u_1 = u_2 = 0$ and $u_{n+3} - 3u_{n+1} + 2u_n = 0$.

4. Show that the generating function for the sequence (u_n) defined by the recursion $u_0 = 1$, $u_{n+1} - 2u_n = 4^n$ for $n \geq 0$, is
$$U(x) = \frac{1-3x}{(1-2x)(1-4x)}.$$
Hence show that $u_n = 2^{2n-1} + 2^{n-1}$.