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A remark on Hofmann and Wolke's additive decompositions of the set of primes

By

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Abstract. We improve upon Hofmann and Wolke's bounds on the conceivable additive decompositions of the set of primes. We also prove that there cannot be a decomposition, where one set is a shifted copy of the other. This answers an open question of Hofmann and Wolke.

We improve upon Hofmann and Wolke's bounds (see [2]) on the conceivable additive decompositions of the set of primes. Their result corresponds to the special case $r = 2$ of our new bounds. We overcome a difficulty that arose in the application of the large sieve method in their work. We also prove that there cannot be a decomposition, where one set is a shifted copy of the other. This answers an open question of Hofmann and Wolke.

Theorem 1. *Let $\mathscr{A}$ and $\mathscr{B}$ be sets of integers with $|\mathscr{A}|, |\mathscr{B}| \geqq 2$. Suppose that $\mathscr{A} + \mathscr{B} = \{a + b : a \in \mathscr{A}, b \in \mathscr{B}\} = \mathscr{P}'$, where $\mathscr{P}'$ coincides with the set of primes $\mathscr{P}$ for large elements. Then the following bounds on the counting functions hold for all $r \geqq 2$, for suitable positive constants c_r and for sufficiently large $x > x_r$:*

$$\exp\left(c_r \frac{\log x}{\log_r x}\right) \ll A(x) \ll x \exp\left(-c_r \frac{\log x}{\log_r x}\right).$$

Here $\log_r x$ denotes the r-th iterated logarithm. The same applies to $B(x)$.

Theorem 2. *There does not exist an additive decomposition $\mathscr{A} + \mathscr{B} = \mathscr{P}'$ with $\mathscr{B} = \{a + k : a \in \mathscr{A}\}$, where k is any fixed constant.*

We will first outline the method of Hofmann and Wolke.

Suppose that $\mathscr{P} \cap [x_0, \infty] = \mathscr{P}' \cap [x_0, \infty]$. Let $a_1 \in \mathscr{A}$ with $a_1 > x^{1/2}$. If $a_1 + b_1 = p_1 > x_0$, then p_1 is prime and $a_1 \not\equiv -b_1 \bmod p$ for all primes $p < p_1$. Hence, for all primes $x_0 < p \leqq x^{1/2}$ the set $\mathscr{A}$ avoids the negatives of those residue classes modulo p that occur in $\mathscr{B}$. So one can use a sieve method with the number of forbidden classes $\omega(p) = B(p)$ to obtain an upper bound on $A(x)$.

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Lemma 1.

$$A(x) \leqq \frac{2x}{L} + x^{\frac{1}{2}}, \textit{ where}$$

$$L = \sum_{q \leqq x^{1/2}} \mu^2(q) \prod_{p|q} \frac{\omega(p)}{p - \omega(p)} \geqq \max_{m \in \mathbb{N}} \exp\left(m \log\left(\frac{1}{m} \sum_{p \leqq x^{1/(2m)}} \frac{\omega(p)}{p}\right)\right).$$

The proof is immediate when applying Montgomery's large sieve (see [1], Theorem 5.3.2) to the interval $[\sqrt{x}, x]$, as was done in [2]. For the evaluation of L see [3]. Hence an existing lower bound on $B(x)$ leads to an upper bound on $A(x)$. On the other hand $\pi(x) \ll A(x)B(x)$ shows that an upper bound on $A(x)$ also implies a lower bound on $B(x)$. One starts with $|\mathcal{B}| \geqq 2$, hence one can use an upper bound sieve with $\omega(p) = 2$ leading to $A(x) \ll \frac{x}{(\log x)^2}$. This implies the lower bound $B(x) \gg \log x$ etc. Now Hofmann and Wolke's method can be summarized in the following table.

$B(x) \gg$	$\omega(p) =$	m	$A(x) \ll$
$\lvert\mathcal{B}\rvert \geqq 2$	2	–	$x(\log x)^{-2}$
$\log x$	$[c_0 \log p]$	$[(\log x)^{\frac{1}{2}}]$	$x \exp(-c_1 (\log x)^{\frac{1}{2}})$
$\exp(\tilde{c}_1 (\log x)^{\frac{1}{2}})$	$[c_1' \exp(\tilde{c}_1 (\log p)^{\frac{1}{2}})]$	$\left[\frac{\log x}{(\log_2 x)^2}\right]$	$x \exp\left(-c_2 \frac{\log x}{\log_2 x}\right)$

They mention that an application of Montgomery's sieve does not yield new results for $\omega(p) \geqq d_1 \exp\left(d_2 \frac{\log p}{\log_2 p}\right)$. In contrast to this we will apply Montgomery's sieve successfully with $\omega(p) = [d_1 \exp\left(d_2 \frac{\log p}{\log_r p}\right)]$ for all $r \geqq 2$.

Lemma 2. *For given $r \geqq 2$, let $\omega(p) = [d_1 \exp\left(d_2 \frac{\log p}{\log_r p}\right)]$. Then $L \gg \exp\left(c \frac{\log x}{\log_{r+1} x}\right)$ for a suitable positive constant c.*

Proof.

$$\sum_{p \leqq y} \frac{\omega(p)}{p} \geqq \sum_{y/2 \leqq p \leqq y} \frac{\omega(p)}{p} \geqq (\pi(y) - \pi(y/2)) \frac{d_1 \exp\left(d_2 \frac{\log(y/2)}{\log_r (y/2)}\right)}{y}$$

$$\gg \exp\left(d_3 \frac{\log y}{\log_r y}\right).$$

With $m = \left[\frac{\log x}{(\log_2 x)^2}\right]$ we have $\log_r x^{1/(2m)} \sim \log_{r-1} \frac{(\log_2 x)^2}{2} \leqq 2 \log_{r+1} x$ and $\sum_{p \leqq x^{1/(2m)}} \frac{\omega(p)}{p} \gg \exp\left(d_4 \frac{\log x}{m \log_{r+1}}\right)$. Hence

$$\log L \geqq m \log\left(\frac{1}{m} \sum_{p \leqq x^{1/(2m)}} \frac{\omega(p)}{p}\right) \geqq m\left(-\log m + d_4 \frac{\log x}{m \log_{r+1} x}\right) \geqq c \frac{\log x}{\log_{r+1} x}.$$

□

Lemma 2 allows to continue the sieve iteration of [2].

$B(x) \gg$	$\omega(p) =$	m	$A(x) \ll$
$\exp\left(\tilde{c}_2 \frac{\log x}{\log_2 x}\right)$	$\left[c_2' \exp\left(\tilde{c}_2 \frac{\log p}{\log_2 p}\right)\right]$	$\left[\frac{\log x}{(\log_2 x)^2}\right]$	$x \exp\left(-c_3 \frac{\log x}{\log_3 x}\right)$
....	....	....	....
$\exp\left(\tilde{c}_r \frac{\log x}{\log_r x}\right)$	$\left[c_r' \exp\left(\tilde{c}_r \frac{\log p}{\log_r p}\right)\right]$	$\left[\frac{\log x}{(\log_2 x)^2}\right]$	$x \exp\left(-c_{r+1} \frac{\log x}{\log_{r+1} x}\right)$

It is in fact possible to let r grow with x. But since the constants implied behave like $c_r \approx \frac{1}{2^r}$ we see that r can only increase so extremely slowly that we refrain from stating a new result.

Hofmann and Wolke also considered the problem whether an additive decomposition of the set of primes exists with $\mathscr{B} = \{a+1 : a \in \mathscr{A}\}$. They proved that in this situation one has the bounds

$$\frac{x^{\frac{1}{2}}}{(\log x)^{\frac{1}{2}}} \ll A(x) \ll x^{\frac{1}{2}} \text{ and } \frac{x^{\frac{1}{2}}}{(\log x)^{\frac{1}{2}}} \ll B(x) \ll x^{\frac{1}{2}}.$$

I would like to thank Dr. Puchta (Freiburg) for drawing my attention to this problem. He later informed me that he independently settled this problem.

Lemma 3. *Suppose that $\mathscr{A} + \mathscr{B} = \mathscr{P}'$. Then the large elements of one of the sets $\mathscr{A}$ and $\mathscr{B}$ lie in precisely one residue class modulo* 3, *and the large elements of the other set lie in precisely two residue classes modulo* 3.

Theorem 2 is then immediate, since a shift does not change the number of occurring classes modulo 3.

Proof of the lemma. If in one of the sets $\mathscr{A}, \mathscr{B}$ all three residue classes modulo 3 were represented infinitely often, then the class $0 \bmod 3$ would contain infinitely many elements of $\mathscr{P}' = \mathscr{A} + \mathscr{B}$, contradicting the definition of $\mathscr{P}'$. The same conclusion holds, if we assume that in each of $\mathscr{A}$ and $\mathscr{B}$ two residue classes modulo 3 were represented infinitely often. Note that any two such classes are consecutive. If, on the other hand, each of $\mathscr{A}$ and $\mathscr{B}$ has infinitely many elements in only one residue class modulo 3, then in at least one of the two classes 1 or 2 modulo 3 we would have only $\ll A(x) + B(x) \ll x/(\log x)^2$ elements. This would contradict the Prime Number Theorem for arithmetic progressions. □

Finally we remark that the bounds on $A(x)B(x)$ in [2] can be given explicitly: For sufficiently large x and for an arbitrary $\varepsilon > 0$ one has

$$\frac{x}{\log x} \leqq A(x)B(x) \leqq \left(\frac{\pi^4}{9} + \varepsilon\right)x.$$

The lower bound follows from $\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right)$. The upper bound follows from the large sieve inequality $S(x) \leqq 2x/L$ and $\sum_{q \leqq x} \mu^2(q) \sim \frac{6}{\pi^2}x$. It should be pointed out that the very same constants can be obtained from an unpublished manuscript (see [4]) which Prof. Wirsing presented at the Oberwolfach conference in 1972.

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References

[1] J. Brüdern, Einführung in die analytische Zahlentheorie. Berlin 1995.
[2] A. Hofmann and D. Wolke, On additive decompositions of the set of primes. Arch. Math. **67,** 379–382 (1996).
[3] R. C. Vaughan, Some applications of Montgomery's sieve. J. Number Theory **5,** 64–79 (1973).
[4] E. Wirsing, On the additive decomposibility of the set of primes. Unpublished manuscript, 1972/1998.

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