On conjectures of T. Ordowski and Z.W. Sun concerning primes and quadratic forms

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On the occasion of Helmut Maier's 60th birthday.
With admiration
for his beautiful results on the distribution of primes.

Abstract We discuss recent conjectures of T. Ordowski and Z.W. Sun on limits of certain coordinate-wise defined functions of primes in $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, Let $p \equiv 1 \bmod 4$ be a prime and let $p = a_p^2 + b_p^2$ be the unique representation with positive integers $a_p > b_p$. Then the following holds:

$$\lim_{N\to\infty} \frac{\sum_{p\le N, p\equiv 1 \bmod 4} a_p^k}{\sum_{p\le N, p\equiv 1 \bmod 4} b_p^k} = \frac{\int_0^{\pi/4} \cos^k(x)\,dx}{\int_0^{\pi/4} \sin^k(x)\,dx}.$$

For $k = 1$ this proves, but for $k = 2$ this disproves the conjectures in question. We shall also generalise the result to cover all positive definite, primitive, binary quadratic forms. In addition we will discuss the case of indefinite forms and prove a result that covers many cases in this instance.

1 Primes and quadratic forms

Tomasz Ordowski (see Sun [12], section 6) conjectured:

Conjecture 1. Let $p \equiv 1 \bmod 4$ be a prime and let $p = a_p^2 + b_p^2$ be the unique representation with positive integers $a_p > b_p$.

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$$a) \qquad \lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv 1 \bmod 4} a_p}{\sum_{p\le N,\, p\equiv 1 \bmod 4} b_p} = 1+\sqrt{2},$$

$$b) \qquad \lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv 1 \bmod 4} a_p^2}{\sum_{p\le N,\, p\equiv 1 \bmod 4} b_p^2} = \frac{9}{2}.$$

Z.W. Sun [12] stated a number of related conjectures for other quadratic forms, including the following:

Conjecture 2 (Sun). Let $p \equiv 1 \bmod 3$ be a prime and let $p = x_p^2 + x_p y_p + y_p^2$ be the unique representation with positive integers $x_p > y_p$.

$$a) \qquad \lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv 1 \bmod 3} x_p}{\sum_{p\le N,\, p\equiv 1 \bmod 3} y_p} = 1+\sqrt{3},$$

$$b) \qquad \lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv 1 \bmod 3} x_p^2}{\sum_{p\le N,\, p\equiv 1 \bmod 1} y_p^2} = \frac{52}{9}.$$

This conjecture can also be found in the comments on sequence A218585 in [10]. Sun further remarked that numerically

$$\lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv 1 \bmod 3} x_p^3}{\sum_{p\le N,\, p\equiv 1 \bmod 1} y_p^3} \approx 11.15 \quad \text{and} \quad \lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv 1 \bmod 3} x_p^4}{\sum_{p\le N,\, p\equiv 1 \bmod 1} y_p^4} \approx 20.6.$$

In this paper we will show that these limits exist and how one can evaluate them. Cases a) of both conjectures turn out to be true, but cases b) of both conjectures are wrong. It seems to us that the authors guessed the values based on some experimental data, but without theoretical justification. For the two quadratic forms above we will determine the nature and value of the sums with arbitrary $k \in \mathbb{N}$.

Theorem 1. *Let $p \equiv 1 \bmod 4$ be a prime and let $p = a_p^2 + b_p^2$ be the unique representation with positive integers $a_p > b_p$. Then*

$$I_k := \lim_{N\to\infty} \frac{\sum_{p\le N, p\equiv 1 \bmod 4} a_p^k}{\sum_{p\le N, p\equiv 1 \bmod 4} b_p^k} = \frac{\int_0^{\pi/4} \cos^k(x)\,dx}{\int_0^{\pi/4} \sin^k(x)\,dx}.$$

For $k=1$ this value is indeed $1+\sqrt{2}$. For $k=2$ the value is $\frac{\pi+2}{\pi-2}$ which is about 4.50388. Here is a small table of exact and numerical values.

k	exact value	approx. numerical value
1	$1+\sqrt{2}$	2.41421
2	$\dfrac{\pi+2}{\pi-2}$	4.50388
3	$\frac{5}{7}(5+4\sqrt{2})$	7.61204
4	$\dfrac{3\pi+8}{3\pi-8}$	12.2298
5	$(43\sqrt{2})/(64-43\sqrt{2})=\frac{43}{199}(43+32\sqrt{2})$	19.0701
6	$\dfrac{15\pi+44}{15\pi-44}$	29.1700
7	$(177\sqrt{2})/(256-177\sqrt{2})=\frac{177}{1439}(177+128\sqrt{2})$	44.0371
8	$\dfrac{21\pi+64}{21\pi-64}$	65.8612

Remark 1. The method of proof can be easily adapted to prove a slightly more general result:
Let $0 \le C_1 < C_2$ be nonnegative constants. Let $p \equiv 1 \bmod 4$ be a prime and let $p = a_p^2 + b_p^2$ be a representation with positive integers $C_1 a_p < b_p < C_2 a_p$. Then (counting representations with multiplicity if $C_1 < 1 < C_2$)

$$I_k(C_1,C_2) := \lim_{N\to\infty} \frac{\sum_{p\le N, p\equiv 1 \bmod 4} a_p^k}{\sum_{p\le N, p\equiv 1 \bmod 4} b_p^k} = \frac{\int_{\arctan C_1}^{\arctan C_2} \cos^k(x)\,dx}{\int_{\arctan C_1}^{\arctan C_2} \sin^k(x)\,dx}.$$

The special case $C_1 = 0, C_2 = 1$ immediately gives Theorem 1.

Remark 2. It is possible to investigate the number theoretic properties of the integers in the related sequences. For example the coefficients of π in the expressions

$$\int_0^{\pi/4} \cos^k(x)\,dx = \begin{cases} \frac{2+\pi}{8} & k=2 \\ \frac{8+3\pi}{32} & k=4 \\ \frac{11}{48}+\frac{10\pi}{128} & k=6 \\ \frac{5}{24}+\frac{35\pi}{512} & k=8 \end{cases}$$

etc. leads to the sequence $1,3,10,35,126,\dots$ which is well studied, with a lot of further comments and connections stated in the Online encyclopeadia of integer sequences, sequence, A001700, [10].

On the other hand, for $k=2\ell$ we find that $I_k = (A_\ell\pi + B_\ell)/(A_\ell\pi - B_\ell)$ where A_ℓ is the sequence

$$1,3,15,21,315,3465,45045,15015,765765,14549535\dots$$

which is, at the time of writing, not in the OEIS. However, the sequence is very closely related to sequence A025547, Least common multiple of $\{1,3,5,...,2n-1\}$: $1,3,15,105,315,3465,45045,45045,765765,14549535,14549535,\dots$ The relation to this sequence is quite natural by the recursive nature of the integrals. (Expanding the fractions would enlarge 21 to 105 and 15015 to 45045). We do not follow any of these paths further.

We now study two properties of the values of I_k:

Corollary 1. *Let* $I_k = \lim_{N\to\infty} \frac{\sum_{p\le N, p\equiv 1 \bmod 4} a_p^k}{\sum_{p\le N, p\equiv 1 \bmod 4} b_p^k}$, *then* $I_k \in \mathbb{Q}(\sqrt{2})$, *with* I_k *irrational, when k is odd, and* $I_k \in \mathbb{Q}(\pi)$, *with* I_k *transcendental, when* $k \ge 2$ *is even.*

The following result states the asymptotic growth of I_k:

Theorem 2. *As k tends to infinity, we have the following estimate:*

$$I_k \sim \left(\frac{\pi k}{2}\right)^{\frac{1}{2}} 2^{k/2}. \tag{1}$$

We now come to Conjecture 2. A consequence of our main theorem below (Theorem 4) is the following which we state as a result in its own right.

Theorem 3. *Let* $p \equiv 1 \bmod 3$ *be a prime and let* $p = x_p^2 + x_p y_p + y_p^2$ *be the unique representation with positive integers* $x_p > y_p$. *Then*

$$J_k = \lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv 1 \bmod 3} x_p^k}{\sum_{p\le N,\, p\equiv 1 \bmod 3} y_p^k} = \frac{U(k)}{V(k)},$$

where

$$U(k) = \int_0^{\pi/6} (\sqrt{3}\cos x - \sin x)^k \, dx$$

and

$$V(k) = 2^k \int_0^{\pi/6} \sin^k x \, dx.$$

In particular this gives

k	exact value of J_k	approx. numerical value
1	$1+\sqrt{3}$	2.73205
2	$\dfrac{2\pi}{2\pi - 3\sqrt{3}}$	5.78012
3	$\frac{1}{13}(67+45\sqrt{3})$	11.1494
4	$\dfrac{2(\sqrt{3}-2\pi)}{7\sqrt{3}-4\pi}$	20.5927
5	$\frac{7}{709}(1837+1113\sqrt{3})$	37.1698
6	$\dfrac{9\sqrt{3}-10\pi}{18\sqrt{3}-10\pi}$	66.2204

Again, part a) of the conjecture is correct, while b) is not.

Theorems 1 and 3 above, and many others, are consequences of the following general result. Before stating this we need to make some comments on the uniqueness of representations. In the examples above the representations are unique, but it is easy to find forms which give two representations. For Theorem 3 if we had considered instead $Q(x,y) = x^2 - xy + y^2$ so that $Q(x,y) = Q(x, x-y)$ and thus $0 < y < x$ is equivalent to $0 < x-y < x$. We thus obtain two representations for every representable prime. Similarly, if instead of $x^2 + y^2$ we considered $(x-y)^2 + y^2 = x^2 - 2xy + 2y^2$. We note that $x^2 - 3xy + 3y^2$ gives four different representations with $0 < y < x$ of primes $p \equiv 1 \bmod 6$. The worst case forms are $ax^2 + bxy + cy^2$ with $4ac - b^2 = 3$ and b large and negative. For these we have 5 or 6 representations. For example, with $Q(x,y) = x^2 - 7xy + 13y^2$ we have

$$\begin{aligned}
109 &= Q(8,5) = Q(16,7) = Q(27,5) = Q(33,7) = Q(41,12) = Q(43,12) \\
&\quad \text{(6 representations)}, \\
103 &= Q(17,2) = Q(25,9) = Q(35,11) = Q(38,9) = Q(42,11) \\
&\quad \text{(5 representations)}.
\end{aligned}$$

We therefore need to state our general theorem carefully to take this into account. The reason for the numbers of solutions occuring will become apparent in the proof of the theorem.

Theorem 4. *Let $Q(x,y) = ax^2 + bxy + cy^2$ be a positive definite, primitive (i.e. $\gcd(a,b,c) = 1$), binary quadratic form with integer coefficents. Write $D = 4ac - b^2$, so $D > 0$ as the form is positive definite. Let $\delta = \sqrt{D}$, and*

$$\beta = \begin{cases} \frac{1}{2}\pi & \text{if } a + 2b = 0, \\ \arctan(\delta/(b+2a)) & \text{otherwise}, \end{cases}$$

with $\arctan x \in [0, \pi]$. Then, for primes p represented by $Q(x,y)$ with $0 < y < x$ we let x_p, y_p denote a solution to $0 < y_p < x_p, Q(x_p, y_p) = p$. For all other primes we write $x_p = y_p = 0$. When there is more than one pair x_p, y_p we assume all pairs are counted in the expressions that follow. We define x_n, y_n similarly for any positive integer n. We then have

$$R_k = \lim_{N\to\infty} \frac{\sum_{p\le N} x_p^k}{\sum_{p\le N} y_p^k} = \lim_{N\to\infty} \frac{\sum_{n\le N} x_n^k}{\sum_{n\le N} y_n^k} = \frac{S(k)}{T(k)},$$

where

$$S(k) = \int_0^\beta (\delta\cos\theta - b\sin\theta)^k \, d\theta\,,$$

and

$$T(k) = (2a)^k \int_0^\beta \sin^k\theta \, d\theta\,.$$

We give one further corollary to illustrate the general result.

Corollary 2. *Let $n \in \mathbb{Z}, n \ge 2$. Then, in the notation of Theorem 4 with $Q(x,y) = x^2 + ny^2$, we have*

$$R_1 = 1+\sqrt{n+1},$$
$$R_2 = \frac{n(\sqrt{n}+(1+n)\arctan\sqrt{n})}{-\sqrt{n}+(1+n)\arctan\sqrt{n}},$$
$$R_3 = \frac{(3+2n)(2+3n+2(1+n)^{3/2})}{3+4n}.$$

This follows by taking $a=1, b=0, c=n$ in Theorem 4 which gives $\delta = 2\sqrt{n}, \beta = \arctan(\sqrt{n})$. It can also be observed that if $n+1$ is a square, then some of these limits are indeed rational numbers.

Z.W. Sun also made a conjecture about the form $u_p^2+3u_pv_p+v_p^2$ for prime $p \equiv \pm 1 \bmod 5$:

Conjecture 3 (Sun). Let $p \equiv \pm 1 \bmod 5$ be a prime and let $p = u_p^2+3u_pv_p+v_p^2$ be the unique representation with positive integers $u_p > v_p$.

$$\lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv \pm 1 \bmod 5} u_p}{\sum_{p\le N,\, p\equiv \pm 1 \bmod 5} v_p} = 1+\sqrt{5}.$$

He also remarked that it seems that

$$\lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv \pm 1 \bmod 5} u_p^2}{\sum_{p\le N,\, p\equiv \pm 1 \bmod 5} v_p^2} \approx 8.185.$$

While Theorem 4 is only valid for positive definite forms we remark that a formal application of the integrals, with $a=1, b=3, D=\sqrt{-5}$ leads to $S(1) = -3+\sqrt{5}$ and $T(1) = 2-\sqrt{5}$ which predicts indeed

$$\lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv \pm 1 \bmod 5} u_p}{\sum_{p\le N,\, p\equiv \pm 1 \bmod 5} v_p} = 1+\sqrt{5}.$$

Moreover, for $k=2$ we find $S(2) = -i(\sqrt{5}-2\,\mathrm{artanh}(\frac{1}{\sqrt{5}}))$ and $T(2) = -\frac{i}{2}(\sqrt{5}-4\,\mathrm{artanh}(\frac{1}{\sqrt{5}}))$, which predicts

$$\lim_{N\to\infty} \frac{\sum_{p\le N,\, p\equiv \pm 1 \bmod 5} u_p^2}{\sum_{p\le N,\, p\equiv \pm 1 \bmod 5} v_p^2} = \frac{2\sqrt{5}-4\,\mathrm{artanh}(\frac{1}{\sqrt{5}})}{\sqrt{5}-4\,\mathrm{artanh}(\frac{1}{\sqrt{5}})} \approx 8.18483.$$

Now there are several potential difficulties when looking at the problem for indefinite forms. For example, in the most general case there is not the control of variable size in terms of prime size that we have in the positive definite case. At first sight there could be issues working in fields with infinitely many units, in particular how we count the number of representations. Moreover, the results we need from Coleman's work [1, 2] are not explicitly stated for indefinite forms, although careful study shows that he does indeed prove the theorem we require. We thus are able to prove the following result which covers the above conjecture and many other cases.

Theorem 5. *Let $Q(x,y) = ax^2+bxy+cy^2$ be an indefinite, primitive, binary quadratic form with integer coefficents. Write $D = b^2-4ac$, which we assume is not a perfect square. Let $\delta = \sqrt{D}$ and we assume that $0 < \delta/(b+2a) < 1$, for if this condition fails there will be infinitely many representations of a prime by $Q(x,y)$ with $0 < y < x$. Put $\kappa = \operatorname{artanh}(\delta/(b+2a))$ (so this is well-defined by the previous condition). Then, for primes p represented by $Q(x,y)$ with $0 < y < x$ we define x_p, y_p by $0 < y_p < x_p, Q(x_p,y_p) = p$. When there is more than one pair x_p, y_p we assume all pairs are counted in the expressions that follow. For all other primes we write $x_p = y_p = 0$. We define x_n, y_n similarly for any positive integer n. We then have*

$$R_k = \lim_{N\to\infty} \frac{\sum_{p\le N} x_p^k}{\sum_{p\le N} y_p^k} = \lim_{N\to\infty} \frac{\sum_{n\le N} x_n^k}{\sum_{n\le N} y_n^k} = \frac{U(k)}{V(k)},$$

where

$$U(k) = \int_0^{\kappa} (\delta\cosh\theta - b\sinh\theta)^k \, d\theta\,,$$

and

$$V(k) = (2a)^k \int_0^{\kappa} \sinh^k\theta \, d\theta\,.$$

Remark 3. We will indicate in the proof how one gets multiple representations of primes and why representations are unique for some forms like $u_p^2+3u_pv_p+v_p^2$.

We give a corollary working out these integrals for a particular family:

Corollary 3. *Let $n \in \mathbb{Z}, b \ge 3$. Then, in the notation of Theorem 5 with $Q(x,y) = x^2+bxy+y^2$, we have*

$$R_1 = 1+\sqrt{b+2}.$$

$$R_2 = \frac{(b-1)\sqrt{b^2-4} - 4\operatorname{artanh}\sqrt{(b-2)/(b+2)}}{\sqrt{b^2-4} - 4\operatorname{artanh}\sqrt{(b-2)/(b+2)}}.$$

2 Proofs: The Simplest Case

We begin by quoting a simple consequence of Coleman's work for Gaussian primes which we will use to prove Theorem 1 directly and which motivated the whole investigation. The reader will see that the proof of Theorem 4 is a straightforward generalisation of this, though the necessary terminology may at first make it look more complicated.

Lemma 1 (Coleman, Theorem 2.1 of [1]). *Let $0 \le \varphi_1 \le \varphi_2 \le 2\pi$, and $0 \le y \le x$. We can define $S = S(x,y,\varphi_1,\varphi_2) = \{\mathbf{z} \in \mathbb{Z}[i] : x-y < |\mathbf{z}|^2 \le x, \varphi_1 \le \varphi \le \varphi_2\}$, where $\varphi = \arg(\mathbf{z}/|\mathbf{z}|)^4$. Let $P = P(x,y,\varphi_1,\varphi_2) = \{\mathbf{p} \in S : |\mathbf{p}|^2 = p,\ prime\}$.*

Let $\varepsilon > 0$ be given. We have the asymptotic result,

$$\sum_{\mathbf{p}\in P(x,y,\varphi_1,\varphi_2)} 1 = \frac{(\varphi_2-\varphi_1)y}{2\pi\log x}\left(1+O\left(\frac{1}{\log x}\right)\right),$$

for $\varphi_2-\varphi_1 > x^{-5/24+\varepsilon}$, $y > x^{19/24+\varepsilon}, x > x_\varepsilon$.

Let us first outline the idea: for asymptotically evaluating $\sum_{p\le N, p\equiv 1 \bmod 4} a_p^k$ it suffices to dissect the sector with radius N from $0<\varphi<\pi/4$ into polar boxes. Coleman's result says that one can dissect this into fine (but not too fine) boxes, so that the number of primes, corresponding to $p=a^2+b^2$, is asymptotically the right number, with some error of smaller order. We can therefore replace summation by integration with negligible error. The same is true for the sum in the denominator. Hence, we get:

$$\lim_{N\to\infty}\frac{\sum_{p\le N, p\equiv 1 \bmod 4} a_p^k}{\sum_{p\le N, p\equiv 1 \bmod 4} b_p^k} = \frac{\int_0^{\pi/4}\cos^k(x)\,dx}{\int_0^{\pi/4}\sin^k(x)\,dx}.$$

The reader should note that this is exactly the same relation we would get if p ran over all numbers representable as the sum of two squares.

Proof (Proof of Theorem 1). We now describe this in more detail. We write $T = 2[\log_2 N], \nu=\sqrt{N}$. Let us first concentrate on Gaussian primes with modulus in the interval $(\frac{\nu}{2},\nu]$, we later sum up over intervals of type $(\frac{\nu}{2^i},\frac{\nu}{2^{i-1}}]$. For $1\le s\le 3T/4, 1\le t\le T/2$ we then define polar boxes

$$\begin{aligned} B_{s,t} &= \{\mathbf{a}=re^{i\varphi}: 1-s/T < (r/\nu)^2 \le 1-(s-1)/T, (t-1)/T < 2\varphi/\pi \le t/T\} \\ &= \left\{\mathbf{a}=re^{i\varphi}: \sqrt{N}\sqrt{1-s/T} < r \le \sqrt{N}\sqrt{1-(s-1)/T}, \frac{\pi(t-1)}{2T} < \varphi \le \frac{\pi t}{2T}\right\}. \end{aligned}$$

We note that a polar box $\{(r,\theta): R_2^2 < r \le R_1^2, \theta' \le \theta \le \theta'+\phi\}$ has area $\frac{1}{2}(R_1^2 - R_2^2)\phi$. It follows (something we will need later when we convert sums to integrals) that the box $B_{s,t}$ has area $N\pi/(4T^2)$. Write $\eta=(\log N)^{-1}$. Now $\log M = \log N + O(1)$ for $N/4 < M \le N$. Hence, by Lemma 1 (note though that the corresponding polar box in that lemma has an angle four times that of $B_{s,t}$) for each pair s,t we have

$$\sum_{\mathbf{p}\in B_{s,t}} 1 = \frac{N}{T^2\log N}\left(1+O(\eta)\right).$$

In each polar box $\cos\varphi = \cos(t\pi/2T)+O(\eta)$ and similarly for $\sin\varphi$. Also $r = \nu(1-s/T)^{\frac{1}{2}} + O(\nu\eta)$. Hence

$$\begin{aligned}\sum_{\frac{N}{4}<p\le N, p\equiv 1 \bmod 4} a_p^k &= \sum_{s,t}\sum_{\mathbf{p}\in B_{s,t}} a_p^k \\ &= (1+O(\eta))\sum_{s,t}\frac{N}{T^2\log N}\Big(\nu(1-s/T)^{\frac{1}{2}}\cos(t\pi/2T)+O(\eta\nu)\Big)^k \\ &= \frac{4}{\pi}(\eta+O(\eta^2))\sum_{s,t}\int\int_{re^{ix}\in B_{s,t}}(r\cos x+O(\nu\eta))^k r\,dr\,dx \\ &= \frac{4}{\pi}(\eta+O(\eta^2))\int_{\nu/2}^{\nu} r^{k+1}\,dr\int_0^{\pi/4}\cos^k x\,dx.\end{aligned}$$

We remark that it is easy to check the case $k=0$ of the above which must give the number of primes $\equiv 1 \bmod 4$ between $N/4$ and N. This is $\frac{3}{8}N\eta(1+O(\eta))$ and equals the final line of the above display by an elementary calculation.

Similarly,

$$\sum_{\frac{N}{4}<p\le N, p\equiv 1 \bmod 4} b_p^k = \frac{4}{\pi}(\eta+O(\eta^2))\int_{\nu/2}^{\nu} r^{k+1}\,dr\int_0^{\pi/4}\sin^k x\,dx.$$

Adding up over the intervals $(\frac{\nu}{2^i},\frac{\nu}{2^{i-1}}]$, and cancelling common factors one finds that

$$\lim_{N\to\infty}\frac{\sum_{p\le N, p\equiv 1 \bmod 4} a_p^k}{\sum_{p\le N, p\equiv 1 \bmod 4} b_p^k} = \frac{\int_0^{\pi/4}\cos^k(x)\,dx}{\int_0^{\pi/4}\sin^k(x)\,dx}.$$

Proof (Proof of Corollary 1). We now evaluate these integrals, for small exponents, and determine the arithmetic nature of the values. It is well known that

$$\begin{aligned}\int_0^{\pi/4}\sin^n(x)\,dx &= -\frac{\sin^{n-1}x\cos x}{n}\Big|_0^{\pi/4}+\frac{n-1}{n}\int_0^{\pi/4}\sin^{n-2}x\,dx \\ &= -\frac{1}{n2^{n/2}}+\frac{n-1}{n}\int_0^{\pi/4}\sin^{n-2}x\,dx,\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/4}\cos^n(x)\,dx &= \frac{\cos^{n-1}x\sin x}{n}\Big|_0^{\pi/4}+\frac{n-1}{n}\int_0^{\pi/4}\cos^{n-2}x\,dx \\ &= \frac{1}{n2^{n/2}}+\frac{n-1}{n}\int_0^{\pi/4}\cos^{n-2}x\,dx.\end{aligned}$$

We note for the initial values $n=0$ and $n=1$ that

$$\int_0^{\pi/4} \sin^0(x)\,dx = \frac{\pi}{4},$$
$$\int_0^{\pi/4} \sin^1(x)\,dx = 1 - \frac{1}{\sqrt{2}},$$
$$\int_0^{\pi/4} \cos^0(x)\,dx = \frac{\pi}{4},$$
$$\int_0^{\pi/4} \cos^1(x)\,dx = \frac{1}{\sqrt{2}}.$$

From this it is easy, to calculate for small k the limit by hand. For example, let $k = 5$.

$$\int_0^{\pi/4} \sin^3(x)\,dx = -\frac{1}{3\cdot 2^{3/2}} + \frac{2}{3}(1 - \frac{1}{\sqrt{2}}) = \frac{1}{12}(8 - 5\sqrt{2}).$$

$$\int_0^{\pi/4} \sin^5(x)\,dx = -\frac{1}{5\cdot 2^{5/2}} + \frac{4}{5}\cdot\frac{1}{12}(8 - 5\sqrt{2}) = \frac{1}{120}(64 - 43\sqrt{2}).$$

$$\int_0^{\pi/4} \cos^3(x)\,dx = \frac{1}{3\cdot 2^{3/2}} + \frac{2}{3}\frac{1}{\sqrt{2}} = \frac{5\sqrt{2}}{12}.$$

$$\int_0^{\pi/4} \cos^5(x)\,dx = \frac{1}{5\cdot 2^{5/2}} + \frac{4}{5}\cdot\frac{5\sqrt{2}}{12} = \frac{43\sqrt{2}}{120}.$$

Hence

$$I_5 = \lim_{N\to\infty} \frac{\sum_{p\le N, p\equiv 1 \bmod 4} a_p^5}{\sum_{p\le N, p\equiv 1 \bmod 4} b_p^5} = \frac{\frac{43\sqrt{2}}{120}}{\frac{1}{120}(64 - 43\sqrt{2})} = \frac{43\sqrt{2}}{64 - 43\sqrt{2}}.$$

It is clear from the recursion formulae above, that for odd k, $\int_0^{\pi/4} \cos^k x\,dx = r_1\sqrt{2}$, and $\int_0^{\pi/4} \sin^k x\,dx = r_2 + r_3\sqrt{2}$, where $r_1, r_2, r_3 \in \mathbb{Q}\backslash\{0\}$. It follows that $I_k \in \mathbb{Q}(\sqrt{2}) \setminus \mathbb{Q}$. Similarly, as remarked above, when $k = 2\ell$ we find that $I_k = (A_\ell\pi + B_\ell)/(A_\ell\pi - B_\ell)$ with integers $A_\ell, B_\ell > 0$, so the value of $I(k)$ is a rational expression of π, where the π can never cancel, hence $I(k)$ is transcendental, when $k \ge 2$ is even.

Proof (Proof of Theorem 2). We note that

$$\int_0^{\pi/4} \cos^n(x)\,dx = \int_0^{\pi/2} \cos^n(x)\,dx + O\left(2^{-n/2}\right).$$

For n even, say $n = 2m$, we have by the recursion formula

$$\begin{aligned}\int_0^{\pi/2} \cos^n(x)\,dx &= \frac{\pi}{2}\frac{(2m-1)(2m-3)\dots 3}{2m(2m-2)\dots 2}\\ &= \frac{\pi}{2}\frac{(2m)!}{(2^m m!)^2}\\ &\sim \frac{\pi}{2}\frac{(2m/e)^{2m}\sqrt{4\pi m}}{2^{2m}(m/e)^{2m}2\pi m}\\ &\sim \left(\frac{\pi}{2n}\right)^{\frac{1}{2}}.\end{aligned}$$

In the above we have used Stirling's formula to obtain asymptotic formulae for $m!$ and $(2m)!$. Similarly, for odd n we have

$$\int_0^{\pi/2} \cos^n(x)\,dx \sim \left(\frac{\pi(1+n)}{2n^2}\right)^{\frac{1}{2}} \sim \left(\frac{\pi}{2n}\right)^{\frac{1}{2}}.$$

On the other hand, we note that

$$\sin(\pi/4-x) = \frac{1}{\sqrt{2}}\left(1-x+O(x^2)\right).$$

Hence the natural logarithm of the function $e^{kx}2^{k/2}(\sin(\pi/4-x)^k$ is

$$\tfrac{k}{2}\log 2 + kx + k\log(\sin(\pi/4-x)) = O(kx^2).$$

From this we easily obtain

$$(\sin(\pi/4-x))^k = 2^{-k/2}e^{-kx}\left(1+O\left(kx^2\right)\right).$$

Hence, writing $\lambda = \pi/4 - k^{-2/3}, \mu = k^{1/3}$, we have

$$\begin{aligned}\int_0^{\pi/4} \sin^k(x)\,dx &= \int_0^{\lambda} \sin^k(x)\,dx + \int_{\lambda}^{\pi/4} \sin^k(x)\,dx\\ &= O\left(2^{-k/2}e^{-\mu}\right) + \frac{2^{-k/2}}{k}\left(1+O\left(\mu^{-1}\right)\right).\end{aligned}$$

Combining the above results gives (1) as desired.

3 Proofs: The General Positive Definite Case

A number of authors, including Landau, Hecke [3, 4], Rademacher [11], Kubilius [9], Kalnin′ [6] and Coleman [1, 2] established results with regard to equidistribution of prime ideals in certain regions. We shall here need to work with the distribution of prime ideals and the 1-1 correspondence that exists between these ideals and the representation of their norms by poitive definite quadratic forms. A further complication arises that the discriminant of the quadratic form may not be a fundamental discriminant (those given in the next paragraph) and this forces us to use a more general result.

We must now define the notation needed to state Coleman's theorem in its full generality and we will quote this more or less verbatim from [1]. Let $\mathbb{Q}(\sqrt{\Delta})$ be the imaginary quadratic number field with discriminant Δ or 4Δ depending on whether or not $\Delta \equiv 1 \bmod 4$, respectively, and Δ is a negative square-free integer. We use gothic letters $\mathfrak{a}, \mathfrak{f}$ to denote ideals and $\mathfrak{p}$ will represent a prime ideal. We write $N(\mathfrak{a})$ for the norm of $\mathfrak{a}$. Given a non-zero ideal $\mathfrak{f}$, let $g = g(\mathfrak{f})$ be the number of units ε such that $\varepsilon \equiv 1 \bmod \mathfrak{f}$. We write K for an ideal class $\bmod \mathfrak{f}$ and (ξ) for the principal ideal generated by an algebraic integer ξ. For each such class we assume there has been chosen and fixed an ideal $\mathfrak{a}_0 \in K^{-1}$. Then given $\mathfrak{a} \in K$ we can define $\xi_{\mathfrak{a}} \in \mathfrak{a}_0$ by $(\xi_{\mathfrak{a}}) = \mathfrak{a}\mathfrak{a}_0$ and $\xi_{\mathfrak{a}} \equiv 1 \bmod \mathfrak{f}$. This algebraic integer is unique up to multiplication by the units $\varepsilon \equiv 1 \bmod \mathfrak{f}$. We write

$$\lambda(\xi_{\mathfrak{a}}) = \left(\frac{\xi_{\mathfrak{a}}}{|\xi_{\mathfrak{a}}|}\right)^g.$$

By the definition of $g, \arg(\lambda(\xi_{\mathfrak{a}}))$ is unique $\bmod 2\pi$.

Lemma 2 (Coleman, Theorem 2.1 of [1]). *Given $0 \le \varphi_1 \le \varphi_2 \le 2\pi$, $0 \le y \le x$ and an ideal class K $\bmod \mathfrak{f}$. We define $S = S(x,y,\varphi_1,\varphi_2,K) = \{\mathfrak{a} \in K : x-y < N(\mathfrak{a}) \le x, \varphi_1 \le \arg(\lambda(\xi_{\mathfrak{a}})) \le \varphi_2\}$. Let $P = P(x,y,\varphi_1,\varphi_2,K) = \{\mathfrak{p} \in S : N(\mathfrak{p}) = p$, prime$\}$.*

Let $\varepsilon > 0$ be given. We have the asymptotic result,

$$\sum_{\mathfrak{p} \in P(x,y,\varphi_1,\varphi_2,K)} 1 = \frac{(\varphi_2 - \varphi_1)y}{2\pi h(\mathfrak{f}) \log x}\left(1 + O\left(\frac{1}{\log x}\right)\right),$$

for $\varphi_2 - \varphi_1 > x^{-5/24+\varepsilon}$, $y > x^{19/24+\varepsilon}, x > x_\varepsilon$. Here $h(\mathfrak{f})$ is the order of the abelian group of ideal classes $\bmod \mathfrak{f}$.

Proof (Proof of Theorem 4). The proof that

$$\lim_{N\to\infty} \frac{\sum_{n\le N} x_n^k}{\sum_{p\le N} y_n^k} = \frac{S(k)}{T(k)},$$

follows by applying a simple change of variable to map the region

$$0 < y < x, \quad Q(x,y) < N$$

onto the sector

$$0 < r < N^{\frac{1}{2}}, \quad 0 < \theta < \beta.$$

This change of variable will underlie the proof of the more difficult case for primes

$$\lim_{N\to\infty} \frac{\sum_{p\le N} x_p^k}{\sum_{p\le N} y_p^k} = \frac{S(k)}{T(k)},$$

so we shall concentrate on establishing this.

Let $\Delta = -D$, and we suppose initially that this is a fundamental discriminant with $\Delta < -4$. Hence there are just the two units $-1, 1$ in $\mathbb{Q}(\sqrt{\Delta})$. We follow Coleman's argument based on [7] to obtain the 1-1 correspondence between prime ideals and points at which $Q(m,n)$ is prime. In Lemma 1 we take

$$\mathfrak{a}_0 = \left[a, \tfrac{1}{2}(b - i\delta)\right] .$$

The 1-1 correspondence is then

$$N(\mathfrak{p}) = p = Q(m,n)$$

with

$$\xi_{\mathfrak{p}} = ma + n\left(\frac{b - i\delta}{2}\right) .$$

We must remark at this stage that this correspondence is 1-1 between prime ideals and m,n and not necessarily between p and m,n. This is why we may get two repesentations if two distinct prime ideals with norm p fall within the sector we are about to describe (which can be larger than the first quadrant if $b < -2a$).

We note that $|\xi_{\mathfrak{p}}|^2 = ap$. So, if we write

$$\phi = \arg(\xi_{\mathfrak{p}}), \quad r = \sqrt{p},$$

we have

$$m = \frac{r}{\sqrt{a}\delta}\left(\delta\cos\phi + b\sin\phi\right), \qquad n = -2a\frac{r}{\sqrt{a}\delta}\sin\phi .$$

Now writing $\theta = -\phi$ the condition $0 < n < m$ translates to $0 < \theta < \beta$ with

$$m = \frac{r}{\sqrt{a}\delta}\left(\delta\cos\theta - b\sin\theta\right), \qquad n = 2a\frac{r}{\sqrt{a}\delta}\sin\theta .$$

We divide the region $0 < r < N^{\frac{1}{2}}, 0 < \theta < \beta$ into polar boxes as in the proof of Theorem 1 and apply Lemma 2 to the corresponding regions. As in the proof of Theorem 1 we can convert the sums to integrals with smaller order errors. When we divide one sum by the other the integrals over r cancel, as do various constants and the $1/\log N$ factor, leaving just $S(k)/T(k)$ as claimed.

Now if $\Delta = -3$ or -4 we have 6 or 4 units respectively. We recall that $\xi_{\mathfrak{p}}$ is only unique up to multiplication by units. In the above argument we can have up to two distinct prime ideals and $\xi_{\mathfrak{p}}$ could be multiplied by 3 or 2 units leading to different values m,n and still remain within the sector under consideration. This leads to the multiple representations. Of course, for Theorem 1 $\kappa = \pi/4$ constrains $\xi_{\mathfrak{p}}$ to one value, and similarly for Theorem 3 $\kappa = \pi/6$, leading to one value. For the form $x^2 - 7xy + 13y^2$ we have $\Delta = -3$, $\kappa = \arctan(-\sqrt{3}/5) = 2.808\ldots$. We note that $5\pi/6 < \kappa < \pi$ and so there will be either 5 or 6 representations depending on whether or not all 6 possible values for ξ lie in the sector.

Now suppose that $D = -f^2\Delta$ with Δ a fundamental discriminant and continue working in $\mathbb{Q}(\sqrt{\Delta})$. If we repeat the above argument then we would require

$$\xi_{\mathfrak{p}} = ma + n\left(\frac{b - if\delta}{2}\right).$$

We therefore need to restrict ourselves to counting those prime ideals which lead to such $\xi_{\mathfrak{p}}$. This corresponds to restricting the prime ideals to a union of ideal classes mod $\mathfrak{f}$ for a suitable $\mathfrak{f}$. We give the simplest case by way of illustration. Let $Q(x,y) = x^2 + 4y^2$, so $D = 16 = f^2(-\Delta)$ with $f = 2$. Let $\mathfrak{f} = (2)$. Then there are just two ideal classes coprime to $\mathfrak{f}$: $\{(u+vi) : v \text{ even}, u \text{ odd}\}$, $\{(u+vi) : u \text{ even}, v \text{ odd}\}$. Counting only prime ideals in the class with v even will then give $p = x^2 + 4y^2$ as required.

4 The Indefinite Case

We first consider how to describe the geometry in the real case. The natural embedding from $\mathbb{Q}(\delta)$ into $\mathbb{R}^2$ is $a + b\delta \longrightarrow (a, b\delta)$. We write ζ' for the algebraic conjugate of $\zeta \in \mathbb{Q}(\delta)$. We shall discover that the polar boxes of the imaginary case give way to hyperbolic boxes in this new situation. We then note that there is an analogous correspondence between prime ideals and points at which $Q(m,n)$ is prime. However we now need to be much more careful about the number of points (m,n) corresponding to each prime ideal. We need only deal with the fundamental discriminant case as the adaptation to the general case follows as previously. Now we write

$$\begin{aligned}
\xi_{\mathfrak{p}} &= ma + n\left(\frac{b-\delta}{2}\right) \\
&= u + v\delta \quad \text{where } u^2 - Dv^2 = r^2 a \\
&= \sqrt{a}r(\cosh\phi + \sinh\phi) \quad \text{for a uniquely defined value } \phi.
\end{aligned}$$

Note that this gives a 1-1 corespondence between $\xi_{\mathfrak{p}}$ and (m,n). Following the argument of the previous case, if we write $\theta = -\phi$ then $0 < m < n$ becomes

$$m = \frac{r}{\sqrt{a}\delta}\left(\delta\cosh\theta - b\sinh\theta\right), \qquad n = 2a\frac{r}{\sqrt{a}\delta}\sinh\theta,$$

with $0 < \theta < \kappa$. We thus have a hyperbolic box

$$\{(r\cosh\theta, r\sinh\theta) : 0 \le r \le N^{\frac{1}{2}}, 0 < \theta < \kappa\},$$

which we dissect into small hyperbolic boxes in a corresponding manner to our earlier discussion. In particular, where $\xi_{\mathfrak{a}}$ occurs we define $t(\xi_{\mathfrak{a}})$ to be the unique value t such that

$$\xi_{\mathfrak{a}} = \sqrt{N(\mathfrak{a})}(\cosh t + \sinh t) = \sqrt{N(\mathfrak{a})}e^t\,.$$

Now $\xi_{\mathfrak{p}}$ is only unique up to multiplication by units. Let ε_0 be the fundamental unit of $\mathbb{Q}(\delta)$. Multiplying $\xi_{\mathfrak{p}}$ by an even power of ε_0 gives another candidate for $\xi_{\mathfrak{p}}$ but shifts ϕ by $2\log\varepsilon_0$. Hence the maximum number of additional representations of p from this ideal will be strictly less than

$$\frac{\kappa}{2\log\varepsilon_0}\,. \tag{2}$$

However, there is the ideal containing the algebraic conjugate of $\xi_{\mathfrak{p}}$, that is $\xi_{\mathfrak{p}}'$, to consider. Possibly multiplying $\xi_{\mathfrak{p}}'$ by an even power of ε_0 will give a number in the required range. So, writing $t = t(\xi_{\mathfrak{p}})$, we would need $0 < -t < \kappa$ and $0 < t + 2h\log\varepsilon_0 < \kappa$. In the case of $x^2+3xy+y^2$ the value of (2) is exactly $1/2$ and so the representation is unique. On the other hand, for x^2+xy-y^2 the value is 1. There can therefore be no more representations from the original ideal, but if $0 < -t < \kappa$ then we obtain $0 < t + 2\log\varepsilon_0 < \kappa$ giving exactly one other representation. Of course, it is easy to see there will be two representations for this form as the transformation $(x,y) \longrightarrow (x, x-y)$ leaves the expression unchanged. If $0 < \delta/(2a+b) < 1$ fails, then the only restriction on θ is $\theta > 0$ and so we obtain infinitely many solutions by multiplying $\xi_{\mathfrak{p}}$ by any positive even power of ε_0. That is why we cannot prove a result of the required form in this case.

We must now show that Coleman's work in [2] supplies us with the required formula for prime ideals in hyperbolic boxes as above. There Theorem 2 is his main theorem. We remark that Hensley [5, §5] also gives an explicit account for how to count prime ideals in hyperbolic boxes in the case of real quadratic fields. We must first describe Hecke characters in $\mathbb{Q}(\delta)$, and we quote Hecke's original definition almost verbatim from [4]. Given an integer ρ of $\mathbb{Q}(\delta)$ we define

$$\lambda(\rho) = \exp\left(\frac{i\pi}{\log\varepsilon_0}\log\left|\frac{\rho}{\rho'}\right|\right).$$

This is clearly a multiplicative function of ρ which takes the value 1 at all units (as they are powers of ε_0, and dividing a unit by its conjugate gives an even power of ε_0). The Hecke characters are then all integer powers of $\lambda(\cdot)$. If $N(\rho) = r$, and $\rho = r(\cosh t + \sinh t)$, then

$$\rho' = r(\cosh t - \sinh t), \quad \frac{\rho}{\rho'} = e^{2t}\,.$$

We can therefore use the Hecke characters to pick out the condition $\sigma < t < \sigma + \tau$. Since we can investigate the size of the norm using $N(\rho)^s (s \in \mathbb{C})$ just as in the imaginary case, we obtain the following result from [2, Theorem 2]. We continue to use the terminology stated before Lemma 2.

Lemma 3. *Let $\varepsilon > 0$ be given. Given $0 \le \varphi_0 < \varphi_0 + \tau \le 1 < x$, $x^{-1/5+\varepsilon} \le \tau < 1$ and an ideal class K* mod $\mathfrak{f}$. *We define $S = S(x, \varphi_0, \tau, K) = \{\mathfrak{a} \in K : x(1-\tau) < N(\mathfrak{a}) \le$*

$x, \varphi_0 \le t(\xi_{\mathfrak{a}}) \le \varphi_0 + \tau\}$. *Let* $P = P(x, \varphi_0, \tau, K) = \{\mathfrak{p} \in S : N(\mathfrak{p}) = p,$ *prime*$\}$. *We have the asymptotic result,*

$$\sum_{\mathfrak{p} \in P(x,\varphi_0,\tau,K)} 1 = \frac{x\tau^2}{h(\mathfrak{f}) \log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Here $h(\mathfrak{f})$ *is the order of the abelian group of ideal classes* mod $\mathfrak{f}$.

This supplies us with precisely the formula we need for counting prime ideals in a hypebolic box and the proof can then be easily completed.

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