PREOPEN SETS AND RESOLVABLE SPACES *

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Abstract

This paper presents solutions to some recent questions raised by Katetov about the collection of preopen sets in a topological space.

1 Introduction

In a recent paper Mashour et al. [5] presented and studied a series of questions raised by Katetov about the collection of preopen sets in a topological space. These questions are as follows:

K1: Find necessary and sufficient conditions under which every preopen set is open.

K2: Find conditions under which every dense-in-itself set is preopen.

K3: Find conditions under which the intersection of any two preopen sets preopen.

K4: If (X, τ) is a topological space, let τ^* denote the topology on X obtained by taking the collection of all preopen sets of (X, τ) as a subbase. Find conditions under which τ^* is discrete.

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Reilly and Vamanamurthy [7] continued the study of these questions and obtained a complete solution to $K1$ as well as partial solutions to the other questions. The purpose of the present paper is to provide complete solutions to $K3$ and $K4$, thus generalizing some results obtained in [7]. Furthermore, a solution to $K2$ is given.

2 Preliminaries

For any subset A of a topological space (X, τ) we denote the closure of A resp. the interior of A with respect to τ by $\overline{A}$ resp. $intA$. The relative topology on a subset Y of (X, τ) is denoted by $\tau|Y$. If A is a subset of $Y \subseteq X$, the closure of A resp. the interior of A with respect to $\tau|Y$ is denoted by $\overline{A}^Y$ resp. $int_Y A$.

Definition 1 A subset S of (X, τ) is called

(i) an *α–set*, if $S \subseteq int\ \overline{intS}$,

(ii) a *semi–open set*, if $S \subseteq \overline{intS}$,

(iii) a *preopen* set, if $S \subseteq int\overline{S}$.

These notions were introduced by Njastad [6], Levine [4] and Mashhour et al. [5], respectively. We denote the family of all α–sets in (X, τ) by τ^α . Njastad [6] has shown that τ^α is a topology on X satisfying $\tau \subseteq \tau^\alpha$. Reilly and Vamanamurthy [8] proved that a subset of a space (X, τ) is an α–set if and only if it is semi–open and preopen. The families of all semi–open sets and of all preopen sets in (X, τ) are denoted by $SO(X, \tau)$ and $PO(X, \tau)$, respectively. In general one cannot expect the families $SO(X, \tau)$ or $PO(X, \tau)$ to be topologies. Njastad [6] has proved that $SO(X, \tau)$ is a topology if and only if (X, τ) is extremally disconnected. Let us observe that both $SO(X, \tau)$ and $PO(X, \tau)$ are closed under forming arbitrary unions. Hence $PO(X, \tau)$ is a topology if and only if the intersection of any two preopen sets is preopen.

The following simple characterization of preopen sets turns out to be very useful.

Proposition 2.1 For any subset of a space (X, τ) the following are equivalent:

(i) $S \in PO(X, \tau)$.

(ii) There is a regular open set $G \subseteq X$ such that $S \subseteq G$ and $\overline{S} = \overline{G}$.

(iii) S is the intersection of a regular open set and a dense set.

(iv) S is the intersection of an open set and a dense set.

Proof. (i) $\Rightarrow$ (ii) : Let $S \in PO(X, \tau)$, i.e. $S \subseteq int\overline{S}$. Let $G = int\overline{S}$. Then G is regular open with $S \subseteq G$ and $\overline{S} = \overline{G}$.

(ii) $\Rightarrow$ (iii) : Suppose (ii) holds. Let $D = S \cup (X \setminus G)$. Then D is dense and $S = G \cap D$.

(iii) $\Rightarrow$ (iv) : This is trivial.

(iv) $\Rightarrow$ (i) : Suppose $S = G \cap D$ with G open and D dense. Then $\overline{S} = \overline{G}$, hence $S \subseteq G \subseteq \overline{G} = \overline{S}$. $\square$

Obviously open sets and dense sets in (X, τ) are preopen. Proposition 2.1 immediately shows that the intersection of an open set and a preopen set is preopen again. Moreover, since $PO(X, \tau) = PO(X, \tau^\alpha)$ (see Proposition 2.2 below), the intersection of an α–set and any preopen set is preopen.

The following result is due to Jankovic [3].

Proposition 2.2 Let (X, τ) be a topological space. Then

(i) $(\tau^\alpha)^\alpha = \tau^\alpha$,

(ii) $SO(X, \tau) = SO(X, \tau^\alpha)$,

(iii) $PO(X, \tau) = PO(X, \tau^\alpha)$.

Our next result concerning the existence of non–preopen sets can be easily shown. The proof is hence ommitted.

Proposition 2.3 Let (X, τ) be a topological space, $S \in PO(X, \tau)$ and $x \in \overline{S} \setminus int\overline{S}$. Then $S \cup \{x\} \notin PO(X, \tau)$. In particular, if G is regular open in (X, τ) and $x \in \overline{G} \setminus G$ then $G \cup \{x\} \notin PO(X, \tau)$.

3 Resolvable and irresolvable spaces

A space (X,τ) is called *resolvable* if there is a subset D of X such that D and $X \setminus D$ are both dense in (X,τ) . A subset A of X is resolvable if the subspace $(A,\tau|A)$ is resolvable. A space (X,τ) is called *irresolvable* if it is not resolvable. Resolvable spaces have been studied in a paper of Hewitt [2] in 1943. Clearly any resolvable space is dense-in-itself and each open subspace of a resolvable space is also resolvable. A space (X,τ) is said to be *hereditarily irresolvable* if does not contain a nonempty resolvable subset. Dense-in-itself hereditarily irresolvable spaces have been called *SI–spaces* by Hewitt [2] .

Recall that (X,τ) is said to be *submaximal* if each dense subset is open. Reilly and Vamanamurthy ([7], Theorem 4) have shown that a space (X,τ) is submaximal if and only if $PO(X,\tau) = \tau$, thus answering question $K1$. An *MI–space* is a dense-in-itself submaximal space. We have the following implications :

submaximal $\Rightarrow$ hereditarily irresolvable $\Rightarrow$ irresolvable

$MI \Rightarrow$ submaximal

$SI \Rightarrow$ hereditarily irresolvable.

Theorem 3.1 [2] Every topological space (X,τ) can be represented as a disjoint union $X = F \cup G$ where F is closed and resolvable and G is open and hereditarily irresolvable. (X,τ) is resolvable if and only if $G = \emptyset$, and (X,τ) is hereditarily irresolvable if and only if $F = \emptyset$.

Note that it is not necessary to restrict ourselves to the class of dense-in-itself spaces as Hewitt [2] did. In addition, one easily checks that the representation of (X,τ) given by Theorem 3.1 us unique. It will henceforth be called the *Hewitt representation* of (X,τ) .

In 1969, El'kin [1] studied the class of dense-in-itself irresolvable spaces using ultrafilters. It is easily seen that Lemma 1 in [1] is also valid if the hypothesis "dense-in-itself" is dropped. Thus we have

Theorem 3.2 For a space (X,τ) the following are equivalent.

(i) (X,τ) contains an open, dense and hereditarily irresolvable subspace.

(ii) Every open ultrafilter on X is a base for an ultrafilter on X .

(iii) Every nonempty open set is irresolvable.

(iv) For each dense subset D of (X,τ) , $int\ D$ is dense.

(v) For every $A \subseteq X$, if $int\ A = \emptyset$ then A is nowhere dense.

As a consequence of Theorem 3.2 we obtain a characterization of irresolvable spaces which seems to be new.

Theorem 3.3 A space (X,τ) is irresolvable if and only if $\{int\ D : \overline{D} = X\}$ is a filterbase on X .

Proof. Suppose that (X,τ) is resolvable. Then there exist disjoint dense sets $D, E \subseteq X$ such that $X = D \cup E$. Hence $int\ D = \emptyset$, a contradiction.

To prove the converse, let (X,τ) be irresolvable and let $X = F \cup G$ be the Hewitt representation of (X,τ) . Then G is nonempty. If D_1 and D_2 are both dense in (X,τ) , then $D_1 \cap G$ and $D_2 \cap G$ are dense in G , hence nonempty. By Theorem 3.2 (iv) , $G \cap int(D_1 \cap D_2)$ is dense in G . Since F is resolvable, we have $intF = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$ and $intF \subseteq \overline{E_1} \cap \overline{E_2}$. Let $D_3 = G \cap int(D_1 \cap D_2) \cup E_1$. Then D_3 is clearly dense in (X,τ) . Let us show that $intD_3 \subseteq G \cap int(D_1 \cap D_2)$. Let $x \in intD_3$. If $x \in G \cap int(D_1 \cap D_2)$ we are done. If $x \in E_1$, then there is an open neighbourhood U of x such that $U \subseteq F$ and $U \subseteq D_3$. Hence $U = U \cap F \subseteq E_1$, thus $U \cap (X \setminus E_1) = \emptyset$. Since $E_2 \subseteq X \setminus E_1$ we have $U \cap \overline{E_2} = \emptyset$ and thus $U \cap intF = \emptyset$, a contradiction. It follows that $intD_3 \subseteq intD_1 \cap intD_2$ proving that $\{int\ D : \overline{D} = X\}$ is a filterbase on X . □

4 Results

In this section we present solutions to $K2$, $K3$ and $K4$ and study some consequences of the results obtained.

Theorem 4.1 For a space (X, τ) the following are equivalent:

(i) (X, τ) contains an open, dense and hereditarily irresolvable subspace,

(ii) $PO(X, \tau) \subseteq SO(X, \tau)$,

(iii) $\tau^\alpha = PO(X, \tau)$,

(iv) (X, τ^α) is submaximal.

Proof. (i) $\Rightarrow$ (ii) : Let $S \in PO(X, \tau)$. By Proposition 2.1, $S = G \cap D$ where G is open and D is dense. By Theorem 3.2 (iv), $intD$ is dense. Hence $\overline{intD} = \overline{intD \cap G} = \overline{G}$, consequently $S \subseteq G \subseteq \overline{G} = \overline{intS}$. Thus $S \in SO(X, \tau)$.

(ii) $\Rightarrow$ (iii) : Obvious, since $\tau^\alpha = SO(X, \tau) \cap PO(X, \tau)$ by a result of Reilly and Vamanamurthy [8] .

(iii) $\Leftrightarrow$ (iv) : Follows from Proposition 2.2 and Theorem 4 in [7] .

(iii) $\Rightarrow$ (i) : Let $D \subseteq X$ be dense. Then $D \in PO(X, \tau) = \tau^\alpha \subseteq SO(X, \tau)$. Consequently, $D \subseteq \overline{intD}$, hence $intD$ is dense. By Theorem 3.2 , (X, τ) contains an open, dense and hereditarily irresolvable subspace. □

Corollary 4.2 If (X, τ) contains an open, dense and hereditarily irresolvable subspace, then $PO(X, \tau)$ is a topology. In fact, $PO(X, \tau) = \tau^\alpha$.

The following lemma will be useful in the sequel. Its proof is straightforward.

Lemma 4.3 Let H be an open subset of a space (X, τ) and let $S \subseteq H$. Then $S \in PO(X, \tau)$ if and only if $S \in PO(H, \tau | H)$.

As already pointed out, in proving that $PO(X, \tau)$ is a topology it suffices to show that the intersection of any two preopen sets is preopen. Our next result generalizes this observation. Its proof follows from Proposition 2.1 .

Proposition 4.4 For a space (X, τ) , the collection $PO(X, \tau)$ is a topology if and only if the intersection of any two dense sets is preopen.

Theorem 4.5 For a space (X,τ) , let $X = F \cup G$ denote the Hewitt representation of (X,τ) . Then the following are equivalent:

(i) $PO(X,\tau)$ is a topology on X .

(ii) $\overline{G}$ is open and $\{x\} \in PO(X,\tau)$ for each $x \in intF$.

Proof. (i) $\Rightarrow$ (ii) : Since F is resolvable and closed, $F = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$ and $F = \overline{E_1} = \overline{E_2}$. Let $y \in F$, say $y \in E_1$. Since $G \cup E_1$ and $G \cup E_2 \cup \{y\}$ are both dense, $(G \cup E_1) \cap (G \cup E_2 \cup \{y\}) = G \cup \{y\} \in PO(X,\tau)$. By Proposition 2.3 , $\overline{G} \setminus int\overline{G} = \emptyset$, hence $\overline{G}$ is open.

Let $x \in intF$. Since $G \cup \{x\}$ and $intF$ are both preopen, $(G \cup \{x\}) \cap intF = \{x\} \in PO(X,\tau)$.

(ii) $\Rightarrow$ (i) : If $S_1, S_2 \in PO(X,\tau)$ then $S_1 \cap \overline{G}$ and $S_2 \cap \overline{G}$ are preopen, hence preopen in $\overline{G}$ by Lemma 4.3 . Since G is an open, dense and hereditarily irresolvable subspace of $\overline{G}$, $(S_1 \cap \overline{G}) \cap (S_2 \cap \overline{G})$ is preopen in $\overline{G}$ by Theorem 4.1 , hence preopen by Lemma 4.3 . By assumption, $S_1 \cap S_2 \cap intF \in PO(X,\tau)$ and thus $S_1 \cap S_2 = (S_1 \cap S_2 \cap \overline{G}) \cup (S_1 \cap S_2 \cap intF) \in PO(X,\tau)$. Hence $PO(X,\tau)$ is a topology on X . $\square$

Let us consider some applications of Theorem 4.5 .

Corollary 4.6 For a resolvable space (X,τ) the following are equivalent:

(i) $PO(X,\tau)$ is a topology.

(ii) Every subset of X is preopen.

(iii) Every open set is closed.

Proof. (i) $\Rightarrow$ (ii) : If $X = F \cup G$ denotes the Hewitt representation of (X,τ) , then by hypothesis $X = F = intF$. Hence $\{x\} \in PO(X,\tau)$ for every $x \in X$ by Theorem 4.5 .

(ii) $\Leftrightarrow$ (iii) : This is Theorem 5 in [7] .

(ii) $\Rightarrow$ (i) : Trivial. $\square$

Let $X = F \cup G$ denote the Hewitt representation of (X,τ) . Suppose there is some point $x \in intF$ such that $\{x\}$ is closed. By Theorem 4.5 , $PO(X,\tau)$ fails to be a topology since F is dense-in-itself. In particular, we have

Corollary 4.7 Let (X,τ) be a T_1 space. If $PO(X,\tau)$ is a topology, then $PO(X,\tau)=\tau^\alpha$.

Proof. Since (X,τ) is a T_1 space, $intF$ has to be void. Hence $\overline{G}=X$ and $PO(X,\tau)=\tau^\alpha$ by Theorem 4.1 . □

Note however, that if $PO(X,\tau)$ is a topology, then $PO(X,\tau)\neq\tau^\alpha$ in general. For an indiscrete space (X,τ) we have $\tau^\alpha=\tau$ and $PO(X,\tau)=2^X$. Moreover, let X be an infinite set and pick a point $p\in X$. Then $\tau=\{\emptyset,X,\{p\},X\setminus\{p\}\}$ is a topology on X . One easily checks that (X,τ) is irresolvable and $PO(X,\tau)=2^X$. However, for each nonempty proper subset S of $X\setminus\{p\}$ we have $intS=\emptyset$, hence $S\notin\tau^\alpha$.

Corollary 4.8 Let (X,τ) be connected. If $PO(X,\tau)$ is a topology, then (X,τ) is indiscrete or $PO(X,\tau)=\tau^\alpha$.

Proof. Let $X=F\cup G$ be the Hewitt representation of (X,τ) . Since $X=\overline{G}\cup intF$ either $\overline{G}=\emptyset$ or $intF=\emptyset$. If $\overline{G}=\emptyset$ then (X,τ) is resolvable. By Corollary 4.6 , every open set is closed, hence (X,τ) has to be indiscrete. If $intF=\emptyset$ then $PO(X,\tau)=\tau^\alpha$ by Theorem 4.1 . □

Recall that the topology on X having $PO(X,\tau)$ as a subbase is denoted by τ^* .

Lemma 4.9 Let $X=F\cup G$ be the Hewitt representation of (X,τ) and let x be a nonisolated point of G . Then $\{x\}\notin\tau^*$.

Proof. Suppose that $\{x\}\in\tau^*$. By Proposition 2.1 we have $\{x\}=U\cap D_1\cap...\cap D_k$, where U is open and each set D_i is dense in (X,τ) . By Theorem 3.2 , $int(D_1\cap...\cap D_k)\cap G$ is dense in G . Since $x\in V=U\cap G$ is an open neighbourhood of x we have $V\cap int(D_1\cap...\cap D_k)\cap G\neq\emptyset$ and hence $\{x\}=V\cap int(D_1\cap...\cap D_k)$, a contradiction. □

Theorem 4.10 Let $X=F\cup G$ denote the Hewitt representation of (X,τ) . Then the following are equivalent:

(i) τ^* is discrete.

(ii) F is open and $\{x\}$ is open for each $x\in G$.

Proof. (i) $\Rightarrow$ (ii) : If $G = \emptyset$ we are done, so let us assume that $G \neq \emptyset$. By Lemma 4.9 , $\{x\}$ is open for each $x \in G$, hence $G \subseteq D$ for each dense set $D \subseteq X$. Let $x \in F$. Then $\{x\} = U \cap D_1 \cap ... \cap D_k$ with U open and each D_i dense. Since $G \subseteq D_1 \cap ... \cap D_k$, we have $G \cap U = \emptyset$, hence $U \subseteq F$. Thus $x \in intF$, showing that F is open.

(ii) $\Rightarrow$ (i) : If $x \in G$ then clearly $\{x\} \in \tau^*$. Let $x \in F$. Since F is resolvable, $F = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$ and $F = \overline{E_1} = \overline{E_2}$. We may assume that $x \in E_1$. Then $G \cup E_1$ and $G \cup E_2 \cup \{x\}$ are both dense in (X, τ) , hence $(G \cup E_1) \cap (G \cup E_2 \cup \{x\}) = G \cup \{x\} \in \tau^*$. Consequently $(G \cup \{x\}) \cap F = \{x\} \in \tau^*$. □

Corollary 4.11 If a space (X, τ) is resolvable then τ^* is discrete.

Corollary 4.12 Let (X, τ) be connected. Then τ^* is discrete if and only if (X, τ) is resolvable.

Theorem 4.13 For a space (X, τ) the following are equivalent:

(i) τ^* is discrete.

(ii) For each $x \in X$, either $\{x\}$ is open or there is a preopen set S such that $x \in S$ and $intS = \emptyset$.

Proof. (i) $\Rightarrow$ (ii) : Let $X = F \cup G$ be the Hewitt representation of (X, τ) , and let $x \in X$. If $\{x\}$ is not open then $x \in F$ by Theorem 4.10 . Since F is resolvable, $F = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$ and $F = \overline{E_1} = \overline{E_2}$. We may assume that $x \in E_1$. Since F is open and $G \cup E_1$ is dense, $S = E_1 = (G \cup E_1) \cap F$ is preopen. Clearly $x \in S$ and $intS = \emptyset$.

(ii) $\Rightarrow$ (i) : If $\{x\}$ is open then $\{x\} \in \tau^*$. Otherwise pick a preopen set S such that $x \in S$ and $intS = \emptyset$. Since $X \setminus S$ is dense, $S \cap ((X \setminus S) \cup \{x\}) = \{x\} \in \tau^*$. □

Finally let us consider question $K2$. Recall that a subset A of a space (X, τ) is called *perfect* if it is closed and dense-in-itself.

Theorem 4.14 For a space (X, τ) the following are equivalent:

(i) Every dense-in-itself subset is preopen.

(ii) Every perfect subset is open.

Proof. (i) $\Rightarrow$ (ii) : Let $A \subseteq X$ be perfect. By hypothesis, A is preopen, hence $A \subseteq int\overline{A} = intA$. Thus A is open.

(ii) $\Rightarrow$ (i) : Let $A \subseteq X$ be dense-in-itself. Then $\overline{A}$ is perfect. By hypothesis, $\overline{A}$ is open, hence $A \subseteq \overline{A} = int\overline{A}$. Thus A is preopen. □

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