# A NOTE ON STRONGLY LINDELÖF SPACES *

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**Abstract**

Recently a new class of topological spaces, called strongly Lindelöf spaces, has been introduced. In this note several characterizations of such spaces are obtained and the relationship between strongly Lindelöf spaces and strongly compact spaces is examined. In the last section, maximal strongly Lindelöf spaces are considered.

## 1 Introduction and Preliminaries

In a recent paper, Mashhour et al. [9] have considered so-called strongly Lindelöf spaces by requiring each preopen cover of the space in question to have a countable subcover. A subset of a topological space is said to be preopen if it is contained in the interior of its closure. In this note a simple characterization of such spaces is obtained which reveals that every strongly Lindelöf space is a Lindelöf space of very special type. After investigating the relationship between strongly Lindelöf spaces and strongly compact spaces, maximal members of the class of strongly Lindelöf spaces are considered.

Let $S$ be subset of a topological space $(X, \tau)$ . We denote the closure of $S$ and the interior of $S$ with respect to $\tau$ by $clS$ and $intS$ , respectively. The cardinality of a set $X$ will be denoted by $|X|$ . We denote the set of reals and the set of positive integers by $\mathbb{R}$ and

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$\mathbb{N}$, respectively. Throughout this paper no separation axioms are assumed unless explicitly stated.

A subset $S$ of a space $(X,\tau)$ is called *preopen* [8] if $S \subseteq int(clS)$ . Obviously each open set and each dense set in $(X,\tau)$ is preopen. Ganster [4] has pointed out that a subset $S$ of $(X,\tau)$ is preopen if and only if it is the intersection of an open and a dense subset. Following Bourbaki [2] , a space $(X,\tau)$ is called *submaximal* if each dense subset of $(X,\tau)$ is open. Reilly and Vamanamurthy [11] have shown that $(X,\tau)$ is submaximal if and only if each preopen subset of $(X,\tau)$ is open.

# 2 Strongly Lindelöf spaces

**Definition 1** A space $(X,\tau)$ is called *strongly Lindelöf* (resp. *strongly compact*) if every preopen cover of $(X,\tau)$ admits a countable (resp. a finite) subcover.

These concepts have been introduced by Mashhour et al. in [9] . Strongly compact spaces have been investigated by Ganster in [5] , and Jankovic, Reilly and Vamanamurthy in [7] .

It is obvious that each strongly Lindelöf space is a Lindelöf space. By Corollary 2.3 , the real line is a Lindelöf space which fails to be strongly Lindelöf. Each submaximal Lindelöf space is clearly strongly Lindelöf. By considering the rationals on the real line we obtain a strongly Lindelöf space which is not submaximal. Finally, a countable discrete space is a non–compact strongly Lindelöf space.

In order to characterize strongly Lindelöf spaces it will be convenient to define the notion of a $d$–Lindelöf space.

**Definition 2** A space $(X,\tau)$ is called *d–Lindelöf* if every cover of $(X,\tau)$ by dense subsets has a countable subcover.

**Remark 2.1** Each strongly Lindelöf space is $d$–Lindelöf.

The following result is easily established. Its proof is hence omitted.

**Proposition 2.2** For a space $(X,\tau)$ the following are equivalent:

1) $(X,\tau)$ is $d$–Lindelöf.
2) The complement of any dense subset of $(X,\tau)$ is at most countable.
3) $A \setminus intA$ is at most countable for each $A \subseteq X$ .
4) $clA \setminus A$ is at most countable for each $A \subseteq X$ .
5) $X \setminus I_X$ is countable where $I_X$ is the set of isolated points of $(X,\tau)$ .

**Corollary 2.3** Each uncountable dense-in-itself space fails to be $d$–Lindelöf.

**Corollary 2.4** Let $(X,\tau)$ be $d$–Lindelöf. Then the following are equivalent:

1) $(X,\tau)$ is hereditarily Lindelöf.
2) $(X,\tau)$ is separable.
3) $X$ is countable.

Recall that a space $(X,\tau)$ is called *almost Lindelöf* [10] if every open cover of $(X,\tau)$ has a countable subfamily whose union is dense in $(X,\tau)$ . $(X,\tau)$ is called *nearly Lindelöf* [1] if every cover by regular open subsets has a countable subcover, where a subset $V$ of $(X,\tau)$ is called *regular open* if $V = int(clV)$ .

**Theorem 2.5** For a space $(X,\tau)$ the following are equivalent:

1) $(X,\tau)$ is strongly Lindelöf.
2) $(X,\tau)$ is Lindelöf and $d$–Lindelöf.
3) $(X,\tau)$ is almost Lindelöf and $d$–Lindelöf.
4) $(X,\tau)$ is nearly Lindelöf and $d$–Lindelöf.

**Proof.** The implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are obvious.

3) $\Rightarrow$ 4) : Let $X = \bigcup\{V_i : i \in I\}$ where each $V_i$ is regular open. By assumption there exists a countable subset $I_0 \subseteq I$ such that $\bigcup\{V_i : i \in I_0\}$ is dense. By Proposition 2.2 , $X \setminus \bigcup\{V_i : i \in I_0\}$ is at most countable. This shows that $\bigcup\{V_i : i \in I\}$ admits a countable subcover.

4) $\Rightarrow$ 1) : Let $\{S_i : i \in I\}$ be a preopen cover of $(X,\tau)$ . Since $int(clS_i)$ is regular open for each $i \in I$ there exists a countable subset $I_0 \subseteq I$ such that $X = \bigcup\{int(clS_i) : i \in I_0\}$ . By

Proposition 2.2 , $clS_i \setminus S_i$ is countable for each $i \in I$ and so $X$ is the union of $\bigcup\{S_i : i \in I_0\}$ and a countable set. This proves that $(X, \tau)$ is strongly Lindelöf. □

**Remark 2.6** Let $\mathbb{Q}$ denote the rationals and let $Y$ be an uncountable indiscrete space. By Theorem 2.5 , the topological sum $Y \oplus \mathbb{Q}$ is $d$–Lindelöf but not Lindelöf, hence not strongly Lindelöf.

It is obvious that each strongly compact space is strongly Lindelöf. The following example shows that even a compact, strongly Lindelöf space need not be strongly compact.

**Example 2.7** Let $X = \{(0,0)\} \cup \bigcup\{\{1/n\} \times [0,1] \ : \ n \in \mathbb{N}\}$ . A topology $\tau$ on $X$ is defined as follows. Let each point $p \in \{1/n\} \times (0,1]$ be isolated. The basic open neighbourhoods of $p = (1/n, 0)$ , $n \in \mathbb{N}$ are the cofinite subsets of $\{1/n\} \times [0,1]$ containing $p$ . A basic open neighbourhood of $p = (0,0)$ contains all but finitely many points of $\{(1/n, 0) : n \in \mathbb{N}\}$ .

Clearly $(X, \tau)$ is Hausdorff and compact, hence Lindelöf. By Proposition 2.2 5) , $(X, \tau)$ is $d$–Lindelöf and thus strongly Lindelöf. By Theorem 2 in [5] , however, $(X, \tau)$ fails to be strongly compact.

# 3 Maximal strongly Lindelöf spaces

**Definition 3** A space $(X, \tau)$ is called *maximal strongly Lindelöf* if it is strongly Lindelöf and there exists no strictly finer strongly Lindelöf topology on $X$ .

Hdeib and Pareek [6] have called a space *L–closed* if each Lindelöf subset of $(X, \tau)$ is closed. It is very well known (see e.g. [3]) that a Lindelöf space is maximal Lindelöf if and only if it is $L$–closed. As a consequence of Proposition 2.2 and Theorem 2.5 , if $(X, \tau)$ is strongly Lindelöf and $\tau \subseteq \tau^*$ such that $(X, \tau^*)$ is Lindelöf, then $(X, \tau^*)$ is strongly Lindelöf. Thus the following result is obvious.

**Theorem 3.1** A strongly Lindelöf space $(X, \tau)$ is maximal strongly Lindelöf if and only if it is $L$–closed.

A countable discrete space is obviously maximal strongly Lindelöf. It is easily checked that each maximal strongly Lindelöf space has to be submaximal. Note, however, that if $(X,\tau)$ denotes the 1–point–compactification of a countable discrete space, then $(X,\tau)$ is submaximal and even strongly compact but not maximal strongly Lindelöf.

Recall that a space $(X,\tau)$ is called a *P–space* if the intersection of countably many open sets is open. In [6] it is shown that each $L$–closed Lindelöf space is a $P$–space and that a Hausdorff Lindelöf space is $L$–closed if and only if it a $P$–space. Using this result and Theorem 3.1 we immediately obtain

**Theorem 3.2** Let $(X,\tau)$ be Hausdorff and strongly Lindelöf. Then $(X,\tau)$ is maximal strongly Lindelöf if and only if it is a $P$–space.

**Remark 3.3** It follows from a result of Cameron [3] that a Hausdorff maximal strongly Lindelöf space is even a normal $P$–space.

Our final result shows that in Theorem 3.2 it is not possible to drop the assumption that the space in question is Hausdorff.

**Example 3.4** There exists a $T_1$ strongly Lindelöf $P$–space which is not maximal strongly Lindelöf.

**Proof.** Let $X$ be an uncountable set and let $\tau$ be the co-countable topology on $X$ , i.e. the nonempty open sets in $(X,\tau)$ are precisely those sets whose complement is countable. For any subset $A \subseteq X$ we denote by $\tau_A$ the topology on $X$ obtained from $(X,\tau)$ by making $X \setminus A$ an open discrete subspace, i.e. for a subset $V \subseteq X$ we have $V \in \tau_A$ if and only if $V = U \cup K$ for some $U \in \tau$ and $K \subseteq X \setminus A$ . Clearly $(X,\tau_A)$ is $T_1$ and the set of isolated points of $(X,\tau_A)$ is precisely $X \setminus A$ . Now suppose that $A$ is countably infinite. Then $(X,\tau_A)$ is $d$–Lindelöf by Proposition 2.2 . One easily checks that $(X,\tau_A)$ is Lindelöf and thus $(X,\tau_A)$ is strongly Lindelöf by Theorem 2.5 . We now show that $(X,\tau_A)$ is a $P$–space. For each $i \in \mathbb{N}$ , let $V_i \in \tau_A$ and let $V_i = U_i \cup K_i$ where $U_i \in \tau$ and $K_i \subseteq X \setminus A$ . It is routine to verify that $\bigcap\{V_i \ : \ i \in \mathbb{N}\} = \bigcap\{U_i \ : \ i \in \mathbb{N}\} \cup K$ for some $K \subseteq X \setminus A$ . Since $(X,\tau)$ is a $P$–space we have $\bigcap\{V_i \ : \ i \in \mathbb{N}\} \in \tau_A$ and so $(X,\tau_A)$ is a $P$–space.

Now let $B \subseteq A$ such that $B$ is infinite and properly contained in $A$ . By our previous discussion the resulting space $(X, \tau_B)$ will be strongly Lindelöf. Since $\tau_B$ is strictly finer than $\tau_A$ it is clear that $(X, \tau_A)$ is not maximal strongly Lindelöf. $\Box$

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