5

Substitutions, Rauzy fractals and tilings

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5.1 Introduction

This chapter focuses on multiple tilings associated with substitutive dynamical systems. We recall that a substitutive dynamical system (X_σ, S) is a symbolic dynamical system where the shift S acts on the set X_σ of infinite words having the same language as a given infinite word which is generated by powers of a primitive substitution σ. We restrict to the case where the inflation factor of the substitution σ is a unit Pisot number. With such a substitution σ, we associate a multiple tiling composed of tiles which are given by the unique solution of a set equation expressed in terms of a graph associated with the substitution σ: these tiles are attractors of a graph-directed iterated function system (GIFS). They live in $\mathbb{R}^{n-1}$, where n stands for the cardinality of the alphabet of the substitution. Each of these tiles is compact, it is the closure of its interior, it has non-zero measure and it has a fractal boundary that is also an attractor of a GIFS. These tiles are called *central tiles* or *Rauzy fractals*, according to G. Rauzy who introduced them in (Rauzy 1982).

Central tiles were first introduced in (Rauzy 1982) for the case of the Tribonacci substitution $(1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1)$, and then in (Thurston 1989) for the case of the beta-numeration associated with the Tribonacci number (which is the positive root of $X^3 - X^2 - X - 1$). One motivation for Rauzy's construction was to exhibit explicit factors of the substitutive dynamical system (X_σ, S) as translations on compact abelian groups, under the hypothesis that σ is a Pisot substitution.

By extending the seminal construction in (Rauzy 1982), it has been proved that central tiles can be associated with Pisot substitutions (see for instance (Arnoux and Ito 2001) or (Canterini and Siegel 2001b)) as well as

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with beta-numeration with respect to Pisot numbers (*cf.* (Thurston 1989), (Akiyama 1999) and (Akiyama 2002)). They are conjectured to induce tilings in all these cases. The tiling property is known to be equivalent to the fact that the dynamical system (X_σ, S) has pure discrete spectrum (see (Pytheas Fogg 2002, Chapter 7) and (Barge and Kwapisz 2006)) when σ is a unit Pisot irreducible substitution.

We have chosen here to concentrate on tilings associated with substitutions for the sake of clarity. A similar study can be performed in the framework of beta-numeration, with both viewpoints being intimately connected through the notion of beta-substitution. Indeed, a beta-substitution can be associated with any Parry number β (for more details, see Exercise 5.1 and Section 5.11). In the case where β is a Pisot number, the associated substitution can be Pisot reducible as well as Pisot irreducible. The exposition of the theory of central tiles is much simpler when σ is assumed to be Pisot irreducible, even if it extends to the Pisot reducible case. Hence, we will restrict ourselves to the Pisot irreducible case.

There are several approaches for the definition of central tiles. We detail below a construction for unit Pisot substitutions based on a broken line which is defined in terms of the abelianisation of an infinite word generated by σ. Projecting the vertices of this broken line to the contractive subspace of the incidence matrix of σ along its expanding direction and taking the closure of this set yields the central tile. For more details on different approaches, see the surveys in (Pytheas Fogg 2002, Chapters 7 and 8) and (Berthé and Siegel 2005), as well as the discussion in (Barge and Kwapisz 2006) and (Ito and Rao 2006).

The aim of this chapter is to list a great variety of tiling conditions, by focusing on effectivity issues. These conditions rely on the use of various graphs associated with the substitution σ.

This chapter is organised as follows. Section 5.2 gathers all the introductory material. We assume that we are given a unit Pisot irreducible substitution σ. A suitable decomposition of the space $\mathbb{R}^{n-1}$ is first introduced in Section 5.2.1 with respect to the eigenspaces of the incidence matrix $\mathbf{M}_\sigma$ of σ. A definition of the central tile associated with σ as well as its decomposition into subtiles is then provided in Section 5.2.2. We discuss the graph-directed set equation satisfied by the subtiles in Section 5.2.3. Two (multiple) tilings associated with σ are then introduced in Section 5.3. The first one, introduced in Section 5.3.2, is called tiling of the expanding line. This tiling by intervals tiles the expanding line of the incidence matrix $\mathbf{M}_\sigma$ of σ. The second one is *a priori* not a tiling, but a multiple tiling. It is defined on the contracting space of the incidence matrix $\mathbf{M}_\sigma$, and it is made of translated copies of the subtiles of the central tile. It is called the self-replicating multiple tiling. Note that it is conjectured to be a tiling. It will be the main objective of the present chapter to introduce various graphs that provide conditions for this multiple tiling to be a tiling.

The first series of tiling conditions is expressed in geometric terms directly related to properties of the self-replicating multiple tiling. We start in Section 5.4.1 with a sufficient tiling property inspired by the so-called finiteness property (F) (discussed in Section 2.3.2.2). This leads us to introduce successively several graphs in Section 5.4 and Section 5.5, yielding necessary and sufficient conditions. We then discuss in Section 5.6, 5.7 and 5.8 further formulations for the tiling property expressed in terms of the tiling of the expanding line. They can be considered as dual to the former set of conditions. In particular, a formulation in terms of the so-called *overlap coincidence condition* is provided in Section 5.7, as well as, in Section 5.8, a further effective condition based on the notion of *balanced pairs*.

5.2 Basic definitions

We use the terminology of Section 1.4. Let σ be a substitution over the alphabet $A = \{1, 2, \dots, n\}$. *In all that follows σ is assumed to be a unit substitution that is Pisot irreducible.* In particular, σ is primitive by Theorem 1.4.9. Let us recall that a primitive substitution always admits a power that is prolongable (see Definition 1.2.18 and (Queffélec 1987, Proposition V.1)), and which thus generates an infinite word. For the sake of simplicity, we assume that σ generates an infinite word according to Definition 1.2.18, that will be denoted as $u = u_0u_1\cdots$. We will see later that this causes no loss of generality (see Theorem 5.3.16 and Remark 5.3.17). Let us note that u is uniformly recurrent by Proposition 1.4.6, and that $\sigma(u) = u$, *i.e.*, u is a fixed point of σ.

5.2.1 Space decomposition

We want to give a geometric interpretation of the fixed point $u = u_0u_1\cdots$ of the unit Pisot irreducible substitution σ. In the present section we first introduce some algebraic formalism in order to embed u in a subspace of $\mathbb{R}^n$ spanned by the eigenvectors associated with the algebraic conjugates of the Perron–Frobenius eigenvalue of the incidence matrix of σ (see Theorem 1.4.2). Since σ is Pisot irreducible, this subspace turns out to be a hyperplane. We define a suitable projection of $\mathbb{R}^n$ onto this hyperplane. The closure of the projections of the abelianised subwords $\mathbf{P}(u_0u_1\cdots u_{N-1})$, for

$N \in \mathbb{N}$, will comprise the so-called *central tile* or *Rauzy fractal* that will be defined in Section 5.2.2.

Eigenvectors and eigenvalues. Let σ be a unit Pisot irreducible substitution. We want to decompose $\mathbb{R}^n$ with respect to certain eigenspaces of the incidence matrix $\mathbf{M}_\sigma$ of σ. Let β be the Perron–Frobenius eigenvalue of $\mathbf{M}_\sigma$. According to our assumptions β is a Pisot unit and n is the algebraic degree of β.

Let $r-1$ be the number of real conjugates of β (distinct from β). They are denoted by $\beta^{(2)}, \ldots, \beta^{(r)}$. Each corresponding eigenspace has dimension one according to Perron–Frobenius' theorem (Theorem 1.4.2). Let $2s$ be the number of complex conjugates of β. They are denoted by $\beta^{(r+1)}$, $\overline{\beta^{(r+1)}}$, $\ldots$, $\beta^{(r+s)}$, $\overline{\beta^{(r+s)}}$. Each pair of a complex eigenvector together with its complex conjugate generates a two-dimensional plane. One has $n = r + 2s$ since σ is Pisot irreducible.

Let $\mathbf{v}_\beta$ be a left eigenvector of $\mathbf{M}_\sigma$ (*i.e.*, an eigenvector of ${}^t\mathbf{M}_\sigma$) associated with the eigenvalue β having positive entries contained in $\mathbb{Z}[\beta]$. Such a vector exists by Perron–Frobenius' theorem. Let $\mathbf{u}_\beta$ be the right eigenvector of $\mathbf{M}_\sigma$ associated with β, and normalised by $\langle \mathbf{v}_\beta, \mathbf{u}_\beta \rangle = 1$. The eigenvector $\mathbf{u}_\beta$ is well defined by the above conditions once $\mathbf{v}_\beta$ is given. Again by Perron–Frobenius' theorem, $\mathbf{u}_\beta$ has positive coordinates in $\mathbb{Q}(\beta)$. We obtain left eigenvectors $\mathbf{v}_{\beta^{(i)}}$ for the algebraic conjugates $\beta^{(i)}$ of β by replacing β by $\beta^{(i)}$ in the coordinates of the vector $\mathbf{v}_\beta$. We similarly obtain the right eigenvectors $\mathbf{u}_{\beta^{(i)}}$. Furthermore, the coordinates of $\mathbf{v}_\beta$ are easily seen to be linearly independent over $\mathbb{Q}$. The same holds for the coordinates of $\mathbf{u}_\beta$.

Remark 5.2.1 Note that this normalisation convention for $\mathbf{u}_\beta$ *a priori* does not correspond to the normalised Perron–Frobenius eigenvector of Theorem 1.4.5 and Proposition 10.4.2 whose coordinates give the frequencies of letters in u (in this latter case, the sum of coordinates equals 1). See also the discussion in Section 5.11.

The right and left eigenvectors are easily seen to satisfy the following relations, for $k \geq 2$, $i \geq 2$, $k \neq i$

$$\langle \mathbf{v}_\beta, \mathbf{u}_{\beta^{(k)}} \rangle = 0, \ \langle \mathbf{v}_{\beta^{(i)}}, \mathbf{u}_{\beta^{(k)}} \rangle = 0, \ \langle \mathbf{v}_{\beta^{(k)}}, \mathbf{u}_{\beta^{(k)}} \rangle = 1. \tag{5.1}$$

For more details see (Canterini and Siegel 2001b, Section 2), (Ei, Ito, and Rao 2006, Lemma 2.5), (Baker, Barge, and Kwapisz 2006) or (Siegel and Thuswaldner 2010).

A suitable decomposition of the space. Using the eigenvectors defined above we can decompose $\mathbb{R}^n$ as follows. The *contracting space* of the matrix

$\mathbf{M}_\sigma$ is the subspace $\mathbb{H}_c$ generated by the eigenvectors $\mathbf{u}_{\beta^{(i)}}$ associated with the $n-1$ conjugates of β (each of which has modulus less than one). The *expanding line* of $\mathbf{M}_\sigma$ is the real line $\mathbb{H}_e$ generated by the eigenvector $\mathbf{u}_\beta$. Note that the subscripts c and e stand here as abbreviations for *contracting* and *expanding*, respectively. The space $\mathbb{H}_c$ has dimension $r+2s-1 = n-1$ so that $\mathbb{H}_c \simeq \mathbb{R}^{n-1}$. Moreover, $\mathbb{H}_c$ is orthogonal to $\mathbf{v}_\beta$, according to (5.1).

We denote by $h_\sigma :\ \mathbb{H}_c \to \mathbb{H}_c$ the restriction of $\mathbf{M}_\sigma$ to $\mathbb{H}_c$. The mapping h_σ is a uniform contraction whose eigenvalues are the conjugates of β. Note that it scales down the $(n-1)$-dimensional Lebesgue measure by the factor $|\beta^{(2)} \cdots \beta^{(r)}|\, |\beta^{(r+1)}|^2 \cdots |\beta^{(r+s)}|^2 = 1/\beta$, since β is a unit. This contraction mapping will play a prominent role in the sequel.

In order to make the distinction between elements of the n-dimensional space $\mathbb{R}^n$ and elements of the $(n-1)$-dimensional space $\mathbb{H}_c$, we restrict the use of bold symbols for the vectors and linear mappings of $\mathbb{R}^n$.

We denote by μ_k the k-dimensional Lebesgue measure. In particular, we work with μ_{n-1} on $\mathbb{H}_c$, and with μ_1 on $\mathbb{H}_e$.

Projections on the eigenspaces. Let $\pi_c : \mathbb{R}^n \to \mathbb{H}_c$ be the projection of $\mathbb{R}^n$ onto $\mathbb{H}_c$ along $\mathbb{H}_e$, according to the natural decomposition $\mathbb{R}^n = \mathbb{H}_c \oplus \mathbb{H}_e$. We recall that $\mathbf{P}$ denotes the abelianisation mapping defined in Section 1.4. The relation $\mathbf{P}(\sigma(w)) = \mathbf{M}_\sigma \mathbf{P}(w)$ for all $w \in A^*$ implies the commutation relation

$$\forall\, w \in A^*,\ \pi_c \circ \mathbf{P} \circ \sigma(w) = h_\sigma \circ \pi_c \circ \mathbf{P}(w). \tag{5.2}$$

Relation (5.2) reads as follows: when applying σ to a word w, the abelianisation $\mathbf{P}(w)$ is mapped onto $\mathbf{M}_\sigma \mathbf{P}(w)$, which has *a priori* larger entries since $\mathbf{M}_\sigma$ has non-negative entries, and thus 'moves away' from the origin. However, when considering the projection on $\mathbb{H}_c$ of the abelianisations, the point $\pi_c \circ \mathbf{P}(w)$, which is mapped to the point $h_\sigma \circ \pi_c \circ \mathbf{P}(w)$ when applying σ, gets closer to the origin since h_σ is a uniform contraction. The relation (5.2) will play a key role in the sequel.

We deduce from (5.1) that any element $\mathbf{x} \in \mathbb{R}^d$ admits the decomposition

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{v}_\beta \rangle\, \mathbf{u}_\beta + \sum_{i=2}^{r+2s} \langle \mathbf{x}, \mathbf{v}_{\beta^{(i)}} \rangle \mathbf{u}_{\beta^{(i)}}. \tag{5.3}$$

For more details, see (Canterini and Siegel 2001b, Section 2.1). This implies that the projection of $\mathbf{x}$ onto $\mathbb{H}_e$ along $\mathbb{H}_c$ is equal to $\langle \mathbf{x}, \mathbf{v}_\beta \rangle \mathbf{u}_\beta$. We thus define

$$\pi_e \colon \mathbb{R}^n \to \mathbb{R},\ \ \mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{v}_\beta \rangle. \tag{5.4}$$

The mapping π_e is the projection of $\mathbb{R}^n$ onto the expanding line along the contracting space $\mathbb{H}_c$ followed by a suitable renormalisation that makes it into a mapping with values in $\mathbb{R}$ and not in $\mathbb{H}_e$. One has $\pi_e(\mathbb{H}_c) = 0$. The mapping π_e measures in some sense the distance to the hyperplane $\mathbb{H}_c$. We thus define the *height* of a vector $\mathbf{x} \in \mathbb{R}^n$ as $\langle \mathbf{x}, \mathbf{v}_\beta \rangle = \pi_e(\mathbf{x})$.

One deduces furthermore from simple algebraic considerations applied to (5.3) that

$$\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{Q}^n,\ \pi_c(\mathbf{x}) = \pi_c(\mathbf{y}) \iff \langle \mathbf{x}, \mathbf{v}_\beta \rangle = \langle \mathbf{y}, \mathbf{v}_\beta \rangle \iff \mathbf{x} = \mathbf{y}. \quad (5.5)$$

For more details, see (Canterini and Siegel 2001b, Section 2.1).

5.2.2 Central tile

We first introduce the notion of a broken line associated with the fixed point u of σ.

Definition 5.2.2 The *broken line* L_u associated with the fixed point u of the unit Pisot irreducible substitution σ is defined as the broken line in $\mathbb{R}^n$ whose set of vertices is given by $\{\mathbf{P}(u_0 \cdots u_{N-1}) \mid N \in \mathbb{N}\}$.

We can also describe the broken line as a stair made of a union of segments. More precisely, for $\mathbf{x} \in \mathbb{Z}^n$ and $i \in A$, we denote by $[\mathbf{x}, i]_g$ the segment $\{\mathbf{x} + \theta e_i \mid \theta \in [0,1]\}$. We call such a segment a *basic geometric strand*, according to (Barge and Kwapisz 2006). We will use and develop this terminology in Section 5.6.1. The broken line associated with u is thus the union of the basic geometric strands $[\mathbf{P}(u_0 \cdots u_{N-1}), u_N]_g$, for $N \in \mathbb{N}$, *i.e.*,

$$L_u = \bigcup_{N \in \mathbb{N}} [\mathbf{P}(u_0 \cdots u_{N-1}), u_N]_g.$$

Definition of the central tile. The *central tile* (or *Rauzy fractal*) associated with the unit Pisot irreducible substitution σ is the closure of the projection by π_c onto the contracting space $\mathbb{H}_c$ of the vertices of the broken line L_u associated with the fixed point u of σ, *i.e.*,

$$\mathcal{T}_\sigma := \overline{\{\pi_c \circ \mathbf{P}(u_0 \cdots u_{N-1}) \mid N \in \mathbb{N}\}}.$$

Subtiles of the central tile $\mathcal{T}_\sigma$ are defined according the the letter u_N occurring after the word $u_0 \cdots u_{N-1}$. Indeed, we set for each $i \in A$

$$\mathcal{T}_\sigma(i) := \overline{\{\pi_c \circ \mathbf{P}(u_0 \cdots u_{N-1}) \mid N \in \mathbb{N},\ u_N = i\}}.$$

By definition, the central tile $\mathcal{T}_\sigma$ consists of the finite union of its subtiles, *i.e.*,

$$\mathcal{T}_\sigma = \bigcup_{i \in A} \mathcal{T}_\sigma(i).$$

We will see later (see Corollary 5.2.8, Theorem 5.3.16 and Remark 5.3.17) that the central tile $\mathcal{T}_\sigma$ and the subtiles $\mathcal{T}_\sigma(i)$ do not depend on the choice of u. They only depend on the substitution σ.

Theorem 5.2.3 *Let σ be a unit Pisot irreducible substitution. The central tile $\mathcal{T}_\sigma$ and the subtiles $\mathcal{T}_\sigma(i)$ associated with σ are compact sets.*

Proof Note that the compactness of the subtiles $\mathcal{T}_\sigma(i)$ is a direct consequence of the compactness of $\mathcal{T}_\sigma$, since they are closed subsets of $\mathcal{T}_\sigma$. To prove the compactness of $\mathcal{T}_\sigma$, it is enough to show that the points $\pi_c \circ \mathbf{P}(u_0 \cdots u_{N-1})$, for $N \in \mathbb{N}$, remain at a uniformly bounded distance of the origin in $\mathbb{H}_c$.

In order to prove this, we use a decomposition of the prefixes $u_0 \cdots u_{N-1}$ into images by powers of σ of a finite number of words. Since $\sigma(u) = u$, there exists a unique $L \leq N$ such that $\sigma(u_0 \cdots u_{L-1})$ is a proper prefix of $u_0 \cdots u_{N-1}$, and $u_0 \cdots u_{N-1}$ is a prefix of $\sigma(u_0 \cdots u_L)$. In other words, there exists a proper prefix p of $\sigma(u_L)$, such that

$$u_0 \cdots u_{N-1} = \sigma(u_0 \cdots u_{L-1})\, p \text{ with } \sigma(u_L) = p\, u_N\, s. \tag{5.6}$$

By iterating this process, one gets for every N an expansion of the form

$$u_0 \cdots u_{N-1} = \sigma^K(p_K)\sigma^{K-1}(p_{K-1}) \cdots \sigma(p_1)p_0,$$

where the p_i belong to a finite set of words that only depends on σ. Note that we have obtained a numeration system on words, the so-called Dumont–Thomas numeration (see Sections 9.4.2 and 5.11 for more details). By (5.2), one has

$$\pi_c \circ \mathbf{P}(u_0 \cdots u_{N-1}) = h_\sigma^K \circ \pi_c \circ \mathbf{P}(p_K) + \cdots + h_\sigma \circ \pi_c \circ \mathbf{P}(p_1) + \pi_c \circ \mathbf{P}(p_0).$$

We know that h_σ is a uniform contraction on $\mathbb{H}_c$. As the $\mathbf{P}(p_i)$ take finitely many values, this implies that the points $\pi_c \circ \mathbf{P}(u_0 \cdots u_{N-1})$, for $N \in \mathbb{N}$, remain at a uniformly bounded distance from the origin, which ends the proof. □

5.2.3 A graph-directed iterated function system

We now discuss a key property of the central tile and its subtiles, namely they satisfy a set equation. By the solution of a set equation we mean

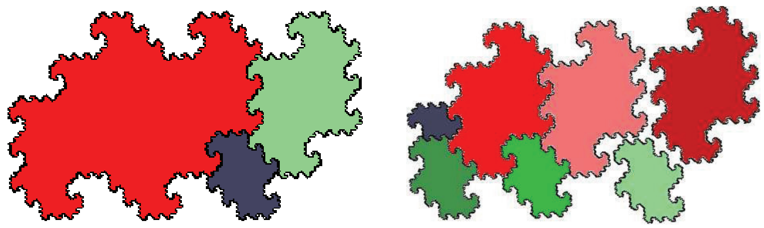

Fig. 5.1. The central tile and its subtiles for the substitution $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 1$ (left), and its decomposition into subtiles (right).

the following. We are given a collection of finitely many compact sets $\{K_1, \dots, K_q\}$. Each set K_i can be decomposed as a union of contracted copies of itself and the other sets K_j. Associated with such a set equation there is a natural graph: its set of vertices is given by $\{K_i \mid 1 \leq i \leq q\}$ and there is an edge e (labelled $i \xrightarrow{e} j$) from K_i to K_j if K_j appears in the decomposition of K_i.

Let us formalise this concept by introducing the notion of *graph-directed iterated function system*. We consider a finite directed graph G with set of vertices $\{1, \dots, q\}$ and set of edges E for which each vertex has at least one outgoing edge. With each edge e of the graph, is associated a contractive mapping $\tau_e : \mathbb{R}^n \to \mathbb{R}^n$. We call $(G, \{\tau_e\}_{e \in E})$ a *graph-directed iterated function system* (*GIFS*, for short, see (Mauldin and Williams 1988)).

It can be shown by a fixed point argument that given a GIFS $(G, \{\tau_e\}_{e \in E})$ there exists a unique collection of non-empty compact sets $K_1, \dots, K_q \subset \mathbb{R}^n$ having the property that

$$K_i = \bigcup_{i \xrightarrow{e} j} \tau_e(K_j),$$

where the union runs over all edges in G leading away from the vertex i. The sets K_i are called *GIFS attractors* or *solutions of the GIFS*. Note that the uniqueness statement does not hold for general sets, but only for non-empty compact sets.

Let us see how to apply this formalism to the subtiles $\mathcal{T}_\sigma(i)$. The graph that will be used is the so-called *prefix-suffix graph*. This graph describes the way images of letters under σ can be decomposed, according to the proof of Theorem 5.2.3. It is the starting point for the construction of several kinds of graphs introduced later in this chapter. For more on this graph, see (Canterini and Siegel 2001a, Canterini and Siegel 2001b).

Definition 5.2.4 (Prefix-suffix graph) Let σ be a substitution over the alphabet A. Let P_σ be the finite set

$$P_\sigma := \{(p,i,s) \in A^* \times A \times A^* \mid \exists\, j \in A,\ \sigma(j) = pis\}. \tag{5.7}$$

The set of vertices of the *prefix-suffix graph* $\mathcal{G}_\sigma$ *of* σ is the alphabet A. There is an edge labelled by $(p,i,s) \in P_\sigma$ from i towards j if, and only if, $pis = \sigma(j)$. We then use the notation $i \xrightarrow{(p,i,s)} j$.

Example 5.2.5 Let us consider as an example the substitution $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 1$, whose central tile is depicted on the left side of Figure 5.1. Its prefix-suffix graph is depicted in Figure 5.2. We recall that ε is the empty word.

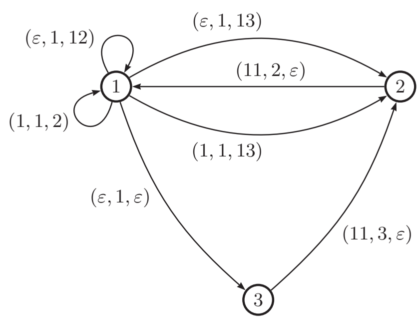

Fig. 5.2. The prefix-suffix graph for $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 1$.

By associating with the edge $e = (p,i,s)$ the contraction mapping

$$\tau_e : \nu \in \mathbb{R}^n \mapsto h_\sigma(\nu) + \pi_c \circ \mathbf{P}(p) \in \mathbb{R}^n,$$

we get the GIFS $(\mathcal{G}_\sigma, \{\tau_e\}_{e \in P_\sigma})$. We now can give explicitly the set equation satisfied by the subtiles of the central tile. This is the content of the following theorem (see (Sirvent and Wang 2002) and also (Ito and Rao 2006)).

Theorem 5.2.6 (Sirvent and Wang 2002) *Let* σ *be a unit Pisot irreducible substitution over the alphabet* A. *The subtiles* $\mathcal{T}_\sigma(i)$ *are the solutions of the GIFS* $(\mathcal{G}_\sigma, \{\tau_e\}_{e \in P_\sigma})$, i.e.,

$$\forall\, i \in A,\ \mathcal{T}_\sigma(i) = \bigcup_{\substack{j \in A, \\ i \xrightarrow{(p,i,s)} j}} h_\sigma(\mathcal{T}_\sigma(j)) + \pi_c \circ \mathbf{P}(p). \tag{5.8}$$

Furthermore, the union in (5.8) is a measure disjoint union.

Before giving a proof of this theorem, let us illustrate it on an example.

Example 5.2.7 We continue with the substitution σ of Example 5.2.5. In order to decompose $\mathcal{T}_\sigma(1)$ by (5.16) we look for the outgoing edges for the vertex 1 in the prefix-suffix graph. Equation (5.8) gives

$$\begin{aligned}
\mathcal{T}_\sigma(1) =& h_\sigma(\mathcal{T}_\sigma(1)) \cup (h_\sigma(\mathcal{T}_\sigma(1)) + \pi_c(\mathbf{e}_1)) \cup h_\sigma(\mathcal{T}_\sigma(2)) \\
& \cup (h_\sigma(\mathcal{T}_\sigma(2)) + \pi_c(\mathbf{e}_1)) \cup h_\sigma(\mathcal{T}_\sigma(3)), \\
\mathcal{T}_\sigma(2) =& h_\sigma(\mathcal{T}_\sigma(1)) + 2\pi_c(\mathbf{e}_1), \\
\mathcal{T}_\sigma(3) =& h_\sigma(\mathcal{T}_\sigma(2)) + 2\pi_c(\mathbf{e}_1).
\end{aligned}$$

Hence, the largest subtile $\mathcal{T}_\sigma(1)$ can be decomposed into two shrunken copies of $\mathcal{T}_\sigma(1)$, two shrunken copies of $\mathcal{T}_\sigma(2)$ and one shrinked copy of $\mathcal{T}_\sigma(3)$. The subtile $\mathcal{T}_\sigma(2)$ is the geometrically similar image of $\mathcal{T}_\sigma(1)$, and $\mathcal{T}_\sigma(3)$ is the image of $\mathcal{T}_\sigma(2)$. This decomposition is illustrated in Figure 5.1 above. Note that the number of pieces in the decomposition of the subtile $\mathcal{T}_\sigma(i)$ is equal to the number of outgoing edges of the vertex i in the prefix-suffix graph.

Proof of Theorem 5.2.6 We fix $i \in A$ and assume that $u_N = i$. By definition, one has $\pi_c \circ \mathbf{P}(u_0 \cdots u_{N-1}) \in \mathcal{T}_\sigma(i)$. By (5.6), there exist L and a decomposition of $\sigma(u_L)$ as $\sigma(u_L) = pu_N s = pis$ such that $\pi_c \circ \mathbf{P}(u_0 \cdots u_{N-1}) = h_\sigma \circ \pi_c \circ \mathbf{P}(u_0 \cdots u_{L-1}) + \pi_c \circ \mathbf{P}(p)$. We thus get $\pi_c \circ \mathbf{P}(u_0 \cdots u_{N-1}) \in h_\sigma(\mathcal{T}_\sigma(u_L)) + \pi_c \circ \mathbf{P}(p)$. As this is true for each N with $u_N = i$, by grouping by the values of u_L and taking the closure, we obtain the decomposition (5.8) for $\mathcal{T}_\sigma(i)$, *i.e.*,

$$\mathcal{T}_\sigma(i) = \bigcup_{(p,j,s),\ \sigma(j)=pis} h_\sigma(\mathcal{T}_\sigma(j)) + \pi_c \circ \mathbf{P}(p).$$

Recall that h_σ scales down the $(n-1)$-dimensional Lebesgue measure μ_{n-1} by the factor $1/\beta$. We deduce from (5.8) that

$$\forall\, i \in A,\ \beta\, \mu_{n-1}(\mathcal{T}_\sigma(i)) \le \sum_{j \in A} m_{ij}\, \mu_{n-1}(\mathcal{T}_\sigma(j)), \tag{5.9}$$

where the coefficients m_{ji} denote the entries of the incidence matrix $\mathbf{M}_\sigma$. As β is the Perron–Frobenius eigenvalue of $\mathbf{M}_\sigma$, Lemma 1.4.4 implies the reverse inequality. We thus get equality in (5.9). This implies that no overlap with positive measure occurs in the union in (5.8). □

Note that (5.8) admits the following k-fold iteration for any $k \in \mathbb{N}$ and

$i \in A$

$$\mathcal{T}_\sigma(i) = \bigcup_{(p,j,s),\ \sigma^k(j)=pis} h_\sigma^k(\mathcal{T}_\sigma(j)) + \pi_c \circ \mathbf{P}(p). \tag{5.10}$$

From the uniqueness of the solution of (5.8) for non-empty compact sets we deduce the following result.

Corollary 5.2.8 *Let σ be a unit Pisot irreducible substitution. The central tiles $\mathcal{T}_\sigma$ and the subtiles $\mathcal{T}_\sigma(i)$, for $i \in A$, do not depend on the choice of the fixed point u of σ.*

We know so far that each subtile $\mathcal{T}_\sigma(i)$ can be decomposed into shrunken copies of the subtiles (namely into sets of the form $h_\sigma(\mathcal{T}_\tau(j))+\pi_c\circ\mathbf{P}(p)$) that are disjoint in measure. To ensure that the subtiles $\mathcal{T}_\sigma(i)$, for $i \in A$, themselves are pairwise disjoint in measure, we introduce the following combinatorial condition on substitutions. For substitutions of constant length this condition goes back to (Dekking 1978), see details in (Pytheas Fogg 2002, Chapter 7).

Definition 5.2.9 (Arnoux and Ito 2001) A substitution σ over the alphabet A satisfies the *combinatorial strong coincidence condition* if for every pair $(j_1, j_2) \in A^2$, there exist $k \in \mathbb{N}$ and $i \in A$ such that $\sigma^k(j_1) = p_1 i s_1$ and $\sigma^k(j_2) = p_2 i s_2$ with $\mathbf{P}(p_1) = \mathbf{P}(p_2)$.

The combinatorial strong coincidence condition is satisfied by every Pisot irreducible substitution over a two-letter alphabet (Barge and Diamond 2002). It is conjectured that every Pisot irreducible substitution satisfies this condition.

The following theorem relates the combinatorial strong coincidence condition to the disjointness of the interiors of the subtiles $\mathcal{T}_\sigma(i)$, $i \in A$.

Theorem 5.2.10 (Arnoux and Ito 2001) *Let σ be a unit Pisot irreducible substitution. If σ satisfies the combinatorial strong coincidence condition, then the subtiles $\mathcal{T}_\sigma(i)$ of the central tile $\mathcal{T}_\sigma$ are measure disjoint.*

Proof The combinatorial strong coincidence condition implies that for every pair of letters (j_1, j_2) there exist a common letter i, a positive integer k and a common abelianised prefix $\mathbf{P}(p)$ such that $h_\sigma^k(\mathcal{T}_\sigma(j_1)) + \pi_c \circ \mathbf{P}(p)$ and $h_\sigma^k(\mathcal{T}_\sigma(j_2)) + \pi_c \circ \mathbf{P}(p)$ both appear in the k-fold iteration (5.10) of the decomposition of $\mathcal{T}_\sigma(i)$ given by (5.8). Theorem 5.2.6 yields that these tiles are disjoint in measure. $\square$

5.3 Tilings

In this section we define a tiling as well as a multiple tiling associated with a unit Pisot irreducible substitution σ. We start with some definitions.

5.3.1 General definitions

Let K_i, $i \in A$, be a finite collection of compact sets of a subspace $\mathbb{H}$ of $\mathbb{R}^n$, with each of the K_i being the closure of its interior. Let p be a positive integer. A *multiple tiling* of degree p of the space $\mathbb{H}$ by the compact sets K_i is a collection of translated copies of the sets K_i of the form $\mathcal{I} := \{K_i + \gamma \mid (\gamma, i) \in \Gamma\}$, where Γ is a subset $\mathbb{H} \times A$, that satisfies the following conditions.

(i) The entire space $\mathbb{H}$ is covered by the elements of $\mathcal{I}$, *i.e.*,

$$\mathbb{H} = \bigcup_{(\gamma,i)\in\Gamma} K_i + \gamma. \tag{5.11}$$

(ii) Each compact subset of $\mathbb{H}$ intersects a finite number of elements of $\mathcal{I}$.

(iii) Almost every point in $\mathbb{H}$ (with respect to the Lebesgue measure) is covered exactly p times.

The set Γ is called the *translation set*. If the union $\{K_i + \gamma \mid (\gamma, i) \in \Gamma\}$ only satisfies (i), it is said to be a *covering* of $\mathbb{H}$. The sets $K_i + \gamma$ are called *tiles*. In other words, a multiple tiling is a union of tiles $\bigcup_{(\gamma,i)\in\Gamma} K_i + \gamma$ that covers the full space $\mathbb{H}$ with possible overlaps in such a way that almost every point belongs to exactly p tiles. This is illustrated in Figure 5.3 for $p = 2$ with an example obtained in the framework of symmetric beta-expansions taken from (Kalle and Steiner 2009). If $p = 1$, then the multiple tiling is called a *tiling*. See also Figure 5.7 for an example of a tiling.

Condition (ii) means that the first coordinate projection of Γ into $\mathbb{H}$ is a *locally finite* subset of $\mathbb{H}$, *i.e.*, each point in $\mathbb{H}$ has a neighbourhood that intersects only finitely many projected elements of Γ. We also say that Γ is a locally finite set.

Dynamical systems can be associated with tilings in close analogy to dynamical systems associated with substitutions. Indeed, the terminology introduced in Chapter 1 concerning words extends in a natural way to tilings. For more on tiling dynamical systems, see (Solomyak 1997) and (Robinson 2004).

Consider a collection of non-empty compact sets $\{K_i + \gamma \mid (\gamma, i) \in \Gamma\}$ (that is not necessarily a covering or a multiple tiling). A set $K_i + \gamma$ is said to *occur* in $\{K_i + \gamma \mid (\gamma, i) \in \Gamma\}$ if $(\gamma, i) \in \Gamma$. A *patch* is defined as

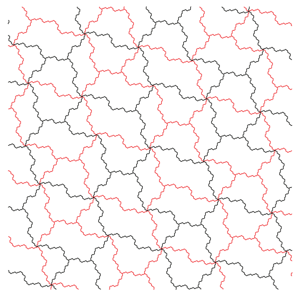

Fig. 5.3. A multiple tiling with $p = 2$.

a finite subset of Γ. It corresponds to a finite union of tiles that occur in $\{K_i + \gamma \mid (\gamma, i) \in \Gamma\}$. The translate of a patch $P = \{(\gamma_1, i_1), \ldots, (\gamma_n, i_n)\}$ by $\nu_0 \in \mathbb{H}$ is defined as $P + \nu_0 := \{(\gamma_1 + \nu_0, i_1), \ldots, (\gamma_n + \nu_0, i_n)\}$. Two patches $P = \{(\gamma_1, i_1), \ldots, (\gamma_n, i_n)\}$ and $P' = \{(\gamma'_1, i'_1), \ldots, (\gamma'_n, i'_n)\}$ are said to be *equivalent* if they coincide up to a translation vector, that is, if there exists $\nu_0 \in \mathbb{H}$ such that $P' = \{(\gamma_1 + \nu_0, i_1), \ldots, (\gamma_n + \nu_0, i_n)\}$.

We now consider a covering of $\mathbb{H}$. We say that a ball $B(\nu, R)$ in $\mathbb{H}$ is contained in a patch $P = \{(\gamma_1, i_1), \ldots, (\gamma_n, i_n)\}$ if $B(\nu, R)$ is a subset of the convex hull of the points γ such that $(\gamma, i) \in P$. We define in a similar way the fact that a patch is contained in a ball. The set Γ is said to be *repetitive* if for any finite patch P, there exists $R > 0$ such that every ball of radius R in $\mathbb{H}$ contains a patch which is equivalent to P. This notion is an analogue of the notion of uniform recurrence for words (see Definition 1.2.9).

A subset of $\mathbb{R}^n$ is said to be a *Delone set* if it is both *uniformly discrete* (there exists $r > 0$ such that any open ball of radius r contains at most one point of this set) and *relatively dense* (there exists $R > 0$ such that every closed ball of radius R contains at least one point of this set). We say by extension that Γ is a *Delone set* if its first coordinate projection on $\mathbb{H}$ is a Delone set. Delone sets have been introduced in the context of point sets and model sets, see *e.g.* (Moody 1997). See also (Lagarias and Pleasants 2002) and (Lagarias and Pleasants 2003) for complexity results on Delone sets that can be compared with analogous results in combinatorics of words on the factor complexity and on the recurrence function.

5.3.2 Tiling of the expanding line

We first associate with σ a tiling by intervals of the expanding half-line $\mathbb{R}_+\mathbf{u}_\beta \subset \mathbb{H}_e$. It is obtained by projecting the broken line L_u associated with u (see Definition 5.2.2) onto the expanding line $\mathbb{H}_e$ along the contracting hyperplane $\mathbb{H}_c$ (see Figure 5.4). This induces a tiling of the half-line $\mathbb{R}_+\mathbf{u}_\beta \subset \mathbb{H}_e$. Using the projection $\pi_e\colon \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{v}_\beta \rangle$, we even get a tiling of $\mathbb{R}_+$ whose tiles are certain translates of the intervals $I_i = [0, \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle]$ for $i = 1, \ldots, n$. In particular, this tiling is obtained by taking the tiles I_{u_0}, $I_{u_1}, \ldots$ adjacent to each other, starting from the origin. The translation set is called the *self-similar translation set* and is equal to

$$\Gamma_e = \{(\pi_e \circ \mathbf{P}(u_0 \cdots u_{N-1}), u_N) \mid N \geq 0\}.$$

Since 0 is an endpoint of a tile and since we have assumed that the coordinates of $\mathbf{v}_\beta$ belong to $\mathbb{Z}[\beta]$, the endpoints of all tiles are contained in $\mathbb{Z}[\beta]$.

We denote the resulting tiling of $\mathbb{R}_+$ by $\mathcal{E}_u$ and refer to it as the *self-similar tiling of the expanding line*. For an illustration, see Figure 5.4. One has

$$\mathcal{E}_u := \{\pi_e[\mathbf{x}, i]_g \mid [\mathbf{x}, i] \in \Gamma_e\}, \tag{5.12}$$

where the basic geometric strand $[\mathbf{x}, i]_g$ is equal to the segment $\{\mathbf{x} + \theta \mathbf{e}_i \mid \theta \in [0, 1]\}$.

The repetitivity of the tiling $\mathcal{E}_u$ is an easy consequence of the fact that u is uniformly recurrent (see Proposition 1.4.6). The terminology 'self-similar' comes from the fact that the set of endpoints of tiles in $\mathcal{E}_u$ is stable by multiplication by β. Sections 5.6, 5.7 and 5.8 rely on this tiling.

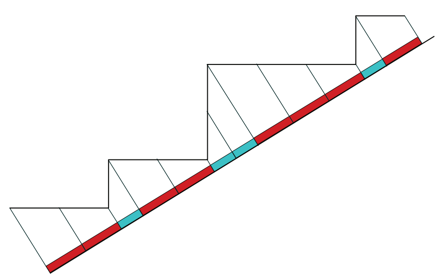

Fig. 5.4. Projecting the broken line L_u. In order to illustrate the relation between the tiling $\mathcal{E}_u$ and the broken line L_u we draw the tiling $\mathcal{E}_u$ parallel to the expanding eigendirection $\mathbf{u}_\beta$ of $\mathbf{M}_\sigma$ and not in the real line, for $\sigma(1) = 112$, $\sigma(2) = 21$.

5.3.3 Self-replicating translation set

We now introduce a multiple tiling associated with the substitution σ. The tiles of this multiple tiling are given by the subtiles $\mathcal{T}_\sigma(i)$, $i \in A$. The corresponding set of translation vectors is obtained by projecting a suitable subset of points of $\mathbb{Z}^n$ on the contracting space $\mathbb{H}_c$. Let us define this set.

Following (Reveillès 1991), we define a notion of discretisation for the hyperplane $\mathbb{H}_c$. The discretised hyperplane is usually called *standard arithmetic discrete hyperplane*. We will use here the shorthand terminology *discrete hyperplane*. We recall that $\mathbb{H}_c$ is the hyperspace orthogonal to the vector $\mathbf{v}_\beta$.

Definition 5.3.1 (Discrete hyperplane) The *discrete hyperplane* associated with $\mathbb{H}_c$ is defined as the set of points $\mathbf{x} \in \mathbb{Z}^n$ that satisfy

$$0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \sum_{i \in A} \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle = ||\mathbf{v}_\beta||_1. \tag{5.13}$$

A discrete hyperplane is a discrete set of points. We now introduce a 'continuous' counterpart to this notion.

Definition 5.3.2 (Stepped hyperplane) The *stepped hyperplane* associated with $\mathbb{H}_c$ is defined as the union of faces of unit cubes whose vertices belong to the discrete hyperplane associated with $\mathbb{H}_c$.

We now want to label the faces contained in a stepped hyperplane. For $\mathbf{x} \in \mathbb{Z}^n$ and $i \in A$, the *face of type i located at* $\mathbf{x}$ is defined as the face orthogonal to the ith canonical vector of the translate of the unit cube located at $\mathbf{x}$, *i.e.*,

$$\mathbf{x} + \{\theta_1 \mathbf{e}_1 + \cdots + \theta_{i-1}\mathbf{e}_{i-1} + \theta_{i+1}\mathbf{e}_{i+1} + \cdots + \theta_n \mathbf{e}_n \mid \theta_j \in [0,1] \text{ for } j \neq i\}.$$

One checks that a face of type i located at $\mathbf{x}$ is a subset of the stepped hyperplane if, and only if, one has

$$0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle. \tag{5.14}$$

For more details, see for instance the references (Berthé and Vuillon 2000), (Arnoux, Berthé, and Ito 2002) or else (Arnoux, Berthé, and Siegel 2004).

Note that a stepped hyperplane is a hypersurface that lives in $\mathbb{R}^n$, whereas a discrete hyperplane is a subset of $\mathbb{Z}^n$. The discrete hyperplane contains all the vertices of the faces contained in the stepped hyperplane, whereas faces of the stepped hyperplane are labelled by pairs $(\mathbf{x}, i)$ that satisfy (5.14). This labelling thus consists in selecting some vertices among all the vertices of the discrete hyperplane according to the value $\langle \mathbf{x}, \mathbf{v}_\beta \rangle$, hence the difference between the right-hand sides of Inequalities (5.13) and (5.14).

We now project the faces of the stepped hyperplane by π_c.

Proposition 5.3.3 *The collection of projections of the faces of the stepped hyperplane,* i.e.,

$$\{\pi_c([\mathbf{x}, i]_g) \mid \mathbf{x} \in \mathbb{Z}^n,\ i \in A,\ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}$$

is a polyhedral tiling of $\mathbb{H}_c$ *by* n *types of projected faces.*

For an explicit proof, see (Berthé and Vuillon 2000) or (Arnoux, Berthé, and Ito 2002). A piece of a stepped hyperplane together with its projection by π_c is depicted in Figure 5.5.

Note that in the Pisot reducible case, $\mathbb{H}_c$ is no longer a hyperplane and the projections of faces do overlap. There is no universal construction known to obtain an analogue polyhedral tiling (for special cases where this is possible, see (Ei and Ito 2005, Ei, Ito, and Rao 2006)). Nevertheless, one obtains a polyhedral covering.

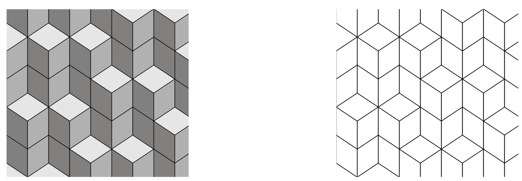

Fig. 5.5. A stepped hyperplane and its projection on $\mathbb{H}_c$ as a polyhedral tiling.

We are now going to replace in this polyhedral tiling projected faces by corresponding subtiles (see Figure 5.7). We will see in Section 5.3.5 that this will yield a multiple tiling (Theorem 5.3.13) which is conjectured to be a tiling. This multiple tiling will be called the *self-replicating multiple tiling*.

With each face of type i located at $\mathbf{x}$ included in the stepped hyperplane, we associate a copy of the tile $\mathcal{T}_\sigma(i)$ located at $\pi_c(\mathbf{x})$ in the contracting space $\mathbb{H}_c$. The *self-replicating translation set* Γ_c is defined as

$$\Gamma_c = \{(\gamma, i) \in \pi_c(\mathbb{Z}^n) \times A \mid \gamma = \pi_c(\mathbf{x}),\ \mathbf{x} \in \mathbb{Z}^n,\ \ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}. \tag{5.15}$$

An element of the form $(\gamma, i) \in \pi_c(\mathbb{Z}^n) \times A$ is called a *tip*. We denote it by $[\gamma, i]^*$. Tips can be considered as symbolic representations of projections of faces. We denote by $[\gamma, i]^*_g$ the projection by π_c of the face of type i located at $\mathbf{x}$, with $\gamma = \pi_c(\mathbf{x})$, *i.e.*,

$$[\gamma, i]^*_g := \pi_c([\mathbf{x}, i]^*) \text{ with } \gamma = \pi_c(\mathbf{x}).$$

We thus make the distinction, thanks to the subscript g, between the projected face $[\pi_c(\mathbf{x}), i]^*_g$ and the tip $[\pi_c(\mathbf{x}), i]^*$. The definition of the graphs and the formalism introduced in Sections 5.4 and 5.5 will illustrate the importance of working with symbolic representations.

Note that the discretisation process underlying Definition 5.3.1 is in some sense 'dual' to the notion of broken line (see Definition 5.2.2), hence, the superscript '$*$' in the notation $[\mathbf{x}, i]^*_g$. Projected faces and segments can also be considered as 'dual'. The use of the symbol '$*$' allows us to make the distinction between the notation used for segments and tips. We will develop this duality idea in Section 5.6.1.

Before stating and proving Proposition 5.3.6 below, we need a density result (see Corollary 5.3.5). This density result will be a direct consequence of Kronecker's theorem that we recall here without proof (a proof of this theorem can be found for instance in (Hardy and Wright 1985)).

Theorem 5.3.4 (Kronecker's theorem) *Let $r \geq 1$ and let $\alpha_1, \dots, \alpha_r$ be real numbers such that $1, \alpha_1, \dots, \alpha_r$ are rationally independent. For every $\varepsilon > 0$ and for every $(x_1, \dots, x_r) \in \mathbb{R}^r$, there exist an element $N \in \mathbb{N}$ and $(p_1, \dots, p_r) \in \mathbb{Z}^r$ such that*

$$\forall i \in \{1, \dots, r\},\ |N\alpha_i - p_i - x_i| < \varepsilon.$$

The proof of the following corollary of Kronecker's theorem can be easily adapted from the proof of (Akiyama 1999, Proposition 1) where it is given in the framework of the beta-numeration, by recalling that the coordinates of $\mathbf{v}_\beta$ are rationally independent. A similar argument can be found in (Canterini and Siegel 2001b, Section 3) stated in terms of minimality of a toral addition.

Corollary 5.3.5 *Let σ be a unit Pisot irreducible substitution. The set $\pi_c(\{\mathbf{z} \in \mathbb{Z}^n \mid \langle \mathbf{z}, \mathbf{v}_\beta \rangle \geq 0\})$ is dense in $\mathbb{H}_c$.*

Proposition 5.3.6 *Let σ be a unit Pisot irreducible substitution. Then the following assertions are true.*

(i) *The set Γ_c is a Delone set.*
(ii) *The union $\{\mathcal{T}_\sigma(i) + \gamma \mid [\gamma, i]^* \in \Gamma_c\}$ is a covering of $\mathbb{H}_c$.*

Proof By Proposition 5.3.3, the projections of the faces of the stepped hyperplane by π_c form a polyhedral tiling of $\mathbb{H}_c$ with translation set Γ_c, which implies (i).

Let us prove (ii). Let $\mathbf{z} \in \mathbb{Z}^n$ with $\langle \mathbf{z}, \mathbf{v}_\beta \rangle \geq 0$. There exists $N \in \mathbb{N}$

such that if we set $\mathbf{x} := \mathbf{P}(u_0 \cdots u_{N-1})$, then $\langle \mathbf{x}, \mathbf{v}_\beta \rangle \leq \langle \mathbf{z}, \mathbf{v}_\beta \rangle < \langle \mathbf{x}, \mathbf{v}_\beta \rangle + \langle \mathbf{e}_{u_N}, \mathbf{v}_\beta \rangle$, where $\mathbf{e}_{u_N} = \mathbf{P}(u_N)$. One deduces that $\mathbf{z} - \mathbf{x}$ satisfies (5.14) with $i = u_N$. As $\mathbf{z} = \mathbf{x} + (\mathbf{z} - \mathbf{x})$ this implies that $\pi_c(\mathbf{z}) \in \mathcal{T}_\sigma(i) + \gamma$ for $[\gamma, i]^* \in \Gamma_c$ with $\gamma = \pi_c(\mathbf{z} - \mathbf{x})$ and $i = u_N$.

Let $\nu \in \mathbb{H}_c$. By Corollary 5.3.5 there exists a sequence $(\pi_c(\mathbf{z}_k))_{k \in \mathbb{N}}$ with $\langle \mathbf{z}_k, \mathbf{v}_\beta \rangle \geq 0$ for all k that converges to ν. Furthermore, we have seen that for each k, there exists $[\gamma_k, i_k]^* \in \Gamma_c$ such that $\pi_c(\mathbf{z}_k) \in \mathcal{T}_\sigma(i_k) + \gamma_k$. Since the subtiles $\mathcal{T}_\sigma(i)$, for $i \in A$, are bounded and Γ_c is uniformly discrete, there are infinitely many k for which (γ_k, i_k) takes the same value, say (γ, i). We thus get $\nu \in \mathcal{T}_\sigma(i) + \gamma$, which implies the covering property. This ends the proof of (ii). □

Corollary 5.3.7 *The subtiles $\mathcal{T}_\sigma(i)$, for $i \in A$, have non-empty interior.*

Proof Since the set Γ_c is countable and according to Proposition 5.3.6 (ii), we deduce from Baire's theorem that there exists $i \in A$ such that the interior of $\mathcal{T}_\sigma(i)$ is not empty. We then deduce from the GIFS equation (5.8) and from the primitivity of σ which implies that the prefix-suffix graph $\mathcal{G}_\sigma$ is strongly connected that all subtiles have non-empty interior. □

5.3.4 Tip substitutions

It remains to prove that the collection $\mathcal{I}_\sigma := \{\mathcal{T}_\sigma(i) + \gamma \mid [\gamma, i]^* \in \Gamma_c\}$ yields a multiple tiling of $\mathbb{H}_c$. This will be the content of Theorem 5.3.13 in Section 5.3.5. In order to prove this theorem, we first need to highlight the self-replicating properties of Γ_c. Indeed, Γ_c is stabilised by an inflation mapping acting on $\pi_c(\mathbb{Z}^n) \times A$. This inflation mapping is nothing but a substitution on tips (or, equivalently, on faces of cubes), that is inspired by the GIFS equation (5.8) satisfied by the subtiles. We explain this more precisely in the present section.

Definition 5.3.8 (GIFS substitution) The (n-dimensional) *GIFS substitution on tips* associated with the (one-dimensional) substitution σ, denoted by $\mathbf{E}_1^*$, is defined on patches of tips by

$$\mathbf{E}_1^*\{[\gamma, i]^*\} = \bigcup_{(p,j,s),\ \sigma(j)=pis} \{[h_\sigma^{-1}(\gamma + \pi_c \circ \mathbf{P}(p)), j]^*\}, \tag{5.16}$$

$$\mathbf{E}_1^*(X_1) \cup \mathbf{E}_1^*(X_2) = \mathbf{E}_1^*(X_1 \cup X_2).$$

We will use the notation $\mathbf{E}_1^*([\gamma, i]^*)$ for $\mathbf{E}_1^*\{[\gamma, i]^*\}$.

Note that we use here the assumption that σ is a unimodular substitution (*i.e.*, its incidence matrix $\mathbf{M}_\sigma$ has determinant ± 1) to ensure that h_σ^{-1} maps

$\pi_c(\mathbb{Z}^n)$ onto $\pi_c(\mathbb{Z}^n)$. Indeed, we use the fact that $h_\sigma \circ \pi_c = \pi_c \circ \mathbf{M}_\sigma$ (see (5.2)), and that the first coordinate γ of a tip belongs to $\pi_c(\mathbb{Z}^n)$.

There is a deep relation between the GIFS substitution $\mathbf{E}_1^*$ and the GIFS equation (5.8), which can indeed be rewritten as

$$\forall\, [\gamma, i]^* \in \Gamma_c,\ \mathcal{T}_\sigma(i) + \gamma = \bigcup_{[\eta,j]^* \in \mathbf{E}_1^*([\gamma,i]^*)} h_\sigma(\mathcal{T}_\sigma(j) + \eta). \tag{5.17}$$

This formalism will thus be a particularly convenient way to describe the GIFS equation (5.8) in the graph constructions of Section 5.5. It has been introduced by (Arnoux and Ito 2001) and (Sano, Arnoux, and Ito 2001) (under the name *generalised substitutions* with the notation $E_1^*(\sigma)$). We omit here the reference to σ for the sake of simplicity in the notation $\mathbf{E}_1^*$. The subscript of $\mathbf{E}_1^*$ stands for the codimension of faces (in the present chapter, they are codimension one faces of hypercubes), while the superscript of $\mathbf{E}_1^*$ indicates that it is the dual mapping of some mapping $\mathbf{E}_1$, that we will introduce in Section 5.6.1. Examples of generalised substitutions are given in (Pytheas Fogg 2002, Chapter 8). Extensions to more general spaces based on faces of hypercubes having higher codimension have also been provided in (Sano, Arnoux, and Ito 2001).

Example 5.3.9 We continue with the substitution $\sigma(1) = 112, \sigma(2) = 113, \sigma(3) = 1$ considered in Examples 5.2.5 and 5.2.7.

In order to compute $\mathbf{E}_1^*([0,i]^*)$ by (5.16) we look for the occurrences of the letter 1 in $\sigma(1)$, $\sigma(2)$ and $\sigma(3)$. This yields

$$\mathbf{E}_1^*([0,1]^*) = [0,1]^* \cup [0,2]^* \cup [0,3]^* \cup [h_\sigma^{-1} \circ \pi_c \circ \mathbf{P}(1), 1]^* \cup [h_\sigma^{-1} \circ \pi_c \circ \mathbf{P}(1), 2]^*.$$

We similarly compute

$$\mathbf{E}_1^*([0,2]^*) = [h_\sigma^{-1} \circ \pi_c \circ \mathbf{P}(11),\ 1]^*, \qquad \mathbf{E}_1^*(0,3]^*) = [h_\sigma^{-1} \circ \pi_c \circ \mathbf{P}(11), 2]^*.$$

By applying the commutation relation $\mathbf{h}_c^{-1} \circ \pi_c = \pi_c \circ \mathbf{M}_\sigma^{-1}$ (see (5.2)), one gets $h_\sigma^{-1} \circ \pi_c \circ \mathbf{P}(1) = h_\sigma^{-1} \circ \pi_c(\mathbf{e}_1) = \pi_c(\mathbf{M}_\sigma^{-1}(\mathbf{e}_1)) = \pi_c(\mathbf{e}_3)$. We thus deduce the following relations

$$\begin{aligned}
\mathbf{E}_1^*([0,1]^*) &= [0,1]^* \cup [0,2]^* \cup [0,3]^* \cup [\pi_c(\mathbf{e}_3), 1]^* \cup [\pi_c(\mathbf{e}_3), 2]^* \\
\mathbf{E}_1^*([0,2]^*) &= [2\pi_c(\mathbf{e}_3), 1]^* \\
\mathbf{E}_1^*([0,3]^*) &= [2\pi_c(\mathbf{e}_3), 2]^*.
\end{aligned}$$

These images are depicted in Figure 5.6, by representing tips as projected faces. Compare with the computation of the decomposition of the subtiles given in Example 5.2.7 which is illustrated in Figure 5.1.

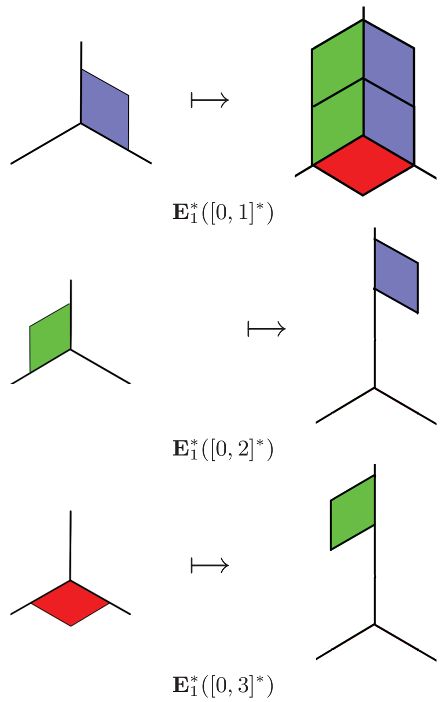

Fig. 5.6. An illustration of the images of the tips $[0,i]^*$, for $i = 1,2,3$, under $\mathbf{E}_1^*$ for the substitution $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 1$. We represent here the tip $[\pi_c(\mathbf{x}), i]^*$ by its projection $[\pi_c(\mathbf{x}), i]_g^*$.

Important properties of $\mathbf{E}_1^*$ are subsumed in the following theorem. For a proof, see (Arnoux and Ito 2001) and see also (Arnoux, Berthé, and Siegel 2004).

Theorem 5.3.10 (Arnoux and Ito 2001) *Let σ be a unit Pisot irreducible substitution. Let $\mathbf{E}_1^*$ be its associated GIFS substitution.*

(i) *The images of two different tips in Γ_c under $\mathbf{E}_1^*$ share no tip in common.*
(ii) *The translation set Γ_c is stable under the action of the mapping $\mathbf{E}_1^*$.*
(iii) *The substitution $\mathbf{E}_1^*$ maps Γ_c onto Γ_c,* i.e., $\mathbf{E}_1^*(\Gamma_c) = \Gamma_c$.

According to Assertion (iii) of Theorem 5.3.10, the set of positions of tiles (given by Γ_c) is stable under the action of an inflation rule, namely the mapping $\mathbf{E}_1^*$, which plays the role of the multiplication by β acting on the tiling of the expanding line introduced in Section 5.3.2. In other words, Γ_c can be seen as the fixed point of a multidimensional combinatorial transformation, namely $\mathbf{E}_1^*$. This explains why the set Γ_c is called *self-replicating* translation set. Note that we use the term 'self-replicating' and not 'self-similar' since the mapping h_σ is possibly not a similarity.

This was the first step towards the proof of the multiple tiling property of $\{\mathcal{T}_\sigma(i) + \gamma \mid (\gamma, i) \in \Gamma_c\}$. Before detailing the proof of this property, let us state the following fundamental result.

Proposition 5.3.11 *Let σ be a unit Pisot irreducible substitution and $\mathbf{E}_1^*$ be its associated GIFS substitution. If $[\eta_1, j_1]^*, [\eta_2, j_2]^* \in {\mathbf{E}_1^*}^N[\gamma, i]^*$ holds for some $[\gamma, i]^* \in \Gamma_c$ and some N, then*

$$\mu_{n-1}((\mathcal{T}_\sigma(j_1) + \eta_1) \cap (\mathcal{T}_\sigma(j_2) + \eta_2)) = 0.$$

Proof As the GIFS equation (5.17) can be iterated, we obtain the following N-fold iteration of the decomposition of $\mathcal{T}_\sigma(i)$, *i.e.*,

$$\forall\, [\gamma, i]^* \in \Gamma_c,\ \mathcal{T}_\sigma(i) + \gamma = \bigcup_{[\eta, j]^* \in {\mathbf{E}_1^*}^N([\gamma, i]^*)} h_\sigma^N(\mathcal{T}_\sigma(j) + \eta). \tag{5.18}$$

According to Theorem 5.2.6 we know that all pairs of pieces in the union on the right-hand side intersect on a set with zero measure.

Thus $[\eta_1, j_1]^*, [\eta_2, j_2]^* \in {\mathbf{E}_1^*}^N[\gamma, i]^*$ implies that the intersection $h_\sigma^N(\mathcal{T}_\sigma(j_1) + \eta_1) \cap h_\sigma^N(\mathcal{T}_\sigma(j_2) + \eta_2)$ has zero measure, which yields that the intersection $(\mathcal{T}_\sigma(j_1) + \eta_1) \cap (\mathcal{T}_\sigma(j_2) + \eta_2)$ has measure zero, too. □

This proposition can be read as follows: the GIFS equation (5.17) implies that tiles translated by vectors issued from the tips in ${\mathbf{E}_1^*}^N([\gamma, i]^*)$ cannot intersect. This property will be exploited all through this chapter.

5.3.5 Self-replicating multiple tiling

We are now going to prove the multiple tiling property. First we need the following statement on subtiles, whose proof follows the proofs of (Praggastis 1999, Proposition 1.1) and (Sing 2006, Proposition 4.99).

Theorem 5.3.12 *Let σ be a unit Pisot irreducible substitution. The boundary of the central tile $\mathcal{T}_\sigma$ as well as the boundary of each of its subtiles $\mathcal{T}_\sigma(i)$ has zero measure. Moreover, $\mathcal{T}_\sigma$ as well as each of its subtiles is the closure of its interior.*

Proof One has $\tau_e(\partial X) = \partial(\tau_e(X))$, for every $e \in P_\sigma$ and every set X, since in the GIFS equation (5.8) defining the subtiles $\mathcal{T}_\sigma(i)$ the mappings τ_e are homeomorphisms. One has furthermore $\partial(A \cup B) \subseteq \partial A \cup \partial B$. We use the same notation as in the proof of Theorem 5.2.6. From (5.8) we deduce that

$$\beta\, \mu_{n-1}(\partial \mathcal{T}_\sigma(i)) \leq \sum_{j \in A} m_{ij}\, \mu_{n-1}(\partial \mathcal{T}_\sigma(j)). \tag{5.19}$$

Similarly as in the proof of Theorem 5.2.6, we obtain equality in (5.19). As the union in (5.8) is measure disjoint, the same is true for the sets $\partial \mathcal{T}_\sigma(i)$, for $i \in A$. In particular, they have either all positive measure, or all zero measure. Assume that they have all positive measure. By Corollary 5.3.7, the subtiles $\mathcal{T}_\sigma(i)$ have non-empty interior. Let $i \in A$. Take an open ball B included in the interior of $\mathcal{T}_\sigma(i)$. We consider (5.18) applied to $\mathcal{T}_\sigma(i)$, *i.e.*,

$$\mathcal{T}_\sigma(i)) = \bigcup_{[\eta,j]^* \in \mathbf{E}_1^{*\,N}([0,i]^*)} h_\sigma^N(\mathcal{T}_\sigma(j) + \eta). \tag{5.20}$$

We then take N large enough for $\tau_e^N(\mathcal{T}_\sigma(j)) \subseteq B$, for some j such that $\sigma^N(j) = pis$ and $e = (p,i,s) \in P_{\sigma^N}$. Here $e = (p,i,s)$ is an edge of the prefix-suffix graph associated with σ^N. One has $\tau_e(\nu) = h_\sigma^N(\nu) + \pi_c \circ \mathbf{P}(p)$ for $\nu \in \mathbb{H}_c$. Note also that such an integer N exists since the mappings τ_e are contractions. This implies that $\partial(\tau_e^N(\mathcal{T}_\sigma(j))) \cap \partial \mathcal{T}_\sigma(i) = \emptyset$. We also assume N to be large enough for $m_{ij}^N > 0$ (here we use the primitivity of $\mathbf{M}_\sigma$ and m_{ij}^N are the entries of $\mathbf{M}_\sigma^N$). We deduce from (5.20) that

$$\partial \mathcal{T}_\sigma(i) \subseteq \bigcup_{[\eta,k]^* \in \mathbf{E}_1^{*\,N}([0,i]^*),\ [\eta,k]^* \neq [\pi_c \circ \mathbf{P}(p), j]^*} \partial h_\sigma^N(\mathcal{T}_\sigma(k) + \eta). \tag{5.21}$$

This implies that

$$\mu_{n-1}(\partial \mathcal{T}_\sigma(i)) < \beta^{-N} \sum_{k \in A} m_{ik}^N\, \mu_{n-1}(\partial \mathcal{T}_\sigma(k)),$$

by recalling that the sets in the union on the right-hand side of (5.21) are disjoint in measure. However, this contradicts with the Nth iteration of (5.19) (where the inequality has been proved to be an equality). We thus have proved that the boundary of each subtile has measure zero.

Let us prove now that each subtile is the closure of its interior. Let $i \in A$ and let $\nu \in \mathcal{T}_\sigma(i)$. Let B be an open ball with centre ν. We use as previously the Nth decomposition formula (5.18) for N large enough, and obtain $\nu \in \tau_e^N(\mathcal{T}_\sigma(j)) \subseteq B$, for some j such that $\sigma^N(j) = pis$ and $e = (p,i,s) \in P_{\sigma^N}$. By Corollary 5.3.7, $\mathcal{T}_\sigma(j)$ has non-empty interior, and so does $\tau_e^N(\mathcal{T}_\sigma(j))$. Hence, B contains interior points of $\mathcal{T}_\sigma(i)$. We thus have proved that any open ball centred at ν contains interior points of $\mathcal{T}_\sigma(i)$. Since ν was an arbitrary element of $\mathcal{T}_\sigma$, we conclude that $\mathcal{T}_\sigma(i)$ is the closure of its interior. As $\mathcal{T}_\sigma(i) = \bigcup_{i \in A} \mathcal{T}_\sigma(i)$ the same is true for $\mathcal{T}_\sigma$. □

We now have gathered all prerequisites to be able to prove the following theorem (see (Sirvent and Wang 2002), (Berthé and Siegel 2005) and (Ei, Ito, and Rao 2006)).

Theorem 5.3.13 *Let σ be a unit Pisot irreducible substitution. The collection $\mathcal{I}_\sigma = \{\mathcal{T}_\sigma(i) + \gamma \mid [\gamma, i]^* \in \Gamma_c\}$ is a multiple tiling of $\mathbb{H}_c$. Moreover, Γ_c is repetitive.*

Proof We subdivide the proof into three parts.

The translation set Γ_c is locally finite. As $\mathcal{T}_\sigma(i)$ is compact for each $i \in A$ and as Γ_c is a uniformly discrete set according to Proposition 5.3.6, the collection $\mathcal{I}_\sigma$ is *locally finite*, *i.e.*, there exists a positive integer p such that each point of $\mathbb{H}_c$ is covered at most p times.

The translation set Γ_c is repetitive. We have to prove that for each patch P, there exists $R > 0$ such that each ball of radius R contains a translate of P. Let us fix a finite patch $P = \{[\pi_c(\mathbf{z}_k), i_k]^* \mid 1 \le k \le \ell\}$ of Γ_c. Let R_P be chosen in a way that the ball $B(0, R_P)$ contains the patch P.

We introduce the notion of *slice* above $\mathbb{H}_c$. We denote by $L[a, b] = \{\mathbf{x} \in \mathbb{Z}^n \mid a \le \langle \mathbf{x}, \mathbf{v}_\beta \rangle < b\}$ the set of points whose height is between a and b. Recall that the set Γ_c corresponds to the projection of points in $L[0, ||\mathbf{v}_\beta||_\infty]$.

By (5.15), there exists $\varepsilon_k > 0$ such that $\mathbf{z}_k$ belongs to the slice $L[0, (1 - \varepsilon_k)\langle \mathbf{e}_{i_k}, \mathbf{v}_\beta \rangle]$ for each $k \in \{1, \ldots, \ell\}$. Set $\varepsilon := \frac{1}{2} \min_k \langle \varepsilon_k \mathbf{e}_{i_k}, \mathbf{v}_\beta \rangle$ (note that $\varepsilon > 0$ since P is finite). Still by the definition of Γ_c, we deduce that for every $\mathbf{x} \in \mathbb{Z}^n$, assuming $\mathbf{x} \in L[0, \varepsilon]$ implies that the patch $\pi_c(\mathbf{x}) + P$ belongs to Γ_c.

It now remains to prove that there exists $R > 0$ such that any ball of radius R in $\mathbb{H}_c$ contains a point $\pi_c(\mathbf{x})$ with $\mathbf{x} \in L[0, \varepsilon]$. Recall that the coordinates of $\mathbf{v}_\beta$ are rationally independent. By Kronecker's theorem (Theorem 5.3.4), there exists $\mathbf{x}_0 \in \mathbb{Z}^n$ such that $\mathbf{x}_0 \in L[0, \varepsilon/2]$. Let us divide the slice $L[0, ||\mathbf{v}_\beta||_\infty]$ into $N = \lceil ||\mathbf{v}_\beta||_\infty 2/\varepsilon \rceil$ slices $L[j\varepsilon/2, (j+1)\varepsilon/2]$ of height $\varepsilon/2$. Since $0 < \langle \mathbf{x}_0, \mathbf{v}_\beta \rangle < \varepsilon/2$, each slice can be translated into $L(0, \varepsilon)$: for all $j \le N$, there exists m_j such that $m_j \mathbf{x}_0 + L[j\varepsilon/2, (j+1)\varepsilon/2] \subset L[0, \varepsilon]$.

Let us fix a point ν in $\mathbb{H}_c$. We use the fact that Γ_c is a Delone set, and in particular, that it is relatively dense (see Proposition 5.3.6). Let $R' > 0$ such that every ball of radius $R' > 0$ contains the image by π_c of a point of the discrete hyperplane (see Definition 5.3.1). In particular, the ball $B(\nu, R')$ contains a point $\pi_c(\mathbf{x})$ with $\mathbf{x} \in L[0, ||\mathbf{v}_\beta||_\infty]$. There exists j such that the point $\mathbf{x}$ belongs to one slice $L[j\varepsilon/2, (j+1)\varepsilon/2]$, hence there exists m_j such that $\mathbf{x} + m_j\mathbf{x}_0 \in L[0, \varepsilon]$. From above, this implies that $\pi_c(\mathbf{x} + m_j\mathbf{x}_0) + P$ occurs in Γ_c.

We deduce that the ball centred at ν with radius $R := R' + \max_k ||m_k\mathbf{x}_0|| + R_P$ contains a copy of the initial patch P up to translation. As $\nu \in \mathbb{H}_c$ was arbitrary this proves the repetitivity of Γ_c.

The collection $\mathcal{I}_\sigma$ is a multiple tiling. Suppose that this is wrong. Since the boundary of each subtile has measure zero by Theorem 5.3.12, the union of the boundaries of all elements of $\mathcal{I}_\sigma$ also has measure zero. Thus there are $\nu_1, \nu_2 \in \mathbb{H}_c$, positive integers $\ell_1 \neq \ell_2$ and $\varepsilon > 0$ such that $B(\nu_j, \varepsilon)$ is covered exactly ℓ_j times by the collection $\mathcal{I}_\sigma$, for $j = 1, 2$, *i.e.*, the points contained in $B(\nu_j, \varepsilon)$ belong to exactly ℓ_j tiles of the collection $\mathcal{I}_\sigma$. More precisely, there are patches $P_1, P_2 \subset \Gamma_c$ of cardinality ℓ_1 and ℓ_2, respectively, such that $B(\nu_j, \varepsilon) \subset \bigcap_{[\gamma,i]^* \in P_j} (\mathcal{T}_\sigma(i) + \gamma)$, for $j = 1, 2$. Moreover, $B(\nu_j, \varepsilon)$ has empty intersection with each tile of $\mathcal{I}_\sigma$ that is not contained in P_j. We assume w.l.o.g. that $\ell_1 < \ell_2$.

Consider now the inflated family $h_\sigma^{-m}\mathcal{I}_\sigma$ (we recall that the inverse of h_σ is an expansive mapping). By the arguments above each point in $h_\sigma^{-m}B(\nu_1, \varepsilon)$ is contained in exactly ℓ_1 tiles of $h_\sigma^{-m}\mathcal{I}_\sigma$. By Theorem 5.3.10 (iii), each tile of $h_\sigma^{-m}\mathcal{I}_\sigma$ has the shape $h_\sigma^{-m}(\mathcal{T}_\sigma(i) + \gamma)$, with $[\gamma, i]^* \in \Gamma_c$. By (5.17) and Proposition 5.3.11, such a tile can be decomposed as a finite union of tiles in $\mathcal{I}_\sigma$ which are pairwise disjoint in measure. Thus almost each point in $h_\sigma^{-m}B(\nu_1, \varepsilon)$ is contained in exactly ℓ_1 tiles of the family $\mathcal{I}_\sigma$.

Since the translation set Γ_c is repetitive, we can choose m so large that $h_\sigma^{-m}B(\nu_1, \varepsilon)$ contains a translated copy $P_2 + \gamma$ of the patch P_2. This means that $B(\nu_2, \varepsilon) + \gamma$ is contained in $h_\sigma^{-m}B(\nu_1, \varepsilon)$ for a large enough m. Recall that $B(\nu_2, \varepsilon)$ is covered exactly ℓ_2 times by $\mathcal{I}_\sigma$. There is *a priori* no reason for $B(\nu_2, \varepsilon) + \gamma$ to be covered exactly ℓ_2 times by $\mathcal{I}_\sigma$. Indeed other tiles might 'invade' $B(\nu_2, \varepsilon) + \gamma$. Nevertheless, it is covered by each element of the patch $P_2 + \gamma$ which implies that $B(\nu_2, \varepsilon) + \gamma$ is covered at least ℓ_2 times by elements of $\mathcal{I}_\sigma$. This yields a contradiction since almost every point in $h_\sigma^{-m}B(\nu_1, \varepsilon)$ is contained in exactly ℓ_1 tiles of $\mathcal{I}_\sigma$, and $\ell_1 < \ell_2$. □

Definition 5.3.14 Let σ be a unit Pisot irreducible substitution. We call the multiple tiling $\mathcal{I}_\sigma$ defined in Theorem 5.3.13 the *self-replicating multiple tiling* associated with σ.

For all known examples of unit Pisot irreducible substitutions the self-replicating multiple tiling is indeed a tiling, as illustrated in Figure 5.7, for instance.

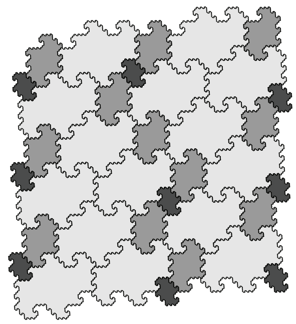

Fig. 5.7. The self-replicating (multiple) tiling for $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 1$. This multiple tiling is indeed a tiling for this substitution.

Definition 5.3.15 (Tiling property) A unit Pisot irreducible substitution σ satisfies the *tiling property* if the self-replicating multiple tiling is a tiling.

The *Pisot conjecture* states that as soon as σ is a unit Pisot irreducible substitution, the tiling property holds. Let us note that the Pisot conjecture has been proved to hold for unit Pisot irreducible substitutions over a two-letter alphabet in (Hollander and Solomyak 2003). The proof strongly relies on the fact that the combinatorial strong coincidence condition is satisfied by every Pisot irreducible substitution over a two-letter alphabet (Barge and Diamond 2002), although the combinatorial strong coincidence condition does not imply the tiling property for a general alphabet.

Note that an immediate reformulation of the tiling property is that $\mu_{n-1}((\mathcal{T}(i) + \gamma) \cap (\mathcal{T}(j) + \eta)) = 0$, for every pair of distinct tiles $\{\mathcal{T}(i) + \gamma, \mathcal{T}(j) + \eta\}$ of the self-replicating multiple tiling.

Note also that in view of the following theorem the assumption that u is generated by σ causes no loss of generality.

Theorem 5.3.16 *Let σ be a unit Pisot irreducible substitution. Let k, ℓ be two positive integers. One has $\mathcal{T}_{\sigma^k} = \mathcal{T}_{\sigma^\ell}$. Furthermore, the substitution σ^k satisfies the tiling property if, and only if, σ^ℓ satisfies the tiling property.*

Proof Let us note that $h_{\sigma^k} = h_\sigma^k$ for all k. According to (5.10), the central tiles $\mathcal{T}_{\sigma^k}$ and $\mathcal{T}_{\sigma^\ell}$ are seen to satisfy

$$\forall\, i \in A,\ \mathcal{T}_{\sigma^k}(i) = \bigcup_{j,(p,i,s),\ \sigma^{k\ell}(j)=pis} h_\sigma^{k\ell}(\mathcal{T}_{\sigma^k}(j)) + \pi_c \circ \mathbf{P}(p) \quad \text{and}$$

$$\forall\, i \in A,\ \mathcal{T}_{\sigma^\ell}(i) = \bigcup_{j,(p,i,s),\ \sigma^{k\ell}(j)=pis} h_\sigma^{k\ell}(\mathcal{T}_{\sigma^\ell}(j)) + \pi_c \circ \mathbf{P}(p),$$

respectively. One deduces that $\mathcal{T}_{\sigma^k}(i)$ and $\mathcal{T}_{\sigma^\ell}(i)$ satisfy the same GIFS equation, and thus, that they coincide. Furthermore, the set Γ_c only depends on $\mathbf{v}_\beta$, which is a common left eigenvector for σ^k and σ^ℓ. This concludes the proof. □

Remark 5.3.17 Let σ be a unit Pisot irreducible substitution that is possibly not prolongable. Assume that σ^k is prolongable for some k (such a k always exists by primitivity of σ). Let u be generated by σ^k with $\sigma^k(u) = u$. We define the *central tile* associated with σ as $\mathcal{T}_\sigma := \bigcup_{i\in A} \mathcal{T}_\sigma(i)$, where the non-empty compact sets $\mathcal{T}_\sigma(i)$ are uniquely determined by the following GIFS equation

$$\forall\, i \in A,\ \mathcal{T}_\sigma(i) = \bigcup_{j,(p,i,s),\ \sigma^j=pis} h_\sigma(\mathcal{T}_\sigma(j)) + \pi_c \circ \mathbf{P}(p).$$

By taking the k-fold iteration of this equation and by uniqueness of its solution, we deduce that $\mathcal{T}_\sigma(i) = \mathcal{T}_{\sigma^k}(i)$, for every $i \in A$.

5.4 Ancestor graphs and tiling conditions

In the remaining part of this chapter we present various conditions for the self-replicating multiple tiling to be a tiling. Recall that the substitution σ satisfies the tiling property if, and only if, each intersection of distinct tiles in the self-replicating multiple tiling has zero measure. In the present section we focus on effective ways to control the measure of intersections of tiles. In Section 5.4.1 we introduce a sufficient condition for the tiling property. In Section 5.4.2 we define a graph that provides an effective way to check this sufficient condition. This leads us to introduce a more intricate graph in Section 5.4.3. This graph yields a necessary and sufficient condition for the tiling property.

5.4.1 Finiteness properties

We have already gained information on intersections of subtiles with zero measure. Indeed, Theorem 5.2.6 states that the shrunken copies of subtiles occurring in the decomposition of each subtile $\mathcal{T}_\sigma(i)$ are disjoint in measure. Moreover, by Theorem 5.2.10, the subtiles $\mathcal{T}_\sigma(i)$, $i \in A$, are disjoint if the substitution σ satisfies the combinatorial strong coincidence condition. We now define a sufficient condition that allows this information to be spread on zero measure intersections throughout the self-replicating multiple tiling, and thus, to exhibit a sufficient condition for the multiple tiling to be indeed a tiling.

Let U denote the patch

$$U := [0,1]^* \cup [0,2]^* \cup \cdots \cup [0,n]^*. \tag{5.22}$$

It is easy to see that $U \subset \Gamma_c$ by (5.15). Pursuing the analogy between tips and their geometric representations in terms of projected faces (see Section 5.3.3) we call U by slight abuse of language the *lower unit cube*.

One easily checks that U is contained in $\mathbf{E}_1^*(U)$. Indeed, for any $j \in A$, $[0, j^*]$ is contained in $\mathbf{E}_1^*([0,i]^*)$, where i is the first letter of $\sigma(j)$. Hence, we deduce from Theorem 5.3.10 (ii) that the sequence of patches $(\mathbf{E}_1^{*m}(U))_{m\geq 0}$ is an increasing sequence of subsets of Γ_c with respect to inclusion. A specific case occurs when $\mathbf{E}_1^{*m}(U)$ eventually covers the entire self-replicating translation set Γ_c if m tends to infinity. As an illustration, the set $\mathbf{E}_1^{*m}(U)$ is depicted in Figures 5.8 and 5.9, in each case for a specific m, for the substitutions $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 1$ and $\tau(1) = 2$, $\tau(2) = 3$, $\tau(3) = 12$. These pictures indicate that $\mathbf{E}_1^{*m}(U)$ eventually covers the whole self-replicating translation set Γ_c in the case of σ, but not in the case of τ.

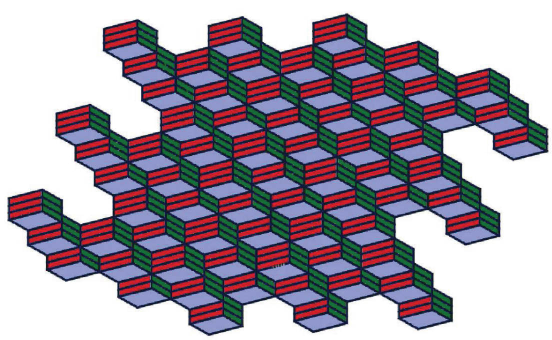

Fig. 5.8. The patch $\mathbf{E}_1^{*5}(U)$ for the substitution $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 1$.

Definition 5.4.1 (Geometric finiteness property) Let σ be a unit Pisot irreducible substitution and $\mathbf{E}_1^*$ be its associated GIFS substitution

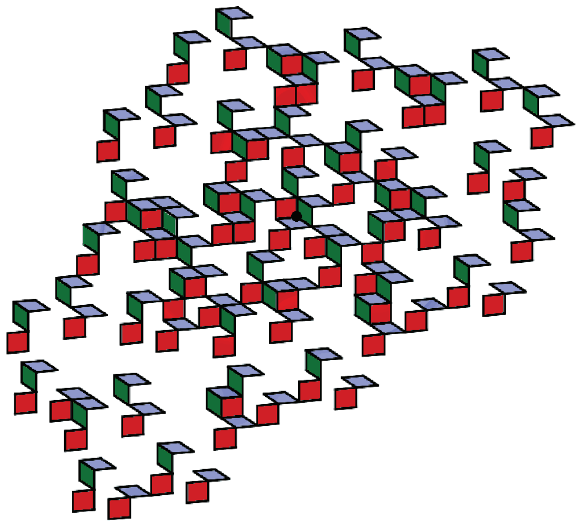

Fig. 5.9. The patch $\mathbf{E}_1^{*15}(U)$ for the substitution $\tau(1) = 2$, $\tau(2) = 3$, $\tau(3) = 12$.

on tips. We say that σ satisfies the *geometric finiteness property* if

$$\Gamma_c = \bigcup_{m \in \mathbb{N}} \mathbf{E}_1^{*m}(U). \tag{5.23}$$

Let us see how to propagate information on zero measure intersections inside the subtiles $\mathcal{T}_\sigma(i)$ to all intersections occurring in the self-replicating tiling when σ satisfies the geometric finiteness property.

Theorem 5.4.2 *Let σ be a unit Pisot irreducible substitution. If σ satisfies both the geometric finiteness property and the combinatorial strong coincidence condition, then the self-replicating multiple tiling is a tiling.*

Proof Let us consider two tiles in the self-replicating multiple tiling, namely $\mathcal{T}_\sigma(i_1) + \gamma_1$ and $\mathcal{T}_\sigma(i_2) + \gamma_2$. By the geometric finiteness property and by (5.17), there exist N, j_1, j_2 such that $\mathcal{T}_\sigma(i_1)+\gamma_1$ (respectively $\mathcal{T}_\sigma(i_2)+\gamma_2$) is a piece of the Nth level decomposition (5.18) of a subtile $\mathcal{T}_\sigma(j_1)$ (respectively $\mathcal{T}_\sigma(j_2)$). If $j_1 = j_2$, we are done because we fall into the assumptions of Proposition 5.3.11. If $j_1 \neq j_2$, we know from Theorem 5.2.10 and the combinatorial strong coincidence assumption that $\mathcal{T}_\sigma(j_1)$ and $\mathcal{T}_\sigma(j_2)$ are disjoint up to a set of zero measure. □

A more restrictive condition for tiling is the following *superfiniteness property* (compare with Definition 5.4.1).

Definition 5.4.3 (Geometric superfiniteness property) Let σ be a unit Pisot irreducible substitution and $\mathbf{E}_1^*$ be its associated GIFS substitution on tips. We say that σ satisfies the *geometric superfiniteness property* if there exists $i \in A$ such that

$$\Gamma_c = \bigcup_{m \in \mathbb{N}} \mathbf{E}_1^{*m}([0,i]^*).$$

Note that in this case, the proof of Theorem 5.4.2 applies without requiring the assumption of the combinatorial strong coincidence.

5.4.2 The ancestor graph

We will now discuss how to check in an effective way whether the geometric finiteness property holds. The idea is to prove that the geometric finiteness property is satisfied if, and only if, an explicit finite patch (depending on σ) is eventually covered by the iterations of $\mathbf{E}_1^*$ on the lower unit cube U.

As a consequence of Theorem 5.3.10, every tip has a unique pre-image under the action of $\mathbf{E}_1^*$. We will refer to this pre-image as *ancestor*.

Definition 5.4.4 (Ancestor of a tip) The *ancestor* of $[\eta, j]^* \in \Gamma_c$ is the unique tip $[\gamma, i]^* \in \Gamma_c$ for which $[\eta, j]^* \in \mathbf{E}_1^*([\gamma, i]^*)$.

We have worked so far with the Euclidean norm in $\mathbb{H}_c$. We now introduce a more convenient norm based on (5.3). Let $||\cdot||_c$ denote the maximum norm on $\mathbb{H}_c$ with respect to vectors $\mathbf{u}_{\beta^{(i)}}$ for $i \geq 2$, *i.e.*,

$$\forall\, \nu \in \mathbb{H}_c,\ ||\nu||_c = \max\{|\langle \nu, \mathbf{v}_{\beta^{(i)}} \rangle| \mid i = 2, \ldots, r+s\}. \tag{5.24}$$

Let $\beta_{\max} := \max\{|\beta^{(j)}| \mid j \geq 2\}$. One has

$$||h_\sigma(\nu)||_c \leq \beta_{\max}\, ||\nu||_c \text{ for all } \nu \in \mathbb{H}_c. \tag{5.25}$$

We denote by $B_c(\nu, R)$ the ball centred at ν of radius R with respect to this norm. Let $M_\sigma := \max\{||\pi_c \circ \mathbf{P}(p)||_c \mid (p, a, s) \in P_\sigma\}$ (see (5.7) for the definition of P_σ).

Definition 5.4.5 (Seed patch) The *seed patch* V_σ associated with the substitution σ is defined as

$$V_\sigma := \left\{ [\gamma, i]^* \in \Gamma_c \mid ||\gamma||_c \leq \frac{M_\sigma}{1 - \beta_{\max}} \right\}. \tag{5.26}$$

Remark 5.4.6 Note that $M_\sigma/(1 - \beta_{\max})$ is an upper bound for the diameter of the tiles $\mathcal{T}_\sigma(i)$, according to the proof of Corollary 5.2.8. Thus $0 \in \mathcal{T}_\sigma(j) + \gamma$ implies that $[\gamma, j]^* \in V_\sigma$.

Theorem 5.4.7 *Let σ be a unit Pisot irreducible substitution. One has*

$$\Gamma_c = \bigcup_{m\in\mathbb{N}} \mathbf{E}_1^{*m}(V_\sigma).$$

Proof The definition of $\mathbf{E}_1^*$ yields that if $[\gamma, i]^*$ is the ancestor of the tip $[\eta, j]^*$, then

$$\gamma = h_\sigma(\eta) - \pi_c \circ \mathbf{P}(p), \tag{5.27}$$

where p is a prefix of $\sigma(i)$. We fix $[\eta, j]^* \in \Gamma_c$. Following Definition 5.4.4, let $[\gamma_k, i_k]^*$ be the successive ancestors of $[\eta, j]^*$, *i.e.*, $[\eta, j]^* \in \mathbf{E}_1^*([\gamma_1, i_1]^*)$ and $[\gamma_k, i_k]^* \in \mathbf{E}_1^*([\gamma_{k+1}, i_{k+1}]^*)$ for all $k \geq 1$. By (5.27), one has $\gamma_{k+1} = h_\sigma(\gamma_k) - \pi_c \circ \mathbf{P}(p)$ where p is a prefix of $\sigma(i_k)$. Therefore we have by (5.25)

$$||\gamma_{k+1}||_c \leq \beta_{\max}||\gamma_k||_c + M_\sigma. \tag{5.28}$$

Let $\alpha \in (\beta_{\max}, 1)$. Then, if $a \geq M_\sigma/(\alpha - \beta_{\max})$, one has $\beta_{\max}a + M_\sigma \leq \alpha a$. This implies

$$||\gamma_k||_c \geq \frac{M_\sigma}{\alpha - \beta_{\max}} \implies ||\gamma_{k+1}||_c \leq \alpha ||\gamma_k||_c. \tag{5.29}$$

Let $V^{(\alpha)} := \left\{[\eta, j]^* \in \Gamma_c \mid ||\gamma||_c < \frac{M_\sigma}{\alpha - \beta_{\max}}\right\}$. All the $V^{(\alpha)}$ are finite patches since Γ_c is uniformly discrete. We also notice that

$$\bigcap_{\beta_{\max} < \alpha < 1} V^{(\alpha)} = V_\sigma.$$

Therefore, there exists $\alpha_0 < 1$ such that $V^{(\alpha_0)} = V_\sigma$ for all $\alpha_0 \leq \alpha < 1$.

By iterating (5.29) we deduce that there is $k \in \mathbb{N}$ such that the kth ancestor $[\gamma_k, i_k]^*$ of $[\eta, j]^*$ satisfies $[\gamma_k, i_k]^* \in V_{\alpha_0}$. As $V^{(\alpha_0)} = V_\sigma$, this implies that $\Gamma_c = \bigcup_{m\in\mathbb{N}} \mathbf{E}_1^{*m}(V_\sigma)$. □

Theorem 5.4.7 is based on the fact that β is a Pisot number. Analogous statements appear in various frameworks, see for instance the references (Akiyama 2000), (Arnoux, Berthé, and Siegel 2004), (Barge and Diamond 2002), (Barge and Kwapisz 2006), (Fernique 2006), (Fuchs and Tijdeman 2006) or (Ito and Rao 2006).

Remark 5.4.8 According to (5.28), one checks that V_σ contains the ancestors of all its elements (but note also that V_σ is not stable under the action of $\mathbf{E}_1^*$). Furthermore, the seed patch V_σ is easily seen to be effectively computable. Note also that $U \subseteq V_\sigma$.

We deduce from Theorem 5.4.7 the following corollary.

Corollary 5.4.9 *If there exists $m \geq 1$ such that ${\mathbf{E}_1^*}^m(U)$ contains V_σ, then the geometric finiteness property holds. In this case, we can effectively exhibit such an m.*

The proof of Theorem 5.4.7 mostly relies on the notion of ancestor. In order to obtain an algorithmic way to check the geometric finiteness property, we construct a directed graph based on the notion of ancestor and on the seed patch.

Definition 5.4.10 (Ancestor graph) The vertices of the *ancestor graph* are the tips that occur in the seed patch V_σ introduced in Definition 5.4.5. There is an edge from $[\eta, j]^*$ to $[\gamma, i]^*$ if $[\gamma, i]^*$ is the ancestor of $[\eta, j]^*$, *i.e.*, $[\eta, j]^* \in \mathbf{E}_1^*([\gamma, i]^*)$.

The computation of the ancestor graph is straightforward. First, list the tips that belong to the seed patch V_σ. Then, for every $[\gamma, i]^* \in V_\sigma$, compute the tips $[\eta, j]^* \in \mathbf{E}_1^*([\gamma, i]^*)$, and draw an edge from every $[\eta, j]^*$ to $[\gamma, i]^*$, if $[\eta, j]^* \in V_\sigma$.

Remark 5.4.11 The choice of orientation we have made here for the ancestor graph (which consists in following the ancestor relation) might seem to be counter-intuitive at first sight, and in contradiction with the orientation of edges in the prefix-suffix graph. Note that a converse choice has been made in (Siegel and Thuswaldner 2010) for similar graph constructions. Their purpose was to study the boundary of subtiles and thus, to be able to zoom inside the subtiles. On the opposite, here we want to be able to zoom outside the subtiles in order to cover the self-replicating multiple tiling, hence, to trace back ancestors.

By uniqueness of the ancestor and by stability of V_σ with respect to ancestors, every vertex in the ancestor graph admits exactly one outgoing egde. This implies that every sufficiently long path in this finite graph reaches a cycle and cannot exit from it. By a *cycle*, we mean a closed directed path. Note furthermore that a tip of the form $[0, j]^*$ admits a unique outgoing edge which is also of the form $[0, i]^*$. Indeed, the (unique) ancestor of $[0, j]^*$ is the tip $[0, i]^*$ where i is the first letter of $\sigma(j)$. Hence, once a cycle contains a tip of U, it contains only elements of U. We say that it is *contained* in U. This provides a simple effective condition to check the geometric finiteness property.

Proposition 5.4.12 *Let σ be a unit Pisot irreducible substitution. The geometric finiteness property is satisfied if, and only if, all cycles in the ancestor graph are contained in U.*

Proof If the geometric finiteness property holds, then any sufficiently long path in the ancestor graph contains a tip of U, and thus any cycle in the ancestor graph is contained in U. Conversely, assume that any cycle is contained in U. Any tip in V_σ admits in its sequence of successive ancestors one element that belongs to a cycle, hence to U, which ends the proof in view of Corollary 5.4.9. □

Example 5.4.13 Two examples of ancestor graphs are depicted in Figures 5.10 and 5.11. One can see that the graph corresponding to $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 1$ satisfies the condition of Proposition 5.4.12, whereas the graph corresponding to $\tau(1) = 2$, $\tau(2) = 3$, $\tau(3) = 12$ does not satisfy it. In this second example one can see that the graph, which is made of two connected components, admits two cycles, only one of them made of tips of U.

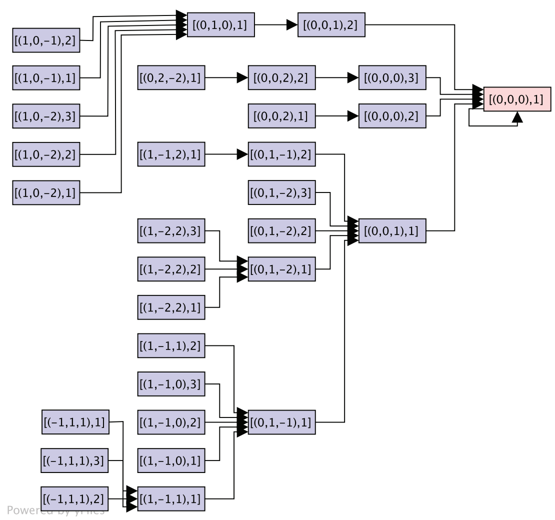

Fig. 5.10. The ancestor graph for the substitution $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 1$. To keep notation simple in the picture of the graph, we omitted the projection π_c in the labels of the vertices.

5.4.3 The two-piece ancestor graph

The geometric finiteness property means that the self-replicating translation set Γ_c can be covered by iterating $\mathbf{E}_1^*$ on the patch U. It turns out that

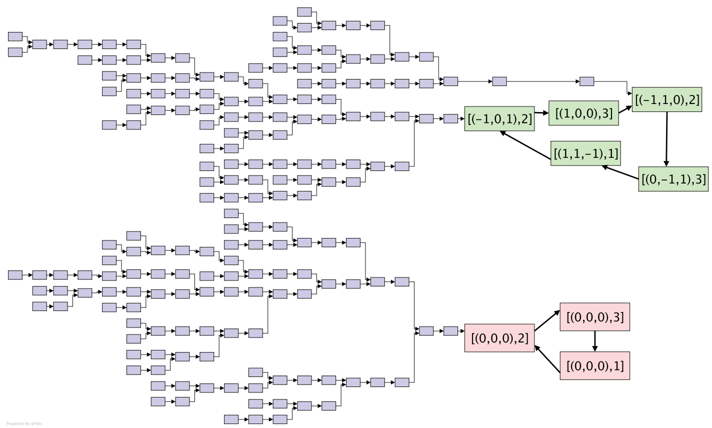

Fig. 5.11. The ancestor graph for the substitution $\tau(1) = 2$, $\tau(2) = 3$, $\tau(3) = 12$. To keep notation simple in the picture of the graph, we omitted the projection π_c in the labels of the vertices.

this condition is not necessary for the tiling property. To overcome this, we will use the repetitivity of Γ_c, and deal with translations of the sets $\mathbf{E}_1^{*m}(U)$. This will lead us to introduce a further graph, inspired by the ancestor graph, which will allow an algorithmic criterion equivalent to the tiling property to be given, and not only a sufficient condition, such as Proposition 5.4.12.

Theorem 5.4.14 (Ito and Rao 2006) *Let σ be a unit Pisot irreducible substitution and $\mathbf{E}_1^*$ be its associated GIFS substitution on tips. The self-replicating multiple tiling is a tiling if, and only if, for every $i \in A$, the radius of the largest ball contained in the union*

$$\bigcup_{[\gamma,j]^* \in \mathbf{E}_1^{*m}([0,i]^*)} [\gamma, j]_g^* \tag{5.30}$$

tends to infinity with m.

Proof Let us note that this statement does not depend on the choice of the norm by the equivalence of norms. Assume that σ satisfies the tiling property. We fix $i \in A$. For $m \in \mathbb{N}$, let $B_c(\delta_m, R_m)$ be the ball (for the norm $||\cdot||_c$) with largest radius contained in $h_\sigma^{-m}(\mathcal{T}_\sigma(i))$. Since $\mathcal{T}_\sigma(i)$ has non-empty interior (Corollary 5.3.7) and since h_σ^{-1} is an expansion, we have that $\lim_{m\to+\infty} R_m = \infty$. The GIFS equation (5.17) yields that $h_\sigma^{-m}\mathcal{T}_\sigma(i)$ is covered by the tiles $\mathcal{T}_\sigma(j) + \gamma$ with $[\gamma, j]^* \in \mathbf{E}_1^{*m}([0, i]^*)$. From the tiling assumption, we deduce that for every tip $[\eta,k]^* \notin \mathbf{E}_1^{*m}([0,i]^*)$, the tile $\mathcal{T}_\sigma(k)+\eta$ is measure disjoint from $B_c(\delta_m,R_m)$. Let C denote the diameter of the central tile $\mathcal{T}_\sigma$, *i.e.*, $C=\sup\{||\nu-\nu'||_c \mid \nu,\nu'\in\mathcal{T}_\sigma\}$. Therefore, every tip $[\gamma,j]^*$ with $||\gamma-\delta_m||_c < R_m - C$ has to belong to $\mathbf{E}_1^{*m}([0,i]^*)$. This implies that the radius of the largest ball contained in the union in (5.30) tends to infinity with m.

Conversely, assume that the radius of the largest ball contained in the union in (5.30) tends to infinity with m. Let P be a patch of Γ_c. By repetitivity (Theorem 5.3.13), P is contained, up to a translation vector, in any large enough ball of Γ_c. Therefore there exist $\nu\in\mathbb{H}_c$ and $m>0$ such that $\nu+P\subset\mathbf{E}_1^{*m}([0,i]^*)$, and thus $P\subset\mathbf{E}_1^{*m}([h_\sigma^m\nu,i]^*)$ by the definition of $\mathbf{E}_1^*$ in (5.16). Proposition 5.3.11 then yields that the tiles $\mathcal{T}_\sigma(j)+\gamma$ with $[\gamma,j]\in P$ have pairwise disjoint interiors. This implies that σ satisfies the tiling property. □

Corollary 5.4.15 *Let σ be a unit Pisot irreducible substitution. The self-replicating multiple tiling is a tiling if, and only if, for every pair of tips $([\eta_1,j_1]^*,[\eta_2,j_2]^*)\in\Gamma_c^2$ there exist $\delta\in\mathbb{H}_c$, $m\geq 0$, and $i\in A$ such that*

$$\delta+\{[\eta_1,j_1]^*,[\eta_2,j_2]^*\}\subset\mathbf{E}_1^{*m}([0,i]^*). \tag{5.31}$$

Proof If this condition is satisfied, Proposition 5.3.11 implies that $\mathcal{T}_\sigma(j_1)+\eta_1$ and $\mathcal{T}_\sigma(j_2)+\eta_2$ do not overlap for arbitrarily chosen $[\eta_1,j_1]^*$ and $[\eta_2,j_2]^*$. Therefore the tiling property is satisfied. Conversely, by Theorem 5.4.14, the tiling property implies that $\mathbf{E}_1^{*m}([0,i]^*)$ contains arbitrarily large balls for large m. Thus, by the repetitivity assertion of Theorem 5.3.13, a translation of each patch $\{[\eta_1,j_1]^*,[\eta_2,j_2]^*\}\subset\Gamma_c$ occurs in $\mathbf{E}_1^{*m}([0,i]^*)$ for some $i\in A$ and some $m\in\mathbb{N}$. □

The formulation of the tiling property given by Corollary 5.4.15 means that every pair of tips belongs to the image of a tip in U up to a common translation vector. In order to check (5.31), we need to trace back ancestors of patches *up to a translation vector*. When dealing with (5.31), we will use the existence of the translation vector δ in order to work only with pairs of tips for which at least one of the elements η_1,η_2 equals 0. We thus introduce the following definition.

Definition 5.4.16 (Two-piece ancestor) Let $\{[\eta_1,j_1]^*,[\eta_2,j_2]^*\}$ be a two-piece patch in Γ_c. A *two-piece ancestor* of this patch is a two-piece patch of the shape $\{[0,i_1]^*,[\gamma,i_2]^*\}\subset\Gamma_c$ for which there exists $\delta\in\mathbb{H}_c$ such that

$$\{[\eta_1,j_1]^*,[\eta_2,j_2]^*\}\subset\delta+\mathbf{E}_1^*\{[0,i_1]^*,[\gamma,i_2]^*\} \tag{5.32}$$

with $\{[\eta_1, j_1]^*, [\eta_2, j_2]^*\} \cap (\delta + \mathbf{E}_1^*[0, i_1]^*) \neq \emptyset$ and $\{[\eta_1, j_1]^*, [\eta_2, j_2]^*\} \cap (\delta + \mathbf{E}_1^*[\gamma, i_2]^*) \neq \emptyset$.

In other words, this means that $\{[\eta_1, j_1]^*, [\eta_2, j_2]^*\}$ appears in the image of the patch $\{[0, i_1]^*, [\gamma, i_2]^*\}$ up to a translated vector, and that the two images of $[0, i_1]^*$ and $[\gamma, i_2]^*$ both have non-empty intersection with $\{[\eta_1, j_1]^*, [\eta_2, j_2]^*\}$. Note that, contrary to the uniquely defined ancestor of a tip (see Definition 5.4.4), a two-piece patch can have several two-piece ancestors. This is due to the freedom given by the translation vector δ. Another important remark is that the tips $[\eta_1, j_1]^*$ and $[\eta_2, j_2]^*$ in Definition 5.4.16 are not required to be different. The same holds for $[0, i_1]^*$ and $[\gamma, i_2]^*$.

In order to check the tiling condition (5.31), we need to recursively check ancestor relations. We thus define a new graph, namely the *two-piece ancestor graph*. To this end we need a new seed patch which is defined as follows.

Definition 5.4.17 (Two-piece seed patch) The *two-piece seed patch* W_σ associated with the substitution σ is defined as

$$W_\sigma := \left\{ [\gamma, i]^* \in \Gamma_c \mid ||\gamma||_c \leq \frac{2M_\sigma}{1 - \beta_{\max}} \right\}. \tag{5.33}$$

Remark 5.4.18 Note that $2M_\sigma/(1 - \beta_{\max})$ is at least twice as large as the diameter of the tiles $\mathcal{T}_\sigma(i)$. Thus $\mathcal{T}_\sigma(i) \cap (\mathcal{T}_\sigma(j) + \gamma) \neq \emptyset$ implies that $[\gamma, j]^* \in W_\sigma$.

Moreover, we represent the pair of tips $\{[0, k]^*, [\gamma, \ell]^*\}$ as $[k, \gamma, \ell]^*$, with $k \leq \ell$ if $\gamma = 0$. The condition $k \leq \ell$ if $\gamma = 0$ simply avoids redundancies.

Definition 5.4.19 (Two-piece ancestor graph) The set of vertices of the *two-piece ancestor graph* is equal to

$$\{[k, \gamma, \ell]^* \mid (k, \gamma, \ell) \in A \times \mathbb{H}_c \times A,\ [\gamma, \ell]^* \in W_\sigma,\ k \leq \ell \text{ if } \gamma = 0\}.$$

There is an edge from $[j_1, \eta, j_2]^*$ to $[i_1, \gamma, i_2]^*$ if the patch $\{[0, i_1]^*, [\gamma, i_2]^*\}$ is a two-piece ancestor of the patch $\{[0, j_1]^*, [\eta, j_2]^*\}$.

One checks that each vertex admits at least one outgoing edge. Nevertheless, there might be several outgoing edges.

Following (Siegel and Thuswaldner 2010), the construction of this graph is straightforward when recalling that similar to V_σ, the two-piece seed patch W_σ can be explicitly computed (see Section 5.4.2). The construction can thus be performed in two steps. One first computes the list of vertices

$[i_1, \gamma, i_2]^*$ of the graph, based on the computation of the two-piece seed patch W_σ. Then, by noticing that $[\eta, j]^* \subset \delta + \mathbf{E}_1^*([\gamma, i]^*)$ if, and only if, there exists a prefix p of $\sigma(j)$ such that $\sigma(j) = pis$ and $\delta = \eta - h_\sigma^{-1}(\gamma + \pi_c \circ \mathbf{P}(p))$, one checks whether condition (5.32) is satisfied for each pair of vertices $([j_1, \eta, j_2]^*, [i_1, \gamma, i_2]^*)$.

We need the following easy lemma.

Lemma 5.4.20 *Let $\{[\gamma_1, i_1]^*, [\gamma_2, i_2]^*\}$ be a patch in Γ_c. Then at least one of the sets $\{[0, i_1]^*, [\gamma_2 - \gamma_1, i_2]^*\}$ and $\{[0, i_2]^*, [\gamma_1 - \gamma_2, i_1]^*\}$ is a patch in Γ_c.*

Proof By (5.5), there exists a unique pair vectors $\{\mathbf{x}_1, \mathbf{x}_2\} \subset \mathbb{Z}^n$ such that $\gamma_i = \pi_c(\mathbf{x}_i)$, for $i = 1, 2$. If $\langle \mathbf{x}_1, \mathbf{v}_\beta \rangle \leq \langle \mathbf{x}_2, \mathbf{v}_\beta \rangle$ then $\{[0, i_1]^*, [\gamma_2 - \gamma_1, i_2]^*\}$ is a patch of Γ_c. If the reverse inequality holds, $\{[0, i_2]^*, [\gamma_1 - \gamma_2, i_1]^*\} \subset \Gamma_c$ and we are done.

□

We now can state the main result of this section.

Theorem 5.4.21 (Two-piece ancestor graph tiling condition) *Let σ be a unit Pisot irreducible substitution. The substitution σ satisfies the tiling condition if, and only if, from any vertex in the two-piece ancestor graph, there exists a path to a vertex of the shape $[i, 0, i]^*$, for $i \in A$.*

Proof The tiling property is equivalent to (5.31), which is itself equivalent to the following condition: for every two-piece patch $\{[0, j_1]^*, [\eta, j_2]^*\}$, there exist $m \in \mathbb{N}$, $i \in A$ and $\delta \in \mathbb{H}_c$ such that $\{[0, j_1]^*, [\eta, j_2]^*\} \subset \delta + \mathbf{E}_1^{*m}([0, i]^*)$. By the definition of the two-piece ancestor this is equivalent to the fact that for each $\{[0, j_1]^*, [\eta, j_2]^*\}$ there is $m \in \mathbb{N}$ and $i \in A$ such that

$$\{[0, i^*], [0, i^*]\} \text{ is an } m\text{th ancestor of } \{[0, j_1^*], [\eta, j_2^*]\}. \tag{5.34}$$

In order to deduce Theorem 5.4.21, it is sufficient to show that we can assume w.l.o.g. that $[\eta, j_2]^* \in W_\sigma$ holds in (5.34).

Suppose on the contrary that $[\eta, j_2]^* \notin W_\sigma$. Then

$$||\eta||_c > \frac{2M_\sigma}{1 - \beta_{\max}}. \tag{5.35}$$

There exists a unique set of two elements $\{[\gamma_1, i_1]^*, [\gamma_2, i_2]^*\} \subset \Gamma_c$ such that

$$[0, j_1]^* \in \mathbf{E}_1^*[\gamma_1, i_1]^* \quad \text{and} \quad [\eta, j_2]^* \in \mathbf{E}_1^*[\gamma_2, i_2]^*. \tag{5.36}$$

By Lemma 5.4.20 one of the sets $\{[0, i_1]^*, [\gamma_2 - \gamma_1, i_2]^*\}$, $\{[0, i_2]^*, [\gamma_1 - \gamma_2, i_1]^*\}$ is contained in Γ_c. Assume that this is true for the first one (the second alternative is handled analogously). Then $\{[0, i_1]^*, [\gamma_2 - \gamma_1, i_2]^*\}$ is a two-piece ancestor of $\{[0, j_1]^*, [\eta, j_2]^*\}$. By (5.27) and (5.36) we have

$$||\gamma_1||_c \leq M_\sigma \quad \text{and} \quad ||\gamma_2||_c \leq \beta_{\max}||\eta||_c + M_\sigma$$

which implies together with (5.35) that $||\gamma_2 - \gamma_1||_c \leq \frac{2M_\sigma}{1-\beta_{\max}} < ||\eta||_c$. Thus, arguing in the same way as in the proof of Theorem 5.4.7 we see that there is a positive integer m' such that $\{[0, j_1]^*, [\eta, j_2]^*\}$ admits an m'th two-piece ancestor $\{[0, k_1]^*, [\gamma', k_2]^*\}$ which satisfies $\gamma' \in W_\sigma$. Thus it suffices to assume $[\eta, j_2]^* \in W_\sigma$ in (5.34) and we are done. □

5.5 Boundary and contact graphs

The tiling condition of Theorem 5.4.21 can be checked in an effective way by constructing the two-piece ancestor graph. However, it turns out that the two-piece ancestor graph is not so easy to handle. Indeed, it can be quite big especially if n is large, and as a second drawback, contrary to the ancestor graph it is not deterministic. We thus introduce two subgraphs of the two-piece ancestor graph, and establish associated tiling conditions, inspired by Proposition 5.4.12 and Theorem 5.4.21.

5.5.1 Boundary graphs

We first state as an immediate consequence of Lemma 5.4.20 the following proposition which shows that we only have to consider intersections between the subtiles $\mathcal{T}(i)$, $i \in A$, and their neighbours in $\mathcal{I}_\sigma$ (see Definition 5.3.14) to check the tiling property.

Proposition 5.5.1 *The tiling property is satisfied if, and only if,* $\mathcal{T}_\sigma(i) \cap (\mathcal{T}_\sigma(j) + \gamma)$ *has zero measure for every* $i \in A$ *and every* $[\gamma, j]^* \in \Gamma_c$ *with* $[\gamma, j]^* \neq [0, i]^*$.

Proof The tiling property holds if, and only if, for any two distinct tips $[\gamma, i]^*, [\eta, j]^* \in \Gamma_c$, we have $\mu_{n-1}((\mathcal{T}_\sigma(i) + \gamma) \cap (\mathcal{T}_\sigma(j) + \eta)) = 0$. But $(\mathcal{T}_\sigma(i)+\gamma) \cap (\mathcal{T}_\sigma(j)+\eta)$ is equal, up to a translation, to $\mathcal{T}_\sigma(i) \cap (\mathcal{T}_\sigma(j)+\eta-\gamma)$, and to $\mathcal{T}_\sigma(j) \cap (\mathcal{T}_\sigma(i)+\gamma-\eta)$. Lemma 5.4.20 implies that either $[\eta-\gamma, j]^* \in \Gamma_c$ or $[\gamma - \eta, i]^* \in \Gamma_c$. This ends the proof. □

As a motivation for the definition of the boundary graph (see Definition 5.5.3 below), let us dwell upon the topological information provided by cycles in the ancestor graph.

Lemma 5.5.2 *A vertex $[\gamma, i]^*$ belongs to a cycle in the ancestor graph if, and only if,* $0 \in \mathcal{T}_\sigma(i) + \gamma$.

Proof We first assume that $[\gamma, i]^*$ belongs to a cycle of the ancestor graph. Thus, there exists $m > 0$ such that $[\gamma, i]^* \in \mathbf{E}_1^{*m}([\gamma, i]^*)$. In view of (5.17) this implies $h_\sigma^m(\mathcal{T}_\sigma(i) + \gamma) \subset \mathcal{T}_\sigma(i) + \gamma$. By iterating this relation and by using the fact that h_σ is a contraction, we deduce that $0 \in \mathcal{T}_\sigma(i) + \gamma$.

Conversely, assume that $0 \in \mathcal{T}_\sigma(i) + \gamma$. We decompose the tile $\mathcal{T}_\sigma(i) + \gamma$ according to (5.17). Since $0 \in \mathcal{T}_\sigma(i) + \gamma$, for every $m \geq 1$ there exists $[\gamma_m, i_m]^* \in \mathbf{E}_1^{*m}([\gamma, i]^*)$ such that $0 \in \mathcal{T}_\sigma(i_m) + \gamma_m$. As, by Remark 5.4.6, we see that $0 \in \mathcal{T}_\sigma(i_m) + \gamma_m$ implies that $\gamma_m \in V_\sigma$, the element $[\gamma_m, i_m]^*$ is a vertex of the ancestor graph. Thus we get the walk

$$[\gamma_m, i_m]^* \to \cdots \to [\gamma_1, i_1]^* \to [\gamma, i]^*$$

in this graph. By the finiteness of V_σ, the sequence $([\gamma_m, i_m]^*)_{m \geq 1}$ takes twice the same value. Let $m_1 < m_2$ be such that $[\gamma_{m_1}, i_{m_1}] = [\gamma_{m_2}, i_{m_2}]$, with $m_1 \neq m_2$. Then, $[\gamma_{m_1}, i_{m_1}]^*$ is contained in a cycle of this graph. Furthermore, there is a walk from $[\gamma_{m_1}, i_{m_1}]^*$ to $[\gamma, i]^*$ in the ancestor graph. Since each vertex in the ancestor graph has a single outgoing edge, this implies that $[\gamma, i]^*$ belongs to the same cycle of the ancestor graph as $[\gamma_{m_1}, i_{m_1}]^*$. □

Lemma 5.5.2 indicates that non-emptyness for solutions of a GIFS equation can be deduced from cycles of the related graph. We now apply this idea together with Proposition 5.5.1 to the two-piece ancestor graph.

Definition 5.5.3 (Boundary graph) Let σ be a unit Pisot irreducible substitution. The *boundary graph* of σ is the subgraph of the two-piece ancestor graph that contains the vertices $[i, \gamma, j]^*$ with $\gamma \neq 0$ or $i \neq j$, for $i, j \in A$, that belong to a cycle, as well as vertices contained in paths leading away from these cycles.

The motivation for this definition will become clearer in the sketch of the proof of next theorem.

Theorem 5.5.4 (Boundary graph tiling condition) *Let σ be a unit Pisot irreducible substitution. The tiling condition is satisfied if, and only if, the spectral radius of the boundary graph (that is, the largest eigenvalue of its adjacency matrix) is strictly smaller than the Perron–Frobenius eigenvalue β of $\mathbf{M}_\sigma$. If this relation holds, the boundary graph provides a GIFS description of the boundary.*

Proof [Sketch] First note that (5.17) implies, assuming the first alternative of Lemma 5.4.20 (the second one can be handled analogously), that

$$\begin{aligned}\mathcal{T}_\sigma(i_1) \cap (\mathcal{T}_\sigma(i_2)+\gamma) &= h_\sigma\left(\bigcup_{\substack{[\eta_1,j_1]^*\in\mathbf{E}_1^*[0,i_1]^*\\ [\eta_2,j_2]^*\in\mathbf{E}_1^*[\gamma,i_2]^*}} ((\mathcal{T}_\sigma(j_1)+\eta_1)\cap(\mathcal{T}_\sigma(j_2)+\eta_2))\right)\\ &= h_\sigma\left(\bigcup_{\substack{[0,j_1]^*\in-\eta_1+\mathbf{E}_1^*[0,i_1]^*\\ [\eta_2-\eta_1,j_2]^*\in-\eta_1+\mathbf{E}_1^*[\gamma,i_2]^*}} ((\mathcal{T}_\sigma(j_1)\cap(\mathcal{T}_\sigma(j_2)+\eta_2-\eta_1))+\eta_1)\right).\end{aligned}$$

Since, in view of Remark 5.4.18, the intersections are non-empty only for $\eta_2-\eta_1\in W_\sigma$, the equation can be rewritten as

$$\mathcal{T}_\sigma(i_1)\cap(\mathcal{T}_\sigma(i_2)+\gamma) = h_\sigma\left(\bigcup_{[j_1,\eta,j_2]^*\to[i_1,\gamma,i_2]^*} ((\mathcal{T}_\sigma(j_1)\cap(\mathcal{T}_\sigma(j_2)+\eta))+\delta)\right) \tag{5.37}$$

where the union is taken over all edges of the two-piece ancestor graph which lead to the vertex $[i_1,\gamma,i_2]^*$. Such an intersection is interesting if $[i_1,\gamma,i_2]^*$ is not of the form $[i,0,i]^*$ for some $i\in A$ since we are only interested in intersections of two different tiles of $\mathcal{I}_\sigma$. If the union on the right-hand side of (5.37) is empty, *i.e.*, if a vertex of the two-piece ancestor graph has no incoming edge, then also the intersection on the left-hand side is empty. Thus we may successively delete all vertices from the two-piece ancestor graph which have no incoming edges. However, as the two-piece ancestor graph is finite these are vertices which either belong to a cycle or to a path that leads away from a cycle. We may also cancel the vertices of the form $[i,0,i]^*$, for $i\in A$. Indeed, an edge from a vertex of the form $[i,0,i]^*$ reaches a vertex which is also of the same form $[j,0,j]^*$ for some $j\in A$. Hence if $[i_1,\gamma,i_2]^*$ is not of the form $[i,0,i]^*$, then no vertex of this form admits an edge to it. The graph obtained when removing all these vertices is exactly the boundary graph.

Since h_σ is a contraction whose application scales down the $(n-1)$-dimensional Lebesgue measure by the factor $1/\beta$, one deduces that the intersection $\mathcal{T}_\sigma(i_1)\cap(\mathcal{T}_\sigma(i_2)+\gamma)$ has zero measure if the largest eigenvalue of the adjacency matrix of the boundary graph is strictly smaller than β.

The proof of the converse, *i.e.*, that the tiling property implies that the spectral radius of the boundary graph is smaller than β is left as an exercise (see Exercise 5.3). □

Note that (5.37) can be regarded as a GIFS equation for the intersections $\mathcal{T}_\sigma(i_1) \cap (\mathcal{T}_\sigma(i_2) + \gamma)$. In view of Proposition 5.5.5 below, this GIFS can be used to describe $\partial\mathcal{T}_\sigma(i)$ if the tiling condition holds, hence the terminology 'boundary graph'. For more details, see (Siegel and Thuswaldner 2010).

Proposition 5.5.5 *Let σ be a unit Pisot irreducible substitution. One has*

$$\partial\mathcal{T}_\sigma(i) \subseteq \bigcup_{[\gamma,j]^* \neq [0,i]^*,\ [\gamma,j]^* \in \Gamma_c} \mathcal{T}_\sigma(i) \cap (\mathcal{T}_\sigma(j) + \gamma),$$

where equality holds if the tiling property is satisfied.

Proof We fix $i \in A$. Let $\nu \in \partial\mathcal{T}_\sigma(i)$ and assume that ν is not contained in an intersection of the form $\mathcal{T}_\sigma(i) \cap (\mathcal{T}_\sigma(j) + \gamma)$, for $\mathcal{T}_\sigma(j) + \gamma \neq \mathcal{T}_\sigma(i)$. Since the tiles $\mathcal{T}_\sigma(j) + \gamma$ are compact, local finiteness implies that the set $\bigcup_{[\gamma,j]^* \neq [0,i]^*,\ [\gamma,j]^* \in \Gamma_c}(\mathcal{T}_\sigma(j) + \gamma))$ is a closed set in $\mathbb{H}_c$, and thus its complement in $\mathbb{H}_c$ is an open set. Since the complement contains ν, there exists an open neighbourhood B of ν which contains only points of $\mathcal{T}_\sigma(i)$, and no point of other tiles of the self-replicating multiple tiling $\mathcal{I}_\sigma$ of Definition 5.3.14. However, since $\nu \in \partial\mathcal{T}_\sigma(i)$, there is some point $\nu' \in B$ which is not contained in $\mathcal{T}_\sigma(i)$. But then ν' is contained in no tile of $\mathcal{I}_\sigma$, which contradicts the covering property of $\mathcal{I}_\sigma$. We thus have proved the inclusion.

Assume now that the tiling property holds. Moreover, assume that there is $\nu \in \mathrm{int}(\mathcal{T}_\sigma(i))$ which is contained in $\mathcal{T}_\sigma(j) + \gamma$ for some $[\gamma, j]^* \in \Gamma_c$. Then, because $\mathcal{T}_\sigma(i)$ as well as $\mathcal{T}_\sigma(j) + \gamma$ is the closure of its interior (Theorem 5.3.12), there is a $\nu' \in \mathrm{int}(\mathcal{T}_\sigma(i)) \cap \mathrm{int}(\mathcal{T}_\sigma(j) + \gamma)$. Thus, there is a small disk around ν' which is covered by both of these tiles, a contradiction to the tiling property. □

Note that we deduce the following interesting corollary. We only give a proof of the 'if' implication. For a proof of the converse implication, see (Siegel and Thuswaldner 2010).

Corollary 5.5.6 *Let σ be a unit Pisot irreducible substitution. The substitution σ satisfies the geometric finiteness property if, and only if, 0 is an inner point of the central tile $\mathcal{T}_\sigma = \bigcup_{i \in A} \mathcal{T}_\sigma(i)$, and 0 belongs to no other tile of the self-replicating tiling.*

Proof By Proposition 5.4.12 and by Lemma 5.5.2, the geometric finiteness property implies that 0 belongs to no tile of the form $\mathcal{T}_\sigma(i) + \gamma$ with $\gamma \neq 0$ and $i \in A$. Since the tiling property holds and by Proposition 5.5.5, the boundary of $\mathcal{T}_\sigma = \bigcup_{i \in A} \mathcal{T}_\sigma(i)$ is exactly described by the intersections of $\mathcal{T}_\sigma = \bigcup_{i \in A} \mathcal{T}_\sigma(i)$ with the other tiles of the tiling $\mathcal{I}_\sigma$. We deduce that 0 does not belong to the boundary of $\mathcal{T}_\sigma(i)$, and thus that 0 is an inner point of $\mathcal{T}_\sigma = \bigcup_{i \in A} \mathcal{T}_\sigma(i)$. □

Remark 5.5.7 Corollary 5.5.6 was already stated in (Akiyama 2002) in the beta-numeration context. The union of cycles in the ancestor graph is called the *zero-expansion graph* in (Siegel and Thuswaldner 2010). Following the proof of Lemma 5.5.2, labels of edges in cycles allow all Dumont–Thomas expansions (see Sections 5.11 and 9.4.2 for more details) that can be obtained for 0 to be computed explicitly, hence the terminology 'zero-expansion'. Compare also with the notion of *zero automaton* in Chapter 2. The zero-expansion graph allows tiles in $\mathcal{I}_\sigma$ that contain 0 (they are related to cycles in the ancestor graph) to be characterised, and thus, to give a further characterisation of the geometric finiteness property.

Let us end this section with the following statement (given without a proof) that is an analogue of Lemma 5.5.2 and which provides an explicit description of the neighbours of a subtile $\mathcal{T}_\sigma(i)$ with respect to the boundary graph. For a proof, see (Siegel and Thuswaldner 2010).

Proposition 5.5.8 *Let σ be a unit Pisot irreducible substitution. Let $i \in A$ and let $[\gamma, j]^* \in \Gamma_c$ with $\gamma \neq 0$ or $i \neq j$. The intersection $\mathcal{T}_\sigma(i) \cap (\mathcal{T}_\sigma(j) + \gamma)$ is non-empty if, and only if, $[i, \gamma, j]^*$ is a vertex of the boundary graph.*

5.5.2 Approximations of the boundary and contact graphs

Even if the boundary graph is much smaller than the two-piece ancestor graph, its computation relies on the pre-computation of the two-piece seed patch W_σ. However, there is another approach which does not require the pre-computation of this patch. The underlying idea is developed in (Thuswaldner 2006), based on polyhedral approximations of the central tile and its subtiles. Recall that with each tip $[\gamma, i]^*$ we associate the compact polyhedron $[\gamma, i]^*_g$ which is obtained by projecting the corresponding face of type i by π_c (see Section 5.3.3 for details).

Definition 5.5.9 Let σ be a unit Pisot irreducible substitution. The *mth polyhedral approximation* of a subtile $\mathcal{T}_\sigma(i)$ is the union of polyhedra given by

$$\mathcal{T}_\sigma^{(m)}(i) := \bigcup_{[\gamma,i]^* \in \mathbf{E}_1^{*m}[0,i]^*} h_\sigma^m([\gamma, i]^*_g).$$

Proposition 5.5.10 *The Hausdorff limit of the polyhedral approximations $\mathcal{T}_\sigma^{(m)}(i)$ for $m \to \infty$ is the subtile $\mathcal{T}_\sigma(i)$.*

Proof This is a direct consequence of the definition of $\mathbf{E}_1^*$, the GIFS equation (5.8) and the uniqueness of the solution of a GIFS. For more details, see (Arnoux and Ito 2001). □

Remark 5.5.11 Figure 5.9 depicts the image of $\bigcup_{i\in A} \mathcal{T}_\tau^{(15)}(i)$ inflated by h_τ^{-15}, for the substitution $\tau(1) = 2$, $\tau(2) = 3$, $\tau(3) = 12$. We have seen in Example 5.4.13 that σ does not satisfy the geometric finiteness property. Note that Figure 5.9 gives some indication of the fact that 0 (which is dotted) is not an interior point of the central tile.

Definition 5.5.12 (Contact graph) Let σ be a unit Pisot irreducible substitution. A 0 *level vertex* of the two-piece ancestor graph is a vertex of the form $[i, \gamma, j]^*$, such that the ($(n-1)$-dimensional) geometric tips $[0, i]_g^*$ and $[\gamma, j]_g^*$ intersect on exactly one face of dimension $n - 2$. The *contact graph* is the subgraph of the two-piece ancestor graph whose vertices are the 0 level vertices together with all vertices that can be reached by a path in the two-piece ancestor graph starting at a 0 level vertex.

The contact graph is described in (Thuswaldner 2006). Among the graphs considered for tiling criteria so far, the contact graph is the one which is easiest to compute. Indeed, it can be computed with a recursive procedure that always terminates and does not require the computation of the two-piece seed patch W_σ. We first compute the set of 0 level vertices. For $(i, j) \in A^2$, we take the triple $[i, \pi_c(\mathbf{e}_j), i]^*$ if $\langle \mathbf{v}_\beta, \mathbf{e}_j\rangle < \langle \mathbf{v}_\beta, \mathbf{e}_i\rangle$, or $[i, \pi_c(\mathbf{e}_j - \mathbf{e}_i), j]^*$, otherwise. Note that this proves that the tips coming from a 0 level vertex belong to W_σ. We then construct recursively the contact graph, by computing successors (in the two-piece ancestor graph) of already computed vertices. The choice of the initial set implies that such a vertex always belongs to the two-piece ancestor graph (note that these computations do not require the knowledge of the two-piece ancestor graph; we only use the definition of its edges to compute the successors). The construction ends when no additional edge can be added to the graph, which always happens according to (Thuswaldner 2006).

It turns out that the contact graph provides also a suitable decomposition of the boundary of the central tile, which in turn provides a very simple tiling condition. For a complete proof, see (Thuswaldner 2006) together with (Siegel and Thuswaldner 2010).

Theorem 5.5.13 (Contact graph tiling condition) *Let σ be a unit Pisot irreducible substitution. The tiling property is satisfied if, and only if, the spectral radius of the contact graph is strictly smaller than the Perron–Frobenius eigenvalue β of $\mathbf{M}_\sigma$.*

Proof [Sketch] In (Thuswaldner 2006) it is proved that all overlaps $(\mathcal{T}_\sigma(k_1)+\gamma_1)\cap(\mathcal{T}_\sigma(k_2)+\gamma_2)$ of the self-replicating multiple tiling $\mathcal{I}_\sigma$ are translations of overlaps of the form $\mathcal{T}_\sigma(i_1)\cap(\mathcal{T}_\sigma(i_1)+\gamma)$, where $[i_1,\gamma,i_2]^*$ is a vertex of the contact graph. Note that the proof heavily relies on the polyhedral tiling of Proposition 5.3.3. Moreover, (Thuswaldner 2006) shows that these intersections are the solution of the GIFS equation

$$\mathcal{T}_\sigma(i_1)\cap(\mathcal{T}_\sigma(i_2)+\gamma) = \bigcup_{[j_1,\eta,j_2]^*\to[i_1,\gamma,i_2]^*} ((\mathcal{T}_\sigma(j_1)\cap(\mathcal{T}_\sigma(j_2)+\eta))+\delta) \tag{5.38}$$

where the union is taken over all edges of the contact graph which lead to the vertex $[i_1,\gamma,i_2]^*$. The proof can now be finished in the same way as the proof of Theorem 5.5.4. □

This tiling condition is definitely the simplest tiling condition that we have considered so far since the contact graph can be computed recursively without the precomputation of the two-piece seed patch W_σ.

It also provides a 'minimal' GIFS for the boundary of the central tile in the sense that it removes from the boundary graph all the intersections that are redundant, *i.e.*, that are included in other intersections. For details and examples of contact graphs for families of substitutions associated with beta-numeration we refer again to (Thuswaldner 2006).

5.6 Geometric coincidences

We consider now a further set of conditions each of which is equivalent to the tiling property. Using the concept of duality, they can be expressed in terms of the tiling of the expanding line.

5.6.1 Strands and duality

Up to now, we have worked with the set Γ_c consisting of tips which correspond to projections of faces by π_c. The notion of tip has allowed us to define the GIFS substitution $\mathbf{E}_1^*$ (see Definition 5.3.8) inspired by the action of the GIFS equations (5.8) that govern the subtiles $\mathcal{T}_\sigma(i)$, for $i\in A$. The tiling property (see Definition 5.3.15) was then expressed in terms of pre-images of tips under $\mathbf{E}_1^*$ (see *e.g.* Theorem 5.4.21).

We now wish to adopt a dual one-dimensional viewpoint. Instead of working with faces of hypercubes, we will work with line segments, and the projection π_e will play the role of the projection π_c. We consider formal strands to represent stairs joining points in the integer grid $\mathbb{Z}^n$. An element of the form $(\mathbf{x}, i) \in \mathbb{Z}^n \times A$ is called a *basic formal strand*. In the sequel, we use the notation $[\mathbf{x}, i]$ instead of $(\mathbf{x}, i)$ for this object. The *basic geometric strand* $[\mathbf{x}, i]_g$ is defined as the segment connecting $\mathbf{x}$ with $\mathbf{x} + \mathbf{e}_i$. A *formal strand* is then defined as a union of basic formal strands. We similarly define a *geometric strand*.

Note that there is no more '$*$' in the notation for strands: $[\mathbf{y}, j]_g$ represents a segment in $\mathbb{R}^n$ while $[\gamma, i]_g^*$ is the projection in $\mathbb{H}_c$ of a face. Faces of hypercubes and segments can be considered as *dual* in the sense of the duality principle of linear algebra (see (Arnoux and Ito 2001)). Note that in (Sano, Arnoux, and Ito 2001) a Poincaré type duality between generalisations of faces and segments is established. As we shall see in the sequel, this duality will allow us to translate properties of the set Γ_c into properties of the self-similar translation set Γ_e, and thus to work with the tiling $\mathcal{E}_u$ of the expanding line.

If we consider a word $w \in A^*$, the point $\mathbf{P}(w) \in \mathbb{Z}^n$ is the abelianisation of w and one builds in a natural way a formal strand and a geometric strand from $\mathbf{0}$ to $\mathbf{P}(w)$ by simply reading the letters in w. Strands allow one to keep track of the combinatorics of a word that would be lost in the abelianisation process.

We now extend the action of σ to unions of basic strands $[\mathbf{x}, i] \in \mathbb{Z}^n \times A$, according to the formalism of (Arnoux and Ito 2001).

Definition 5.6.1 (Geometric realisation of a substitution) Let σ be a unit Pisot irreducible substitution. The *one-dimensional geometric realisation* of σ is defined on the sets of formal strands by

$$\mathbf{E}_1\{[\mathbf{y}, j]\} = \bigcup_{(p,i,s,),\ \sigma(j)=pis} \{[\mathbf{M}_\sigma \mathbf{y} + \mathbf{P}(p), i]\},$$

$$\mathbf{E}_1(Y_1 \cup Y_2) = \mathbf{E}_1(Y_1) \cup \mathbf{E}_1(Y_2).$$

We also use here the notation $\mathbf{E}_1[\mathbf{y}, j]$ for $\mathbf{E}_1\{[\mathbf{y}, j]\}$. With this formalism at hand, the set $\bigcup_{k \geq 0} \mathbf{E}_1^k[\mathbf{0}, u_0]$ generates the broken line L_u, by replacing formal strands by geometric strands. This implies that the broken line L_u is invariant under the action of $\mathbf{E}_1$.

The following lemma enhances the relation between $\mathbf{E}_1$ and the GIFS substitution $\mathbf{E}_1^*$ acting on the translation set Γ_c.

Lemma 5.6.2 (Duality lemma) *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$. *Then*

$$[\pi_c(\mathbf{y}), j]^* \in \mathbf{E}_1^*([\pi_c(\mathbf{x}), i]^*) \iff [-\mathbf{x}, i] \in \mathbf{E}_1([-\mathbf{y}, j]).$$

Proof Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$. Then $[\pi_c(\mathbf{y}), j]^* \in \mathbf{E}_1^*([\pi_c(\mathbf{x}), i]^*)$ if, and only if, there exists p such that $\sigma(j) = pis$ and $\pi_c(\mathbf{y}) = h_\sigma^{-1}(\pi_c(\mathbf{x}) + \pi_c(\mathbf{P}(p))) = \pi_c(\mathbf{M}_\sigma^{-1}\mathbf{x} + \mathbf{P}(p))$. Equivalently by (5.5) we have $\mathbf{y} = \mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{P}(p))$, *i.e.*, $-\mathbf{x} = \mathbf{M}_\sigma(-\mathbf{y}) + \mathbf{P}(p)$, which gives $[-\mathbf{x}, i] \in \mathbf{E}_1([-\mathbf{y}, j])$. □

By defining suitable vector spaces on strands and tips, $\mathbf{E}_1^*$ and $\mathbf{E}_1$ are linked to each other by the duality principle of linear algebra (up to a reverse of the orientation of the space that leads to introduce the '$-$' sign in the statement of Lemma 5.6.2). This is worked out in (Arnoux and Ito 2001). As a geometric interpretation, we can also say that the broken line L_u and the stepped hyperplane are dual to each other.

From Lemma 5.6.2 and the definition of the self-replicating translation set Γ_e, we deduce the following dictionary. The proof is immediate.

Lemma 5.6.3 (Dictionary) *Let* σ *be a unit Pisot irreducible substitution. The following assertions are true.*

(i) Ancestor. *A tip* $[\pi_c(\mathbf{x}), i]^*$ *is the ancestor of* $[\pi_c(\mathbf{y}), j]^*$ *if, and only if,* $[-\mathbf{x}, i]$ *is a segment of the strand* $\mathbf{E}_1[-\mathbf{y}, j]$.

(ii) Common ancestor. *Two tips* $[\pi_c(\mathbf{y}_1), j_1]^*$ *and* $[\pi_c(\mathbf{y}_2), j_2]^*$ *have a common ancestor* $[\pi_c(\mathbf{x}), i]^*$ *if, and only if, the strands* $\mathbf{E}_1[-\mathbf{y}_1, j_1]$ *and* $\mathbf{E}_1[-\mathbf{y}_2, j_2]$ *both contain the segment* $[-\mathbf{x}, i]$.

(iii) Two-piece ancestor Equation (5.32). *There exists* δ *such that* $\{[\pi_c(\mathbf{y}_1), j_1]^*, [\pi_c(\mathbf{y}_1), j_2]^*\} \subset \delta + \mathbf{E}_1^*\{[0, i_1]^*, [\pi_c(\mathbf{x}), i_2]^*\}$ *if, and only if, there exists* $\mathbf{z}$ *such that* $\mathbf{z} + \{[\mathbf{0}, i_1], [-\mathbf{x}, i_2]\} \subset \mathbf{E}_1\{[-\mathbf{y}_1, j_1], \mathbf{E}_1[-\mathbf{y}_2, j_2]\}$.

(iv) Two-piece patch ancestor. *There exists an element* δ *such that* $\{[\pi_c(\mathbf{y}_1), j_1]^*, [\pi_c(\mathbf{y}_2), j_2]^*\} \subset \delta + \mathbf{E}_1^*([0, i]^*)$ *if, and only if, there exists* $\mathbf{z}$ *such that* $[\mathbf{z}, i] \in \mathbf{E}_1[-\mathbf{y}_1, j_1] \cap \mathbf{E}_1[-\mathbf{y}_2, j_2]$.

5.6.2 Super coincidence condition

The relations established in Section 5.6.1 naturally lead to the following definition which extends the notion of combinatorial strong coincidence introduced in Definition 5.2.9. It can be considered as the dual property of having a common ancestor. This condition was first defined in (Ito and Rao 2006).

Definition 5.6.4 (Geometric strong coincidence) We say that the basic strands $[\mathbf{y}_1, j_1]$ and $[\mathbf{y}_2, j_2]$ have *geometric strong coincidence* if there exists a positive integer such that $\mathbf{E}_1^N[\mathbf{y}_1, j_1]$ and $\mathbf{E}_1^N[\mathbf{y}_2, j_2]$ have at least one basic formal strand in common.

Note that the combinatorial strong coincidence condition of Definition 5.2.9 is equivalent to the fact that $[\mathbf{0}, j_1]$ and $[\mathbf{0}, j_2]$ have geometric strong coincidence for each pair $(j_1, j_2) \in A^2$. Indeed, assume that $\mathbf{E}_1^N[\mathbf{0}, j_1]$ and $\mathbf{E}_1^N[\mathbf{0}, j_2]$ have one basic formal strand in common, say $[\mathbf{x}, i]$. This is equivalent to the existence of the decompositions $\sigma^N(j_1) = p_1 i s_1$ and $\sigma^N(j_2) = p_2 i s_2$, with $\pi_e(\mathbf{x}) = \pi_e \circ \mathbf{P}(p_1) = \pi_e \circ \mathbf{P}(p_2)$. We deduce from (5.5) that $\mathbf{P}(p_1) = \mathbf{P}(p_2)$.

As we shall see, one checks that a two-piece patch $\{[\pi_c(\mathbf{y}_1), j_1]^*, [\pi_c(\mathbf{y}_2), j_2]^*\}$ is a patch of Γ_c up to translation (*i.e.*, there exists δ such that $\{[\pi_c(\mathbf{y}_1) + \delta, j_1]^*, [\pi_c(\mathbf{y}_2) + \delta, j_2]^*\} \subset \Gamma_c$) if, and only if, the projections of the segments $[-\mathbf{y}_1, j_1]_g$ and $[-\mathbf{y}_2, j_2]_g$ along the expanding direction intersect on a non-degenerate interval. Before we make this more precise, we introduce the following definition, due to (Barge and Kwapisz 2006).

Definition 5.6.5 (Height) We say that the basic strands $[\mathbf{x}, i]$ and $[\mathbf{y}, j]$ *have the same height* if

$$\mathrm{int}(\pi_e([\mathbf{x}, i])_g) \cap \mathrm{int}(\pi_e([\mathbf{y}, j]_g)) \neq \emptyset.$$

Lemma 5.6.6 *Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$ such that $\langle \mathbf{x}, \mathbf{v}_\beta \rangle \leq \langle \mathbf{y}, \mathbf{v}_\beta \rangle$. Then for each $i \in A$ the following assertions are equivalent.*

- *The tip $[\pi_c(-\mathbf{x} + \mathbf{y}), i]^*$ belongs to Γ_c.*
- *For each $j \in A$, the basic strands $[\mathbf{x}, i]$ and $[\mathbf{y}, j]$ have the same height.*

Proof Let $[\mathbf{x}, i]$ and $[\mathbf{y}, j]$ be two formal strands having the same height with $\langle \mathbf{x}, \mathbf{v}_\beta \rangle \leq \langle \mathbf{y}, \mathbf{v}_\beta \rangle$. By (5.4) we have that

$$\langle \mathbf{x}, \mathbf{v}_\beta \rangle \leq \langle \mathbf{y}, \mathbf{v}_\beta \rangle < \langle \mathbf{x} + \mathbf{e}_i, \mathbf{v}_\beta \rangle.$$

This implies that $0 \leq \langle -\mathbf{x} + \mathbf{y}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle$. We thus deduce that $[\pi_c(-\mathbf{x} + \mathbf{y}), i]^* \in \Gamma_c$. The proof of the converse implication follows along the same lines. □

Definition 5.6.7 (Super coincidence condition) A unit Pisot irreducible substitution σ satisfies the *super coincidence condition* if any two basic strands $[\mathbf{x}, i]$ and $[\mathbf{y}, j]$ have geometric strong coincidence whenever they have the same height.

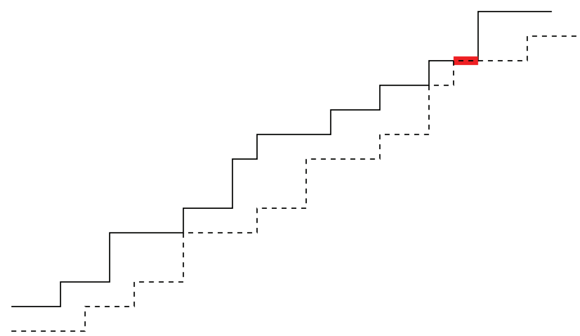

Fig. 5.12. The super coincidence condition.

An illustration of the notion of super coincidence is given in Figure 5.12.

According to (Barge and Kwapisz 2006, Ito and Rao 2006), the tiling condition of Corollary 5.4.15 can be restated as follows.

Theorem 5.6.8 (Ito and Rao 2006) *A unit Pisot irreducible substitution σ satisfies the super coincidence condition if, and only if, the tiling property is satisfied.*

Proof By Corollary 5.4.15, the tiling property is satisfied if, and only if, for any $[\pi_c(\mathbf{y}_1), j_1]^*, [\pi_c(\mathbf{y}_2), j_2]^* \in \Gamma_c$ there is $\delta \in \mathbb{H}_c$ and $i \in A$ such that

$$\{[\pi_c(\mathbf{y}_1), j_1]^*, [\pi_c(\mathbf{y}_2), j_2]^*\} \subset \delta + \mathbf{E}_1^*([0, i]^*).$$

Lemma 5.6.3 (iv) implies that this is equivalent to the fact that there exists $\mathbf{z}$ such that $[\mathbf{z}, i] \in \mathbf{E}_1[-\mathbf{y}_1, j_1] \cap \mathbf{E}_1[-\mathbf{y}_2, j_2]$. Thus, the tiling property is satisfied if, and only if, $[\mathbf{y}_1, j_1]$ and $[\mathbf{y}_2, j_2]$ have geometric strong coincidence for any $[\pi_c(\mathbf{y}_1), j_1]^*, [\pi_c(\mathbf{y}_2), j_2]^* \in \Gamma_c$. In view of Lemma 5.4.20, we may assume w.l.o.g. that $[\pi_c(\mathbf{y}_1) - \pi_c(\mathbf{y}_2), j_1]^* \in \Gamma_c$. Since $[\mathbf{y}_1, j_1]$ and $[\mathbf{y}_2, j_2]$ have geometric strong coincidence if, and only if, $[\mathbf{y}_1 - \mathbf{y}_2, j_1]$ and $[\mathbf{0}, j_2]$ have geometric strong coincidence, the above assertion is equivalent to the fact that $[\mathbf{y}, j_1]$ and $[\mathbf{0}, j_2]$ have geometric strong coincidence if $[\pi_c(\mathbf{y}), j_1]^* \in \Gamma_c$. By Lemma 5.6.6 this is equivalent to the fact that $[\mathbf{y}, j_1]$ and $[\mathbf{0}, j_2]$ have geometric strong coincidence if $[\mathbf{y}, j_1]$ and $[\mathbf{0}, j_2]$ have the same height. However, as $[\mathbf{y}_1, j_1]$ and $[\mathbf{y}_2, j_2]$ have the same height if, and only if, the same is true for $[\mathbf{y}_1 - \mathbf{y}_2, j_1]$ and $[\mathbf{0}, j_2]$, this is equivalent to the super coincidence condition. □

We now define a graph that turns out to be isomorphic to the two-piece ancestor graph. This graph is described in (Barge and Kwapisz 2006).

Similarly as in Section 5.4.3 we introduce the notation $[i_1, \mathbf{x}, i_2]$ for triples $(i_1, \mathbf{x}, i_2) \in A \times \mathbb{Z}^n \times A$ with $[-\pi_c(\mathbf{x}), i_2]^* \in W_\sigma$ where we assume that $i_1 \leq i_2$ if $\mathbf{x} = \mathbf{0}$ to avoid redundancies. The triple $[i_1, \mathbf{x}, i_2]$ represents the pair of tips $[\mathbf{0}, i_1]$ and $[-\mathbf{x}, i_2]$ having the same height by Lemma 5.6.6.

Definition 5.6.9 (Configuration graph) The set of vertices of the *configuration graph* is equal to

$$\{[i_1, \mathbf{x}, i_2] \mid (i_1, \mathbf{x}, i_2) \in A \times \mathbb{Z}^n \times A,\ [-\pi_c(\mathbf{x}), i_2]^* \in W_\sigma,\ i_1 \leq i_2 \text{ if } \mathbf{x} = \mathbf{0}\}.$$

There is an edge from $[j_1, \mathbf{y}, j_2]$ to $[i_1, \mathbf{x}, i_2]$ if there exists $\mathbf{z} \in \mathbb{Z}^n$ such that

$$\mathbf{z} + \{[\mathbf{0}, i_1], [\mathbf{x}, i_2]\} \subset \mathbf{E}_1\{[\mathbf{0}, j_1], [\mathbf{y}, j_2]\}$$

with $\mathbf{z}+[\mathbf{0}, i_1] \cap \mathbf{E}_1\{[\mathbf{0}, j_1], [\mathbf{y}, j_2]\} \neq \emptyset$ and $\mathbf{z}+[\mathbf{x}, i_2] \cap \mathbf{E}_1\{[\mathbf{0}, j_1], [\mathbf{y}, j_2]\} \neq \emptyset$.

Using the duality statements of Lemma 5.6.3 we obtain the following isomorphic graphs.

Proposition 5.6.10 *The configuration graph and the two-piece ancestor graph are isomorphic.*

Proof By definition, $[i_1, \mathbf{x}, i_2]$ is a vertex of the configuration graph if, and only if, $[i_1, -\pi_c(\mathbf{x}), i_2]^*$ is a vertex of the two-piece ancestor graph. By (5.15), we thus get a one-to-one correspondence between the sets of vertices of each graph. Let us consider now edges. According to Lemma 5.6.3 (iii), there exists $\delta \in \mathbb{H}_c$ such that

$$\{[0, j_1]^*, [\pi_c(\mathbf{y}), j_2]^*\} \subset \delta + \mathbf{E}_1^*\{[0, i_1]^*, [\pi_c(\mathbf{x}), i_2]^*\}$$

with

$$\{[0, j_1]^*, [\pi_c(\mathbf{y}), j_2]^*\} \cap (\delta + \mathbf{E}_1^*([0, i_1]^*)) \neq \emptyset \quad \text{and}$$
$$\{[0, j_1]^*, [\pi_c(\mathbf{y}), j_2]^*\} \cap (\delta + \mathbf{E}_1^*([\pi_c(\mathbf{x}), i_2]^*)) \neq \emptyset$$

if, and only if, there exists $\mathbf{z} \in \mathbb{Z}^n$ such that

$$\mathbf{z} + \{[\mathbf{0}, i_1], [-\mathbf{x}, i_2]\} \subset \mathbf{E}_1\{[\mathbf{0}, j_1], [-\mathbf{y}, j_2]\}$$

with

$$(\mathbf{z} + [\mathbf{0}, i_1]) \cap \mathbf{E}_1\{[\mathbf{0}, j_1], [-\mathbf{y}, j_2]\} \neq \emptyset \quad \text{and}$$
$$(\mathbf{z} + [-\mathbf{x}, i_2]) \cap \mathbf{E}_1\{[\mathbf{0}, j_1], [-\mathbf{y}, j_2]\} \neq \emptyset.$$

Hence, also the edges of the configuration graph are in one-to-one correspondence with the edges of the two-piece ancestor graph . □

The following theorem, which has been first proved in (Barge and Kwapisz 2006, Proposition 17.1), is a direct consequence of Proposition 5.6.10 and Theorem 5.4.21.

Theorem 5.6.11 (Barge and Kwapisz 2006, Proposition 17.1) *A unit Pisot irreducible substitution σ satisfies the tiling property if, and only if, from every vertex in the configuration graph, there exists a path to a vertex of the shape $[k, \mathbf{0}, k]$.*

5.7 Overlap coincidences

We are now going to introduce a new viewpoint on configuration graphs and on the super coincidence condition, following the concept of *overlap coincidence*. Overlap coincidence was first used in (Solomyak 1997) for two-dimensional tilings, and later extended to one-dimensional substitution tilings in (Sirvent and Solomyak 2002). As we shall see, this framework allows a graph to be defined that is related to the configuration graph (and the two-piece ancestor graph). Moreover, it provides a simple combinatorial algorithm which allows the tiling property to be decided. In particular, this algorithm avoids having to compute the two-piece seed patch W_σ.

5.7.1 Definitions

So far we have considered pairs of basic strands in $\mathbb{R}^n$ with the same height and checked whether a common basic strand occurs under iterations of $\mathbf{E}_1$. This is the super coincidence condition. By projecting basic strands of the same height by π_e, one recovers intersecting segments, called *overlaps*. The viewpoint used here is to work directly with such intersections. We do not consider all pairs of basic strands with the same height, but we restrict ourselves to basic strands which occur in some translated copies of the broken line L_u.

In order to define suitable translation vectors for the translated copies of L_u, we introduce the set of all possible distances between two tiles of the same type in the self-similar tiling of the expanding line $\mathcal{E}_u = \{\pi_e([\mathbf{x}, i]_g) \mid [\mathbf{x}, i] \in \Gamma_e\}$. Since tiles in the tiling $\mathcal{E}_u$ are ordered according to the fixed point $u = u_0u_1u_2 \cdots$, the set of distances between tiles is described by

$$\begin{aligned}\Xi(\mathcal{E}_u) &= \{\lambda \in \mathbb{R}_+ \mid \lambda = \pi_e \circ \mathbf{P}(u_N \cdots u_{N+m-1}),\ N, m \geq 0,\ u_N = u_{N+m}\}\\ &= \{\lambda \in \mathbb{R}_+ \mid \exists T, T' \in \mathcal{E}_u,\ T' = T + \lambda\}.\end{aligned}$$

One has $\Xi(\mathcal{E}_u) \subseteq \mathbb{Z}[\beta]$, since each coordinate of $\mathbf{v}_\beta$ belongs to $\mathbb{Z}[\beta]$. Note that we get $\beta\,\Xi(\mathcal{E}_u) \subset \Xi(\mathcal{E}_u)$ by using the invariance of u under σ.

We first introduce the notion of overlap which corresponds to the intersection of the projections of basic geometric strands of the same height, following (Sirvent and Solomyak 2002). Let S, T be two tiles occurring in the self-similar tiling $\mathcal{E}_u$. Note that S and T are intervals. For $\lambda \in \mathbb{R}_+$, the triple (T, S, λ) is called *overlap* if $\mathrm{int}(T \cap (S - \lambda)) \neq \emptyset$. See Figures 5.13 and 5.14 for an illustration.

We now restrict ourselves to sets of pairs of tiles T, S in the self-similar tiling $\mathcal{E}_u$ that are separated by a length belonging to $\Xi(\mathcal{E}_u)$, *i.e.*,

$$\mathcal{O}_u = \{(T, S, \lambda) \mid \mathrm{int}(T \cap (S - \lambda)) \neq \emptyset,\ T, S \in \mathcal{E}_u,\ \lambda \in \Xi(\mathcal{E}_u)\}.$$

The reason for this restriction for the set of lengths λ will become clear in Section 5.7.2.

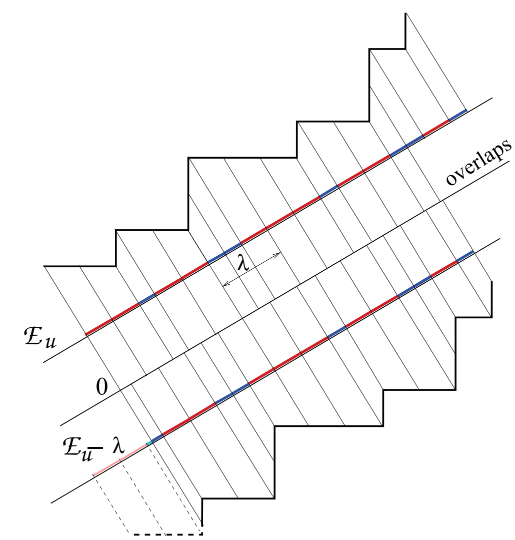

Fig. 5.13. In order to illustrate the relation between the tiling $\mathcal{E}_u$ and the broken lines in $\mathbb{R}^n$ we draw the tilings $\mathcal{E}_u$ and $\mathcal{E}_u - \lambda$ parallel to the expanding eigendirection of M_σ and not in the real line for $\sigma(1) = 112$, $\sigma(2) = 21$.

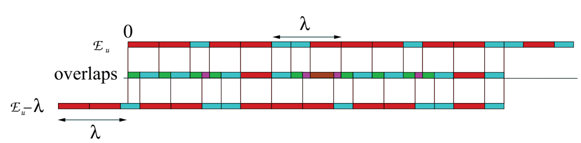

Fig. 5.14. An example of overlaps of a tiling in the real line for $\sigma(1) = 112$, $\sigma(2) = 21$.

The overlaps contained in $\mathcal{O}_u$ can be built in a quite simple geometric way. Indeed, we consider the tiling $\mathcal{E}_u$, and look at the distance λ between two tiles of the same type in $\mathcal{E}_u$. We then shift the tiling $\mathcal{E}_u$ by λ, and take the new tiling by intervals $(\mathcal{E}_u - \lambda) \cap \mathbb{R}_+$, which is not 'synchronised' with $\mathcal{E}_u$, *i.e.*, endpoints of tiles in each tiling do not correspond. We thus define the *synchronised tiling* associated with two tilings by intervals as the tiling by intervals obtained when taking the union of the set of endpoints of both tilings. We use the notation $\mathcal{E} \cap \mathcal{E}'$ for the synchronised tiling associated with $\mathcal{E}$ and $\mathcal{E}'$. We say that we *synchronise* two tilings when we take their synchronised tiling. Synchronising the two tilings $\mathcal{E}_u$ and $\mathcal{E}_u - \lambda$ thus creates new smaller tiles, corresponding to overlaps in $\mathcal{O}_u$. In more geometric terms, we work with the projections of L_u and $L_u - \mathbf{z}$, with $\pi_e(\mathbf{z}) = \lambda$ (see again Figures 5.13 and 5.14).

The number of overlaps in the pair $(\mathcal{E}_u, (\mathcal{E}_u - \lambda) \cap \mathbb{R}^+)$ of self-similar tilings is infinite. We now classify them, up to a translation vector, according to the following equivalence relation: two overlaps (T, S, λ), (T', S', λ') are said to be *equivalent* if there exists $\delta \in \mathbb{R}$ such that $T' = T + \delta$, $S' - \lambda' = S - \lambda + \delta$. The equivalence class of (T, S, λ) is denoted by $[T, S, \lambda]$, *i.e*,

$$[T, S, \lambda] = \{(T', S', \lambda') \in \mathcal{O}_u \mid T' = T + \delta,\ S' - \lambda' = S - \lambda + \delta,\ \delta \in \mathbb{R}\}.$$

The set of equivalence classes of elements in $\mathcal{O}_u$ is denoted by $[\mathcal{O}_u]$. Let us note that $[\mathcal{O}_u]$ obviously depends on the substitution σ. However, it is independent of the choice of the fixed point u (see Exercise 5.5).

Remark 5.7.1 An equivalence class $[T, S, \lambda]$ is clearly determined by the type of T, the type of S and the difference ν of the starting points of T and $S - \lambda$.

For a fixed $\lambda \in \Xi(\mathcal{E}_u)$ let $\mathcal{O}_u(\lambda)$ be the set of all overlaps (T, S, λ) with $S, T \in \mathcal{E}_u$ and denote by $[\mathcal{O}_u(\lambda)]$ the corresponding set of equivalence classes. We introduce the following terminology.

- A class $[T, S, \lambda] \in [\mathcal{O}_u]$ is called an *overlap class*.
- An overlap (T, S, λ) is a *coincidence overlap* if $T = S - \lambda$. This notion extends to overlap classes.
- An overlap (T, S, λ) is a *half-coincidence overlap* if T and $S - \lambda$ have at least one common endpoint. This notion also extends to overlap classes.

A coincidence overlap class is *a fortiori* a half-coincidence overlap class.

Lemma 5.7.2 *Let σ be a unit Pisot irreducible substitution. The set $[\mathcal{O}_u]$ of overlap classes is finite.*

Note that the proof of this lemma uses similar arguments as, for instance, the proof of Proposition 2.3.33 and Lemma 2.4.7.

Proof Let (T, S, λ) be an overlap. There exist basic geometric strands $[\mathbf{x}, i]_g$, $[\mathbf{y}, j]_g$ in L_u such that $T = \pi_e([\mathbf{x}, i]_g)$, $S = \pi_e([\mathbf{y}, j]_g)$. Let $\nu = \pi_e(\mathbf{y} - \mathbf{x}) - \lambda$ be the difference of the starting points of T and $S - \lambda$. In view of Remark 5.7.1 (note that there are finitely many types of tiles), it suffices to show that only finitely many choices of ν are possible. We have

$$T \cap (S - \lambda) = \pi_e(\mathbf{x}) + (\pi_e[\mathbf{0}, i]_g \cap (\pi_e[\mathbf{0}, j]_g + \nu)) \,.$$

Note that (T, S, λ) is an overlap, if, and only if, $-\pi_e(\mathbf{e}_j) < \nu < \pi_e(\mathbf{e}_i)$.

Furthermore, since $\lambda \in \Xi(\mathcal{E}_u)$, there exists $\mathbf{z} \in \mathbb{Z}^n$ such that $\lambda = \langle \mathbf{z}, \mathbf{v}_\beta \rangle$. Recall that we have assumed that the coordinates of $\mathbf{v}_\beta$ all belong to $\mathbb{Z}[\beta]$. This implies that $\nu = \langle \mathbf{y} - \mathbf{x} - \mathbf{z}, \mathbf{v}_\beta \rangle \in \mathbb{Z}[\beta]$, and that the Galois conjugates of ν are given by $\langle \mathbf{y} - \mathbf{x} - \mathbf{z}, \mathbf{v}_{\beta^{(i)}} \rangle$. We shall prove that these Galois conjugates are uniformly bounded. As they are the coordinates of $\pi_c(\mathbf{y} - \mathbf{x} - \mathbf{z})$ in the basis $(\mathbf{u}_{\beta^{(i)}})_{i \geq 2}$, we have to show the boundedness of $\pi_c(\mathbf{y} - \mathbf{x} - \mathbf{z})$. Since $[\mathbf{x}, i]_g$ and $[\mathbf{y}, j]_g$ are segments L_u, we have $\pi_c(\mathbf{x}), \pi_c(\mathbf{y}) \in \mathcal{T}_\sigma$. Thus Theorem 5.2.3 yields that $\pi_c(\mathbf{x})$ and $\pi_c(\mathbf{y})$ are uniformly bounded. Moreover, as $\pi_e(\mathbf{z}) \in \Xi(\mathcal{E}_u)$, we get that $\pi_c(\mathbf{z}) \in \mathcal{T}_\sigma - \mathcal{T}_\sigma$. Using Theorem 5.2.3 again, this implies that $\pi_c(\mathbf{z})$ is uniformly bounded, too. Therefore ν and all its Galois conjugates are uniformly bounded. Since $\nu \in \mathbb{Z}[\beta]$, there are only finitely many possibilities for ν. □

Remark 5.7.3 This proof yields a strong relation between overlaps and tips: we have proved that with each overlap class $[T, S, \lambda]$ we associate the pair of basic strands $([\mathbf{0}, i], [\mathbf{z}, j])$ that have the same height, with $\mathbf{z} \in \mathbb{Z}^n$ being uniquely determined by $\pi_e(\mathbf{y} - \mathbf{x}) - \lambda = \pi_e(\mathbf{z})$, where $T = \pi_e([\mathbf{x}, i]_g)$, $S = \pi_e([\mathbf{y}, j]_g)$. Note that the position of the basic strand $[\mathbf{z}, j]$ is very specific.

With each overlap (T, S, λ) we associate the intersection $T \cap (S - \lambda)$. Recall that by the invariance of the broken line under $\mathbf{E}_1$ we have the self-similarity equation

$$\beta \pi_e([\mathbf{y}, j]_g) = \bigcup_{[\mathbf{x}, i] \in \mathbf{E}_1[\mathbf{y}, j]} \pi_e([\mathbf{x}, i]_g).$$

Applying this equation to the intersection $T \cap (S - \lambda)$ yields the decomposition

$$\beta(T \cap (S - \lambda)) = \bigcup_{T' \subset \beta T,\, S' \subset \beta S, T' \in \mathcal{E}_u, S' \in \mathcal{E}_u} (T' \cap (S' - \beta\lambda)). \qquad (5.39)$$

The sets that occur on the right-hand side of (5.39) are tiles of the synchronised tiling associated with the pair $(\mathcal{E}_u, (\mathcal{E}_u - \beta\lambda) \cap \mathbb{R}_+)$.

The following graph will allow us to formulate a new notion of coincidence that will give rise to a further tiling criterion.

Definition 5.7.4 (Graph of overlaps) The *graph of overlaps*, denoted by $\mathcal{G}_{\mathcal{O}}$, is a directed graph whose set of vertices is the set $[\mathcal{O}_u]$ of overlap classes. There is an edge from $[T, S, \lambda]$ to $[T', S', \lambda']$ if $T' \cap (S' - \lambda')$ is non-empty and appears in the self-similar decomposition (5.39) of $\beta(T \cap (S - \lambda))$.

Note that Lemma 5.7.2 implies that the graph of overlaps $\mathcal{G}_{\mathcal{O}}$ is finite and does not depend on the choice of u (see Exercise 5.5).

The graph of overlaps is very close to the configuration graph (and thus to the two-piece ancestor graph). The main difference is that the configuration graph and the two-piece ancestor graph are defined in terms of tips or basic strands in the two-piece seed patch W_σ, while the construction of the graph of overlaps does not involve W_σ. Indeed, as seen in the proof of Lemma 5.7.2, the graph of overlaps selects pairs of tips and pairs of basic strands according to the set $\Xi(\mathcal{E}_u)$. This leads to a finite number of pairs of tips or of pairs of basic strands and no reduction to the two-piece seed patch is needed any more.

We introduce a new notion of coincidence which is defined in terms of the graph of overlaps.

Definition 5.7.5 (Strong overlap coincidence condition) The unit Pisot irreducible substitution σ satisfies the *strong overlap coincidence condition* if each vertex in the graph of overlaps $\mathcal{G}_{\mathcal{O}}$ admits a path leading to an overlap coincidence.

We will prove in Section 5.7.2 that this condition together with the combinatorial strong coincidence condition is equivalent to the tiling property. However, checking the strong overlap coincidence condition is hard since *a priori* it requires to identify *all* non-empty overlaps provided by the expanding tiling, *i.e.*, to consider all parameters $\lambda \in \Xi(\mathcal{E}_u)$. Thus the first problem is to determine $\Xi(\mathcal{E}_u)$. However, fortunately we will see in Section 5.7.2 that we do not need to work with the whole set $\Xi(\mathcal{E}_u)$ in order to check the strong overlap coincidence condition. Indeed, we can restrict ourselves to an arbitrary *single* element $\lambda \in \Xi(\mathcal{E}_u)$. In particular, we will have to consider only synchronisations for the family $\{\mathcal{E}_u, \mathcal{E}_u - \lambda, \mathcal{E}_u - \beta\lambda, \ldots, \mathcal{E}_u - \beta^m\lambda, \ldots\}$.

Definition 5.7.6 (Graph of overlaps of λ) Let $\lambda \in \Xi(\mathcal{E}_u)$. The *graph* *of overlaps of* λ, denoted by $\mathcal{G}_{\mathcal{O}}(\lambda)$, is the subgraph of the graph of overlaps whose vertices belong to $\bigcup_{i\geq 0}[\mathcal{O}_u(\beta^i\lambda)]$.

As for each $\lambda \in \Xi(\mathcal{E}_u)$ the graph $\mathcal{G}_{\mathcal{O}}(\lambda)$ is a subgraph of the finite graph of overlaps $\mathcal{G}_{\mathcal{O}}$, it is itself a finite graph. Moreover, as a finite graph has only finitely many pairwise non-isomorphic subgraphs there are only finitely many different graphs in the class $\{\mathcal{G}_{\mathcal{O}}(\lambda) \mid \lambda \in \Xi(\mathcal{E}_u)\}$.

To $\mathcal{G}_{\mathcal{O}}(\lambda)$ we relate the following coincidence condition which will turn out to be equivalent to the strong overlap coincidence condition (see Theorem 5.7.13).

Definition 5.7.7 (Weak overlap coincidence) The unit Pisot irreducible substitution σ satisfies the *weak overlap coincidence condition* if there exists $\lambda \in \Xi(\mathcal{E}_u)$ with $\lambda \neq 0$ such that each vertex in its associated graph of overlaps $\mathcal{G}_{\mathcal{O}}(\lambda)$ admits a path to an overlap coincidence.

The advantage of this condition is that it can be checked effectively in an easy way.

The following lemma contains a first result on the relation between the strong and the weak overlap coincidence condition.

Lemma 5.7.8 *The strong overlap coincidence condition is true if, and only if, the weak overlap coincidence condition is true for each* $\lambda \in \Xi(\mathcal{E}_u)$.

Proof Assume that the strong coincidence condition is true and choose $\lambda \in \Xi(\mathcal{E}_u)$ arbitrary. Let $[T, S, \lambda]$ be a vertex of $\mathcal{G}_{\mathcal{O}}(\lambda)$. As $\mathcal{G}_{\mathcal{O}}(\lambda)$ is a subgraph of $\mathcal{G}_{\mathcal{O}}$, there is a path in $\mathcal{G}_{\mathcal{O}}$ from $[T, S, \lambda]$ to a coincidence. However, if a vertex of $\mathcal{G}_{\mathcal{O}}$ is contained in $\mathcal{G}_{\mathcal{O}}(\lambda)$, then all its successors in $\mathcal{G}_{\mathcal{O}}$ are contained in $\mathcal{G}_{\mathcal{O}}(\lambda)$ in view of (5.39). Thus the path from $[T, S, \lambda]$ to a coincidence which is contained in $\mathcal{G}_{\mathcal{O}}$ by assumption is also contained in $\mathcal{G}_{\mathcal{O}}(\lambda)$.

To prove the converse assume that the weak overlap coincidence is true for each $\lambda \in \Xi(\mathcal{E}_u)$. Choose a vertex of $\mathcal{G}_{\mathcal{O}}$. This vertex is of the form $[T, S, \gamma]$ for some $\gamma \in \Xi(\mathcal{E}_u)$. Thus, in $\mathcal{G}_{\mathcal{O}}(\gamma)$ there is a path from $[T, S, \gamma]$ to a coincidence. Since $\mathcal{G}_{\mathcal{O}}(\gamma)$ is a subgraph of $\mathcal{G}_{\mathcal{O}}$, this path also exists in $\mathcal{G}_{\mathcal{O}}$. □

5.7.2 Tiling conditions related to overlap graphs

In this section we have two main aims. First we want to prove that the combinatorial strong coincidence condition together with the strong overlap coincidence condition is equivalent to the tiling property. As mentioned above, the strong overlap coincidence condition is hard to check. Thus, in a second step, we show that strong and weak overlap coincidence are equivalent. Summing up we will arrive at a tiling criterion in terms of combinatorial strong coincidence and weak overlap coincidence.

In all what follows we enumerate the tiles of $\mathcal{E}_u$ starting from the tile next to the origin by $T_0, T_1, \ldots$ One has $T_r = \pi_e[\mathbf{P}(u_0 \cdots u_{r-1}), u_r]_g$ for all $r \in \mathbb{N}$. Moreover, for each $\lambda \in \Xi(\mathcal{E}_u)$, we consider the union of tiles $T \in \mathcal{E}_u$ such that $T + \lambda$ also occurs in $\mathcal{E}_u$, *i.e.*,

$$\mathrm{Occ}(\lambda) := \bigcup_{\{T \in \mathcal{E}_u \mid T+\lambda \in \mathcal{E}_u\}} T.$$

We start with the following lemma (*cf.* (Solomyak 1997, Proposition 6.7) and (Lee, Moody, and Solomyak 2003, Lemma A.8)), which translates the weak overlap coincidence condition in combinatorial terms (see also the related result (Queffélec 1987, Lemma VI.27)).

Lemma 5.7.9 *Let $\lambda \in \Xi(\mathcal{E}_u)$. The graph of overlaps of λ satisfies the weak overlap coincidence condition if, and only if, there exists some constant $c \in (0,1)$ such that*

$$\mathrm{Card}\{T_r \in \mathcal{E}_u \mid r \leq N,\ T_r + \beta^m \lambda \notin \mathcal{E}_u\} \ll N\, c^m \tag{5.40}$$

holds for all $m \in \mathbb{N}$ when N is large enough in terms of m (the implied constant does not depend on N and m).

Proof We first prove that there is a constant $\tilde{c} \in (0,1)$ such that

$$\limsup_{N\to\infty} \frac{\mu_1(\{T_r \mid r \leq N,\ T_r + \beta^m \lambda \notin \mathcal{E}_u\})}{\mu_1(\{T_r \mid r \leq N\})} \ll \tilde{c}^m \tag{5.41}$$

if, and only if, there exists some constant $c \in (0,1)$ such that (5.40) holds for all $m \in \mathbb{N}$ when N is large enough in terms of m.

Assume that (5.40) holds. Let $L_{\min}$ and $L_{\max}$ be the length of the shortest and longest tile in $\mathcal{E}_u$, respectively. One has for N large enough

$$\begin{aligned}\mu_1(\{T_r \mid r \leq N,\ T_r + \beta^m \lambda \notin \mathcal{E}_u\}) &\leq L_{\max}\, \mathrm{Card}\{T_r \mid r \leq N,\ T_r + \beta^m \lambda \notin \mathcal{E}_u\}\\ &\ll c^m N \leq c^m \frac{1}{L_{\min}} \mu_1(\{T_r \mid r \leq N\})\\ &\ll c^m \mu_1(\{T_r \mid r \leq N\}).\end{aligned}$$

Hence,

$$\limsup_{N\to\infty} \frac{\mu_1(\{T_r \mid r \leq N,\ T_r + \beta^m \lambda \notin \mathcal{E}_u\})}{\mu_1(\{T_r \mid r \leq N\})} \ll c^m$$

and (5.41) is true for $\tilde{c} = c$.

Conversely, assume that (5.41) holds for $\tilde{c} \in (0,1)$. Let $c \in (0,1)$ with $\tilde{c} < c$. For N large enough in terms of m, one has

$$\mu_1(\{T_r \mid r \leq N,\ T_r + \beta^m \lambda \notin \mathcal{E}_u\}) \ll c^m \mu_1(\{T_r \mid r \leq N\}).$$

Hence,

$$\begin{aligned}\mathrm{Card}(\{T_r \mid r \leq N,\, T_r + \beta^m \lambda \notin \mathcal{E}_u\}) \\ &\leq \frac{1}{L_{\min}} \mu_1(\{T_r \mid r \leq N,\ T_r + \beta^m \lambda \notin \mathcal{E}_u\}) \\ &\ll c^m \mu_1(\{T_r \mid r \leq N\}) \leq c^m N L_{\max} \ll c^m N,\end{aligned}$$

which ends the proof of the claimed equivalence.

We now prove (5.41). We first assume that the graph of overlaps $\mathcal{G}_{\mathcal{O}}(\lambda)$ of λ satisfies the weak overlap coincidence condition. By the finiteness of $\mathcal{G}_{\mathcal{O}}(\lambda)$ there exists a positive integer ℓ such that each vertex admits a path of length bounded by ℓ to an overlap coincidence.

Using (5.39), one checks that $\mathcal{G}_{\mathcal{O}}(\beta^m \lambda)$ is a subgraph of $\mathcal{G}_{\mathcal{O}}(\lambda)$ whose paths to overlap coincidences do not get longer as the ones in $\mathcal{G}_{\mathcal{O}}(\lambda)$. In particular, $\beta^\ell(T \cap (S - \beta^m \lambda))$ contains an overlap coincidence for each overlap $(T, S, \beta^m \lambda)$. One has $\mu_1(\beta^\ell(T \cap (S - \beta^m \lambda))) \leq \beta^\ell L_{\max}$. Moreover, each overlap $(T, S, \beta^m \lambda)$ produces tiles which belong to $\mathrm{Occ}(\beta^{m+\ell}\lambda)$, and whose union has length bounded from below by $L_{\min}$. Let $b := \left(1 - \frac{L_{\min}}{L_{\max}\beta^\ell}\right)$. Since the definition of Occ immediately implies $\beta \mathrm{Occ}(\beta^m \lambda) \subset \mathrm{Occ}(\beta^{m+1}\lambda)$, and, hence, $\beta^\ell \mathrm{Occ}(\beta^m \lambda) \subset \mathrm{Occ}(\beta^{m+\ell}\lambda)$, we deduce that

$$\begin{aligned}&\limsup_{N\to\infty} \frac{\mu_1\{T_r \in \mathcal{E}_u \mid r \leq N,\ T_r + \beta^{m+\ell}\lambda \notin \mathcal{E}_u\}}{\mu_1\{T_r \in \mathcal{E}_u \mid r \leq N\}} \\ &\leq b \limsup_{N\to\infty} \frac{\mu_1\{T_r \in \mathcal{E}_u \mid r \leq N,\ T_r + \beta^{m}\lambda \notin \mathcal{E}_u\}}{\mu_1\{T_r \in \mathcal{E}_u \mid r \leq N\}}.\end{aligned}$$

Since $b < 1$ does not depend on m and N, by writing $m = k\ell + s$ with $0 \leq s < \ell$ we easily derive (5.41) by iteration.

To prove the converse assume that (5.41) holds and that $\mathcal{G}_{\mathcal{O}}(\lambda)$ does not satisfy the weak overlap coincidence condition. Then there exists an overlap class $[T, S, \beta^m \gamma]$ such that for every $\ell, m > 0$

$$\beta^\ell(T \cap (S - \beta^m \lambda)) \subset \mathbb{R}_+ \setminus \mathrm{Occ}(\beta^m \lambda).$$

By the repetitivity of $\mathcal{E}_u$ this yields

$$\limsup_{N\to\infty} \frac{\mu_1\{T_r \in \mathcal{E}_u \mid r \leq N,\ T_r + \beta^{m+\ell}\lambda \notin \mathcal{E}_u\}}{\mu_1\{T_r \in \mathcal{E}_u \mid r \leq N\}} \gg 1,$$

uniformly in m, which contradicts (5.41). □

We now establish the relation between the tiling property and the strong overlap coincidence condition. The first part of the following lemma is proved in (Lee 2007) in the general context of substitution Delone sets and Meyer sets. Indeed, it can be derived from the *algebraic coincidence condition* introduced in (Lee 2007). For more details, see the notes at the end of the present chapter.

Lemma 5.7.10 *Let σ be a unit Pisot irreducible substitution. If σ satisfies the strong overlap coincidence condition, then the following assertions are true.*

- *There exists $m \in \mathbb{N}$ such that $\beta^m(\Xi(\mathcal{E}_u) - \Xi(\mathcal{E}_u)) \cap \mathbb{R}_+ \subset \Xi(\mathcal{E}_u)$.*
- *There exists $m \in \mathbb{N}$ such that $\beta^m(\Xi(\mathcal{E}_u) + \Xi(\mathcal{E}_u)) \subset \Xi(\mathcal{E}_u)$.*

Proof Let $\lambda_1, \lambda_2 \in \Xi(\mathcal{E}_u)$ with $\lambda_2 - \lambda_1 > 0$ be given. Then Lemma 5.7.9 implies that there exists $m \in \mathbb{N}$ such that for all N large enough and for $i = 1, 2$, one has

$$\mathrm{Card}\{r \leq N \mid T_r + \beta^m \lambda_i \notin \mathcal{E}_u\} < \frac{N}{3}. \tag{5.42}$$

This implies that

$$\mathrm{Card}\{r \leq N \mid T_r + \beta^m \lambda_1 \notin \mathcal{E}_u \text{ or } T_r + \beta^m \lambda_2 \notin \mathcal{E}_u\} < \frac{2N}{3} \tag{5.43}$$

which means that there is some $r \in \mathbb{N}$ such that $T_r + \lambda_1 \beta^m \in \mathcal{E}_u$ and $T_r + \lambda_2 \beta^m \in \mathcal{E}_u$. Thus, $\beta^m(\lambda_2 - \lambda_1) \in \Xi(\mathcal{E}_u)$ which proves the first assertion.

Recall that $L_{\min}$ denotes the length of the shortest tile in $\mathcal{E}_u$. If N is large in terms of m, one has moreover

$$\mathrm{Card}\{\beta^m \lambda_1 / L_{\min} \leq r \leq N \mid T_r - \beta^m \lambda_1 \notin \mathcal{E}_u\} < \frac{N}{3}$$

which implies

$$\mathrm{Card}\{\beta^m \lambda_1 / L_{\min} \leq r \leq N \mid T_r - \beta^m \lambda_1 \notin \mathcal{E}_u \text{ or } T_r + \beta^m \lambda_2 \notin \mathcal{E}_u\} < \frac{2N}{3}.$$

Thus, if N is large in terms of m, there exists $r \leq N$ such that $T_r - \beta^m \lambda_1 \in \mathcal{E}_u$ and $T_r + \lambda_2 \beta^m \in \mathcal{E}_u$. Thus, $\beta^m(\lambda_2 + \lambda_1) \in \Xi(\mathcal{E}_u)$ which proves the second assertion. □

In order to get a relation between the strong overlap coincidence condition and the tiling condition, we need to ensure that all lengths of tiles in $\mathcal{E}_u$ multiplied by some power of β belong to the translation set $\Xi(\mathcal{E}_u)$.

This is realised by assuming the combinatorial strong coincidence condition introduced in Definition 5.2.9.

Lemma 5.7.11 *Let σ be a unit Pisot irreducible substitution that satisfies the combinatorial strong coincidence condition. Then there exists m such that $\beta^m \pi_e \circ \mathbf{P}(i) \in \Xi(\mathcal{E}_u)$ for every letter $i \in A$.*

Proof Let $i \in A$. Let $j \in A$ be such that ij is a factor of the fixed point u. By the combinatorial strong coincidence condition, there exist $m > 0$, a letter k and four words p, q, r, s such that $\sigma^m(i) = pkq$ and $\sigma^m(j) = rks$, with $\mathbf{P}(p) = \mathbf{P}(r)$. Since $\sigma^m(ij)$ is a factor of u, we have that $\pi_e \circ \mathbf{P}(kqr) \in \Xi(\mathcal{E}_u)$. The relation $\mathbf{P}(p) = \mathbf{P}(r)$ yields $\pi_e \circ \mathbf{P}(kqr) = \pi_e \circ \mathbf{P}(pkq) = \pi_e \circ \mathbf{P}(\sigma^m(i)) = \beta^m \pi_e \circ \mathbf{P}(i) \in \Xi(\mathcal{E}_u)$. □

We can now derive the following relation between the strong overlap coincidence condition and the tiling condition.

Theorem 5.7.12 *Let σ be a unit Pisot irreducible substitution. Then σ satisfies the tiling property if, and only if, σ satisfies both the strong overlap coincidence condition and the combinatorial strong coincidence condition.*

Proof We first assume that σ satisfies the tiling property. By Theorem 5.6.8, σ satisfies the super coincidence condition and, in particular, the combinatorial strong coincidence condition. According to Remark 5.7.3, each overlap class $[T, S, \lambda]$ is associated with a pair of basic formal strands $([\mathbf{0}, i], [\mathbf{z}, j])$ having the same height. Since the super coincidence condition holds, there exists m such that $\mathbf{E}_1^m[\mathbf{z}, i]$ and $\mathbf{E}_1^m[0, j]$ contain a common basic formal strand. In other words, one gets an overlap coincidence between $\beta^m T$ and $\beta^m(S - \lambda)$. Therefore the strong overlap coincidence is satisfied.

The converse is slightly more difficult to establish. This is mostly due to the specific positions of the vectors $\mathbf{z}$ associated with overlap classes as noticed in Remark 5.7.3. We assume that σ satisfies both the strong overlap coincidence condition and the combinatorial strong coincidence condition. Let $[\mathbf{x}, i]$ and $[\mathbf{y}, j]$ be a pair of basic formal strands with the same height. Let $T := \pi_e([\mathbf{x}, i]_g)$ and $S := \pi_e([\mathbf{y}, j]_g)$. There exist $\mathbf{z}_1, \mathbf{z}_2 \in \pi_e(\mathbb{Z}^n)$ such that $T \in \mathcal{E}_u - \pi_e(\mathbf{z}_1)$ and $S \in \mathcal{E}_u - \pi_e(\mathbf{z}_2)$. Assume w.l.o.g. that $\pi_e(\mathbf{z}_1) < \pi_e(\mathbf{z}_2)$. Now set $T' := T + \pi_e(\mathbf{z}_1)$ and $S' := S + \pi_e(\mathbf{z}_1)$. Then $T' \in \mathcal{E}_u$ and $S' \in \mathcal{E}_u - \lambda$ with $\lambda := \pi_e(\mathbf{z}_2 - \mathbf{z}_1)$.

By Lemma 5.7.11, the combinatorial strong coincidence condition implies that there exists $m_1 \in \mathbb{N}$ such that $\beta^{m_1} \pi_e \circ \mathbf{P}(k) \in \Xi(\mathcal{E}_u)$ for all $k \in A$. As $\{\mathbf{P}(1), \ldots, \mathbf{P}(n)\} = \{\mathbf{e}(1), \ldots, \mathbf{e}(n)\}$ forms a basis of the lattice $\mathbb{Z}^n$, according to Lemma 5.7.10 for each $\mathbf{z} \in \mathbb{Z}^n$ there exists $m_2 \in \mathbb{N}$ such that $\beta^{m_2}\pi_e(\mathbf{z}) \in \Xi(\mathcal{E}_u)$. Thus there exists $m \in \mathbb{N}$ such that $\lambda \in \Xi(\mathcal{E}_u)$. We thus deduce from the strong overlap coincidence condition that $[\beta^m T, \beta^m S, \beta^m \lambda]$ leads to a coincidence overlap. This implies that the super coincidence condition holds. □

We now turn to the second main result of the present section, the equivalence between the strong and the weak overlap coincidence condition. This is proved in a slightly different context in (Solomyak 1997, Section 6) where it is shown that both conditions are equivalent to the fact that the dynamical system associated with the tiling has pure discrete spectrum (see also (Lee 2007, Section 3) where overlap coincidence is related to pure discrete spectrum). We give a new and direct proof of this result here.

Theorem 5.7.13 *Let σ be a unit Pisot irreducible substitution. Then the following assertions are equivalent.*

(i) *The substitution σ satisfies the weak overlap coincidence condition.*
(ii) *The substitution σ satisfies the strong overlap coincidence condition.*

In view of Lemmas 5.7.8 and 5.7.9 it is clear that Theorem 5.7.13 is a direct consequence of the following lemma. In the proof of this lemma we exploit the fact that weak overlap coincidence implies that the fixed point u of σ is *mean-almost periodic* in the sense of (Queffélec 1987, Definition VI.4).

Lemma 5.7.14 *Let $\lambda_1, \lambda_2 \in \Xi(\mathcal{E}_u)$ with $\lambda_1 \neq 0$. If there exists $c \in (0,1)$ such that*

$$\mathrm{Card}\{T_r \in \mathcal{E}_u \mid r \leq N,\ T_r + \beta^m \lambda_1 \notin \mathcal{E}_u\} \ll N\, c^m \tag{5.44}$$

for all m when N is large enough in terms of m then

$$\mathrm{Card}\{T_r \in \mathcal{E}_u \mid r \leq N,\ T_r + \beta^m \lambda_2 \notin \mathcal{E}_u\} \ll N c^m$$

for all m when N is large enough in terms of m. The implied constants do not depend on m and N.

Proof Let λ_1 be a non-zero element of $\Xi(\mathcal{E}_u)$ such that (5.44) holds and let $\lambda_2 \in \Xi(\mathcal{E}_u)$.

We consider a tile T that occurs in $\mathrm{Occ}(\lambda_2)$. One has $T + \lambda_2 \in \mathcal{E}_u$. By the repetitivity of $\mathcal{E}_u$, there is a set of tiles $W^{(0)} \subset \mathcal{E}_u$ such that the union of tiles of $W^{(0)}$ is a relatively dense set in $\mathbb{R}_+$ (*i.e.*, there exists $L > 0$ such that every interval of length L in $\mathbb{R}_+$ contains at least one point belonging to one tile of $W^{(0)}$) and $T_r + \beta\lambda_2 \in \mathcal{E}_u$ whenever $T_r \in W^{(0)}$.

We now use the self-similarity of $\mathcal{E}_u$ (or equivalently, the fact that $\sigma(u) = u$). We multiply all tiles of $W^{(0)}$ by β^m and subdivide accordingly to arrive again at $\mathcal{E}_u$. Thus, there exist

$$u_1^{(m)} < v_1^{(m)} < u_2^{(m)} < v_2^{(m)} < u_3^{(m)} < v_3^{(m)} < \cdots$$

with

$$\min_i(v_i^{(m)} - u_i^{(m)}) \gg \beta^m,\ \max_i(u_{i+1}^{(m)} - v_i^{(m)}) \ll \beta^m,\ u_1^{(m)} \ll \beta^m \tag{5.45}$$

such that $T_r + \beta^m\lambda_2 \in \mathcal{E}_u$ whenever $T_r \in W^{(m)}$, where $W^{(m)} := \{T_s \mid s \in \bigcup_{i\geq 1}[\![u_i^{(m)}, v_i^{(m)}]\!]\}$. All bounds in (5.45) follow from the fact that

$$\forall a \in A,\ \beta^m \ll |\sigma^m(a)| \ll \beta^m \tag{5.46}$$

(for more details on (5.46) see Section 4.7.3). To get the upper bound for $\max_i(u_{i+1}^{(m)} - v_i^{(m)})$ also the relative denseness of the union of tiles of $W^{(0)}$ has to be used.

Let $m_0 \in \mathbb{N}$ be fixed in a way that $\beta^{m-m_0}\lambda_1 < \min_i(v_i^{(m)} - u_i^{(m)})$ (such a constant exists in view of (5.45)).

For each $K \in \mathbb{N}$ define the 'exceptional set'

$$S_K := \{T_r \in \mathcal{E}_u \mid T_r - k\beta^{m-m_0}\lambda_1 \notin \mathcal{E}_u \ \text{ for some } \ 0 \leq k \leq K\}.$$

Equation (5.44) implies that

$$\mathrm{Card}(S_K \cap \{T_r \mid r \leq N\}) \ll Nc^m \tag{5.47}$$

holds for all m if N large enough (note that the implied constant may depend on K but this is not relevant for us as K will be fixed in a moment). By (5.45) and since $\lambda_1 \neq 0$ we may fix $K \in \mathbb{N}$ in a way that for all $T_r \in \mathcal{E}_u \setminus S_K$ and all $m \in \mathbb{N}$, there exists $k \in [\![0, K]\!]$ such that $T_r - k\beta^{m-m_0}\lambda_1 \in W^{(m)}$.

We set for $k \in [\![0, K]\!]$

$$\mathcal{E}_u^{(k,m)} := \{T_r \in \mathcal{E}_u \mid T_r - k\beta^{m-m_0}\lambda_1 \in W^{(m)}\}.$$

Then

$$\{T_r \mid r \in \mathbb{N}\} = \bigcup_{k=0}^{K} \mathcal{E}_u^{(k,m)} \cup S_K. \tag{5.48}$$

Note that

$$T_r \in \mathcal{E}_u^{(0,m)} \implies T_r + \beta^m\lambda_2 \in \mathcal{E}_u.$$

We now take $k \neq 0$. Let $T_r \in \mathcal{E}_u^{(k,m)}$ such that $T_r + \beta^m\lambda_2 \notin \mathcal{E}_u$. One has $T_r - k\beta^{m-m_0}\lambda_1 \in W^{(m)}$, hence,

$$T_r - k\beta^{m-m_0}\lambda_1 + \beta^m\lambda_2 \in \mathcal{E}_u.$$

We thus have

$$T_r - k\beta^{m-m_0}\lambda_1 + \beta^m\lambda_2 \in \mathcal{E}_u,\ (T_r - k\beta^{m-m_0}\lambda_1 + \beta^m\lambda_2) + k\beta^{m-m_0}\lambda_1 \notin \mathcal{E}_u.$$

Hence, by recalling that $L_{\min}$ is the length of the smallest tile in $\mathcal{E}_u$, we get for $N \in \mathbb{N}$

$$\begin{aligned}&\mathrm{Card}\left\{T_r \in \mathcal{E}_u^{(k,m)} \mid r \leq N,\ T_r + \beta^m\lambda_2 \notin \mathcal{E}_u\right\} \leq \\ &\mathrm{Card}\left\{T_r \mid r \leq N + \frac{\beta^m\lambda_2}{L_{\min}},\ T_r + k\beta^{m-m_0}\lambda_1 \notin \mathcal{E}_u\right\}.\end{aligned}$$

Putting this together with (5.48), we obtain

$$\begin{aligned}&\mathrm{Card}\,\{T_r \in \mathcal{E}_u \mid r \leq N,\ T_r + \beta^m\lambda_2 \notin \mathcal{E}_u\} \\ &\leq \sum_{k=0}^{K} \mathrm{Card}\left\{T_r \in \mathcal{E}_u^{(k,m)} \mid r \leq N,\ T_r + \beta^m\lambda_2 \notin \mathcal{E}_u\right\} \\ &\qquad\qquad + \mathrm{Card}(S_K \cap \{T_r \mid r \leq N\}) \\ &\leq \sum_{k=0}^{K} \mathrm{Card}\{T_r \mid r \leq N + \frac{\beta^m\lambda_2}{L_{\min}},\ T_r + k\beta^{m-m_0}\lambda_1 \notin \mathcal{E}_u\} \\ &\qquad\qquad + \mathrm{Card}(S_K \cap \{T_r \mid r \leq N\}).\end{aligned}$$

We deduce from (5.44), (5.45) and (5.47) that

$$\mathrm{Card}\{T_r \in \mathcal{E}_u \mid r \leq N,\ T_r + \beta^m\lambda_2 \notin \mathcal{E}_u\} \ll \left(N + \frac{K\beta^m\lambda_2}{L_{\min}}\right) c^m + Nc^m.$$

This implies that

$$\mathrm{Card}\{T_r \in \mathcal{E}_u \mid r \leq N,\ T_r + \beta^m\lambda_2 \notin \mathcal{E}_u\} \ll Nc^m$$

holds for all m when N is large in terms of m. □

As mentioned above, Theorem 5.7.13 is an immediate consequence of the previous lemma. According to this theorem it is not necessary to build all of the configuration graph (or of the two-piece ancestor graph) in order to check the tiling property. Indeed, it is enough to build the graph from all pairs of tiles in the broken line that are separated by a fixed vector. We sum this up in the following corollary.

Corollary 5.7.15 *Let σ be a unit Pisot irreducible substitution. Then σ satisfies the tiling property if, and only if, σ satisfies both the* weak *overlap coincidence condition and the combinatorial strong coincidence condition.*

The computation of the pairs of such tiles still requires the knowledge of the language of the fixed point. In Section 5.8, we will take advantage of the weak overlap property in order to obtain a purely combinatorial characterisation of the tiling property. To this end we will need the following corollary which focuses on gaps between overlaps instead of focusing on density of overlaps. Its proof is an immediate consequence of the proof of Lemma 5.7.9.

Corollary 5.7.16 *Let $\lambda \in \Xi(\mathcal{E}_u)$. The weak overlap coincidence condition is satisfied for a $\lambda \in \Xi(\mathcal{E}_u)$ if, and only if, the distance between two successive coincidence overlaps in $(\mathcal{E}_u, (\mathcal{E}_u - \beta^k\lambda) \cap \mathbb{R}_+)$ is bounded uniformly in k.*

5.8 Balanced pair algorithm

This section is devoted to a further effective condition for the tiling property based on the notion of *balanced pairs* introduced in (Michel 1978) and later used *e.g.* by (Livshits 1987), (Queffélec 1987, Chapter VI), (Sirvent and Solomyak 2002) and (Martensen 2004). The starting point for the balanced pair algorithm is Corollary 5.7.16. It states that the tiling property is strongly related to the uniform boundedness (in k) of the length of gaps between successive coincidence overlaps in pairs of tilings of the form $(\mathcal{E}_u, (\mathcal{E}_u - \beta^k\lambda) \cap \mathbb{R}_+))$. According to Theorem 5.7.13, this property has to be checked only for one suitable $\lambda \in \Xi(\mathcal{E}_u)$. We will choose λ of the form $\lambda = \langle \mathbf{P}(w), \mathbf{v}_\beta \rangle$ for some prefix w of u with $u_0 = u_{|w|}$. For this choice we get that the first tile in $\mathcal{E}_u$ coincides with a tile of $(\mathcal{E}_u - \beta^k\lambda) \cap \mathbb{R}_+$ for each $k \in \mathbb{N}$. Indeed, for each k one has $\beta^k\lambda = \langle \mathbf{M}_\sigma^k \mathbf{P}(w), \mathbf{v}_\beta \rangle = \langle \mathbf{P}(\sigma^k(w)), \mathbf{v}_\beta \rangle$ where $\sigma^k(w)$ is a prefix of u. Note that the first overlap and the last overlaps of the part of the synchronised tiling $\mathcal{E}_u \cap (\mathcal{E}_u - \beta^k\lambda)$ that is located between 0 and $\beta^k\lambda$ are half-coincidences.

In applying the balanced pair algorithm, we will start with gaps between half-coincidence overlaps in order to get the uniform boundedness of gaps between successive coincidence overlaps. A gap between two half-coincidence overlaps can be described as an interval that can be decomposed in two ways: firstly, as a union of consecutive tiles of $\mathcal{E}_u$ and secondly, as a union of consecutive tiles of $(\mathcal{E}_u - \lambda\beta^k) \cap \mathbb{R}_+$. The types of these consecutive tiles correspond to two finite subwords v_1 and v_2 of u, respectively. Since the coordinates of $\mathbf{v}_\beta$ are rationally independent, we have $\mathbf{P}(v_1) = \mathbf{P}(v_2)$. This leads us to introduce the following combinatorial definition.

Definition 5.8.1 (Balanced pair) A pair $(v_1, v_2) \in A^* \times A^*$ is said to be a *combinatorial balanced pair*, or for short, a *balanced pair*, if $\mathbf{P}(v_1) = \mathbf{P}(v_2)$. A *one-letter balanced pair* (also called *coincidence balanced pair*) is a balanced pair of the form (a, a), with $a \in A$.

An *irreducible balanced pair* is a pair (v_1, v_2) with the property that no pair (v_1', v_2'), where v_i' is a proper prefix v_i, $i = 1, 2$, is balanced.

Note that when (v_1, v_2) is a balanced pair, $(\sigma(v_1), \sigma(v_2))$ is balanced as well. Obviously, each balanced pair can be split up uniquely into irreducible balanced pairs. This process is called *reduction*.

The set of irreducible balanced pairs obtained after the reduction of a balanced pair of finite words (v_1, v_2) is denoted by $\mathcal{B}(v_1, v_2)$. We say that a balanced pair (v_1, v_2) *leads to a coincidence* if there exists $N \in \mathbb{N}$ such that $\mathcal{B}(\sigma^N(v_1), \sigma^N(v_2))$ contains a one-letter balanced pair.

The following algorithm will lead to a powerful criterion for the tiling property.

Definition 5.8.2 (Balanced pair algorithm) Let I_0 be a non-empty and finite set of balanced pairs. The *balanced pair algorithm* applied to I_0 successively computes the sets

$$I_k := \bigcup_{(v_1,v_2) \in I_{k-1}} \mathcal{B}(\sigma(v_1), \sigma(v_2)).$$

The algorithm is said to *terminate with rank k*, $k \geq 1$, if $I_{k+1} = I_k$ and each balanced pair $(v_1, v_2) \in I_k$ leads to a coincidence. The algorithm is said to *terminate* if it terminates for some rank $k \geq 1$.

Let $I_0 = \{(v_1, v_2)\}$. If the balanced pair algorithm terminates there exist only finitely many different words w which occur between two consecutive one-letter balanced pairs in the pairs $(\sigma^k(v_1), \sigma^k(v_2))$. Thus the length of such gaps is uniformly bounded in k. We formulate this in a more exact way in the following proposition (see also (Sirvent and Solomyak 2002, Theorem 5.6)).

Proposition 5.8.3 *The balanced pair algorithm applied to a non-empty finite set I_0 of balanced pairs terminates if, and only if, the number of letters between two successive one-letter balanced pairs in $(\sigma^k(v_1), \sigma^k(v_2))$ for any $(v_1, v_2) \in \bigcup_k I_k$, is bounded uniformly in k.*

Example 5.8.4 In this example we want to consider the substitution $\sigma(1) = 112$, $\sigma(2) = 13$ and $\sigma(3) = 1$. It is easy to check that this is a unit Pisot irreducible substitution. We want to perform the balanced pair algorithm for this example, starting from $I_0 = \{(12,21),(13,31),(23,32)\}$. Applying σ and reducing yields

$$\begin{aligned}
(12,21) &\to (11213,13112) \to (1,1)(1213,3112),\\
(13,31) &\to (1121,1112) \to (1,1)(1,1)(21,12),\\
(23,32) &\to (131,113) \to (1,1)(31,13).
\end{aligned}$$

Thus in I_1 we have the one-letter pairs $(2,2)$, $(1,1)$ and the new pair $(1213,3112)$. Repeating the procedure for I_1 yields the new reduction

$$(1213,3112) \to (112131121,111211213) \to (1,1)(1,1)(21,12)(31121,11213).$$

Now I_2 contains the new pair $(31121,11213)$. It is treated as follows:

$$\begin{aligned}
(31121,11213) &\to (111211213112,112112131121)\\
&\to (1,1)(1,1)(12,21)(1,1)(12,21)(13,31)(1,1)(12,21).
\end{aligned}$$

This implies that I_3 contains no new pair. Moreover, each of the occurring pairs leads to a one-letter balanced pair. Thus the algorithm terminates.

The notions of balanced pair and reduction also make sense for pairs of infinite words. Let w be a non-empty prefix of the infinite word u. It is not hard to see that the set of irreducible balanced pairs occurring by reducing $(u, S^{|w|}(u))$ (here S denotes the shift) is a finite set (see for instance (Sirvent and Solomyak 2002, Section 3) and Exercise 5.6). Denote this set by $I_0(w)$. We then consider the balanced pairs in $(u, S^{|\sigma(w)|}(u))$, which amounts to applying the reduction process to all balanced pairs in $I_0(w)$. The balanced pair algorithm starting with the set $I_0 = I_0(w)$ is called the *balanced pair algorithm associated with* w.

The balanced pair algorithm is stated in the literature in several different forms. Besides taking the set $I_0(w)$ associated with some prefix w of u as the starting point (Sirvent and Solomyak 2002), the initial set I_0 is defined as $I_0 = \{(ij, ji) \mid i, j \in A,\ i \neq j\}$ in (Barge and Kwapisz 2006). As we will see, both starting sets lead to the same behaviour. Sometimes, it proves to be more convenient to start with $\{(ij, ji) \mid i \neq j\}$ instead of starting with $I_0(w)$ (even if this latter set can be determined in an effective way).

We now relate the balanced pair algorithm to the overlap coincidence condition.

Theorem 5.8.5 *Let σ be a unit Pisot irreducible substitution. Let w be a prefix of u with $u_{|w|} = u_0$ and set $\lambda = \pi_e \circ \mathbf{P}(w) \in \Xi(\mathcal{E}_u)$. The substitution σ satisfies the weak overlap coincidence condition for λ if, and only if, the balanced pair algorithm associated with w terminates.*

Proof Since w is a prefix of u, $(\mathcal{E}_u - \lambda) \cap \mathbb{R}_+$ is obtained from $\mathcal{E}_u$ by deleting the $|w|$ first tiles. This implies that coincidence overlaps between tiles in $(\mathcal{E}_u, (\mathcal{E}_u - \lambda) \cap \mathbb{R}_+)$ are in one-to-one correspondence with one-letter balanced pairs in $(u, S^{|w|}u)$. The same correspondence holds between $(\mathcal{E}_u, (\mathcal{E}_u - \beta^k \lambda) \cap \mathbb{R}_+)$ and $(u, S^{|\sigma^k(w)|}u)$. Now compare a step from $I_k(w)$ to $I_{k+1}(w)$ in the balanced pair algorithm with the symbolic interpretation of the self-similarity equation (5.39). The result now follows from Proposition 5.8.3 and Corollary 5.7.16. □

Corollary 5.7.15 and Theorem 5.8.5 immediately imply the following result (see also (Sirvent and Solomyak 2002, Section 5)).

Corollary 5.8.6 *A unit Pisot irreducible substitution σ satisfies the tiling property if, and only if, the combinatorial strong coincidence condition is satisfied and there exists a prefix w of u with $u_0 = u_{|w|}$ such that the balanced pair algorithm associated with w terminates.*

An even easier criterion for the tiling property can be obtained by starting the balanced pair algorithm with $I_0 = \{(ij, ji) \mid i \neq j\}$ instead of $I_0(w)$. As will be proved below (see Theorem 5.8.8), applying the balanced pair algorithm with this choice of I_0 allows both the combinatorial strong coincidence condition and the weak overlap condition to be checked at once. This yields the purely combinatorial algorithm for tiling discussed in (Barge and Kwapisz 2006). Before stating and proving Theorem 5.8.8, we establish the following auxiliary result.

Lemma 5.8.7 *Let σ be a unit Pisot irreducible substitution. If the balanced pair algorithm starting with $I_0 = \{(ij, ji) \mid i \neq j\}$ terminates, then the balanced pair algorithm starting with any balanced pair $(w, w') \in A^* \times A^*$ terminates.*

Proof We assume that the balanced pair algorithm starting with $I_0 = \{(ij, ji) \mid i \neq j\}$ terminates. Let $w, w' \in A^*$ be such that $\mathbf{P}(w) = \mathbf{P}(w')$. Let ℓ be equal to the common length $|w| = |w'|$ of w and w'. Since $\mathbf{P}(w) = \mathbf{P}(w')$, there exists a permutation ρ of the set $[\![1, \ell]\!]$ such that $w' = w_\rho$ where w_ρ is a shorthand for the word $w_{\rho(1)} \cdots w_{\rho(\ell)}$. We recall that a *transposition* is a permutation that exchanges two elements and that lets the other elements invariant, and that the symmetric group, *i.e.*, the group of permutations of $[\![1, \ell]\!]$ is generated by the permutations τ_i, for $i \in [\![1, \ell - 1]\!]$, where $\tau_i := (i, i+1)$ is the transposition of $[\![1, \ell]\!]$ that exchanges i and $i+1$. We set $T := \{\tau_i \mid i \in [\![1, \ell - 1]\!]\}$.

We have to show that the balanced pair algorithm starting with (w, w_ρ) terminates for each permutation ρ. We will prove this by induction.

In order to establish the induction start we have to consider the balanced pair algorithm starting with (w, w_τ) for some $\tau \in T$. However, since $\mathcal{B}(w, w_\tau) \subset \{(i,i)\} \cup \{(ij, ji) \mid i \neq j\}$ we only have to consider elements of $I_0 = \{(ij, ji) \mid i \neq j\}$ as the starting set to assure termination of the algorithm. Thus the balanced pair algorithm starting with (w, w_τ) terminates by the assumptions of the lemma.

The induction step is proved if we establish the following 'transitivity property'. Let $\tau, \tau' \in T$. If the balanced pair algorithm starting with (w, w_τ) terminates, and if the balanced pair algorithm starting with $(w_\tau, w_{\tau' \circ \tau})$ terminates, then also the balanced pair algorithm starting with $(w, w_{\tau' \circ \tau})$ terminates. To show this, according to Proposition 5.8.3, it is sufficient to prove that the occurrences between successive one-letter balanced pairs in $(\sigma^k(w), \sigma^k(w_{\tau' \circ \tau}))$ are bounded uniformly in k. As the balanced pair algorithm starting with (w, w_τ) as well as with $(w_\tau, w_{\tau \circ \tau'})$ terminates, Proposition 5.8.3 implies that there exists a constant $C > 0$ such that the occurrences between successive one-letter balanced pairs in $(\sigma^k(w), \sigma^k(w_\tau))$ and in $(\sigma^k(w_\tau), \sigma^k(w_{\tau' \circ \tau}))$ are bounded by C for each k.

Let N be large enough such that $|\sigma^N(a)| \geq C + 1$, for every letter $a \in A$. Fix $k \in \mathbb{N}$ and let j be the index of a one-letter balanced pair in the pair of words $(\sigma^k(w), \sigma^k(w_\tau))$, *i.e.*,

$$\sigma^k(w) = pas, \ \sigma^k(w_\tau) = p'as' \text{ with } \mathbf{P}(p) = \mathbf{P}(p') \text{ and } |p| = |p'| = j - 1.$$

Let j' be the index of the first letter of the image $\sigma^N(a)$ of the jth letter in $\sigma^k(w)$ as well as in $\sigma^k(w_\tau)$ under σ^N. This implies that each letter with index $\ell \in \{j', \ldots, j' + C\}$ forms a one-letter balanced pair in the reduction of the pair $(\sigma^{k+N}(w), \sigma^{k+N}(w_\tau))$.

Since by assumption the gaps between one-letter balanced pairs in $(\sigma^{k+N}(w_\tau), \sigma^{k+N}(w_{\tau' \circ \tau}))$ are bounded by C, there is an index $j'' \in \{j', \ldots, j' + C\}$ such that the j''th letters of $\sigma^{k+N}(w_\tau)$ and $\sigma^{k+N}(w_{\tau' \circ \tau})$ coincide. However, as $j'' \in \{j', \ldots, j' + C\}$ also the j''th letters of $\sigma^{k+N}(w)$ and $\sigma^{k+N}(w_\tau)$ coincide. Combining these two assertions we see that the j''th letters of $\sigma^{k+N}(w)$ and $\sigma^{k+N}(w_{\tau' \circ \tau})$ coincide.

Since this argument goes through for every $k \in \mathbb{N}$ and for every index j of a one-letter balanced pair in $(\sigma^k(w), \sigma^k(w_\tau))$, we deduce that the occurrences between successive one-letter balanced pairs in $(\sigma^{k+N}(w), \sigma^{k+N}(w_{\tau' \circ \tau}))$ are bounded uniformly in k by $(C+1)||\sigma^N||$ (recall that the width $\|\sigma\|$ of σ is defined as $\|\sigma\| := \max_{a \in A} |\sigma(a)|$ (see Definition 1.2.20)). This establishes the induction step and the lemma is proved. $\square$

Theorem 5.8.8 *Let σ be a unit Pisot irreducible substitution. The balanced pair algorithm starting with $I_0 = \{(ij, ji) \mid i, j \in A,\ i \neq j\}$ terminates if, and only if, the tiling property is satisfied.*

Proof We first prove that if the balanced pair algorithm starting with $I_0 = \{(ij, ji) \mid i, j \in A,\ i \neq j\}$ terminates, then σ satisfies the combinatorial strong coincidence condition. Let $i, j \in A$. According to Proposition 5.8.3, the distance between two one-letter balanced pairs in $(\sigma^N(i)\sigma^N(j)), \sigma^N(j)\sigma^N(i))$ is uniformly bounded in N. As $|\sigma^\ell(k)| \to \infty$ for $\ell \to \infty$ for each $k \in A$ this yields the existence of $N \in \mathbb{N}$ such that $\sigma^N(i)$ and $\sigma^N(j)$ can be decomposed as $\sigma^N(i) = par$ and $\sigma^N(j) = qas$, with $\mathbf{P}(p) = \mathbf{P}(q)$. Hence, σ satisfies the combinatorial strong coincidence condition.

We now assume that the balanced pair algorithm starting with $I_0 = \{(ij, ji) \mid i, j \in A,\ i \neq j\}$ terminates. Let w be a non-empty prefix of u with $u_{|w|} = u_0$. Let $\lambda = \pi_e \circ \mathbf{P}(w) \in \Xi(\mathcal{E}_u)$. Lemma 5.8.7 implies that the balanced pair algorithm associated with w terminates, since the set $I_0(w)$ is finite. Hence, by Corollary 5.8.5, σ satisfies the weak overlap coincidence condition.

Thus we proved that σ satisfies both the combinatorial strong coincidence condition and the weak overlap coincidence condition. Corollary 5.7.15 now implies that the tiling property is satisfied.

Let us now assume that σ satisfies the tiling property. This implies that both the strong overlap and the combinatorial strong coincidence condition hold. Let $(ij, ji) \in I_0$. Let $T_i = \pi_e([0, i]_g)$, $S_j = \pi_e([\mathbf{e}_i, j]_g)$, $T_j = \pi_e([0, j]_g)$, $S_i = \pi_e([\mathbf{e}_j, i]_g)$. By construction T_i and S_j are adjacent intervals, as well as T_j, S_i, and they cover the same interval $I := T_i \cup S_j = T_j \cup S_i$. This interval can be decomposed as the union of three overlap subintervals O_1, O_2, O_3 that do not necessarily belong to $\mathcal{O}_u$. Assume w.l.o.g. that T_i is longer than S_j. Then these intervals are equal to T_j, $T_i \cap S_i$ and S_j. We deduce from Lemma 5.7.11 that there exists M such that $\beta^M O_1, \beta^M O_2, \beta^M O_3$ can be decomposed into overlaps that belong to $\mathcal{O}_u$. Corollary 5.7.16 then implies that gaps between two coincidence overlaps in $\beta^{k+M} O_1$, $\beta^{k+M} O_2$, $\beta^{k+M} O_3$ are uniformly bounded in k, which implies that gaps between one-letter balanced pairs $(\sigma^k(ij), \sigma^k(ji))$ are also uniformly bounded in k. This implies that the balanced pair algorithm starting with I_0 terminates. □

One may define a graph associated with the balanced pair algorithm in an obvious way. The subgraph of this graph consisting of the coincidences (i, i) corresponds to a slightly modified prefix-suffix graph. If one removes this subgraph, and if the balanced pair algorithm ends, the remaining graph is finite, and its dominant eigenvalue corresponds to the asymptotic order of growth of non-coincidences between two different fixed points of powers of σ.

The balanced pair algorithm only terminates if the substitution satisfies the tiling property. Even if the super coincidence conjecture is true, it may well happen that the algorithm takes a long time before it terminates. We refer the reader to (Sirvent and Solomyak 2002, Section 6) where some examples are presented. They show that even quite simple looking substitutions may lead to quite large sets I_k. This suggests that it might be hard to get an analysis of the complexity of the balanced pair algorithm. Nevertheless, it is a useful and easy to implement tool for checking the tiling property for a given example with a purely combinatorial algorithm.

5.9 Conclusion

As a conclusion, let us summarise the different tiling conditions that we have encountered in the present chapter. The main condition can be formulated as follows: every two-piece patch of Γ_c appears (up to a translation vector) in the iterated image of a tip $[0, i]^*$ with $i \in A$. Several variations around this idea have produced the following graphs.

- The *two-piece ancestor graph* traces back ancestors of patches under a generalised substitution that acts on tips that belong to a finite patch (the two-piece seed patch W_σ) of the self-replicating translation set Γ_c. The tiling property is equivalent to the fact that from every vertex , there exists a path to a vertex with the specific shape $[i, 0, i]^*$, for $i \in A$.
- The *boundary graph* is a subset of the *two-piece ancestor graph*. It describes the tiles in the self-replicating multiple tiling that intersect the subtiles of the central tile. The tiling property can be expressed by computing the spectral radius associated with this graph.
- The *contact graph* is a subgraph of the two-piece ancestor graph that can be computed iteratively without the knowledge of the two-piece seed patch. The tiling property can also be expressed in terms of the spectral radius associated to this graph.
- The *configuration graph* is isomorphic to the two-piece ancestor graph. It checks whether every pair of basic strands eventually contains a common basic strand when applying σ iteratively.
- The *graph of overlaps* follows the same type of construction scheme as the the configuration graph. It is restricted to pairs of strands that are related to the fixed point of the substitution. It has to be combined

with the *combinatorial strong coincidence condition* to provide a tiling property.

- The *balanced pair algorithm* is a simple combinatorial process that describes the growth of gaps between coincidence overlaps and checks whether these gaps are uniformly bounded. It terminates whenever the tiling property is satisfied.

Among these conditions, the balanced pair algorithm is definitively the most combinatorial one. However, it does not always terminate. On the contrary, the contact graph can be computed with a purely algorithmic process that does not require additional computations, and provides a complete tiling characterisation. Note that the relations between the contact graph and the balanced pair algorithm remain unclear so far and deserve specific studies. Note also that the contact graph is not defined in the Pisot reducible case.

5.10 Exercises

Exercise 5.1 Let β be a Parry number (see Definition 2.3.12). The β-substitution σ_β associated with β is defined over the alphabet $\{1, \cdots, n\}$, where n stands for the number of states of the automaton $\mathcal{S}_\beta$ defined in the proof of Proposition 2.3.18, as follows: j is the kth letter of $\sigma_\beta(i)$ if, and only if, there is an arrow in $\mathcal{S}_\beta$ from the state i to the state j labelled by $k-1$. Give the β-substitution σ_β explicitly with respect to $\mathtt{d}_\beta(1)$. Compare its prefix-suffix automaton with the automaton $\mathcal{S}_\beta$. Prove that σ_β is primitive. Show that the Perron–Frobenius eigenvalue of its incidence matrix is equal to β. For more details, see (Lothaire 2002, Chapter 7).

Exercise 5.2 Prove that the first coordinate projection of Γ_c on $\mathbb{H}_c$ is not left invariant by any non-zero translation vector.

Exercise 5.3 Prove that the fact that the self-replicating multiple tiling $\mathcal{I}_\sigma$ is actually a tiling implies that the largest eigenvalue of the adjacency matrix of the boundary graph (as well as of the contact graph) is strictly smaller than the largest eigenvalue of $\mathbf{M}_\sigma$.
Hint: Look at the corresponding result for the contact graph of a self-similar lattice tiling in (Gröchenig and Haas 1994, Section 4).

Exercise 5.4 Prove that if two basic strands have geometric strong coincidence, then they have the same height.

Exercise 5.5 Prove that $[\mathcal{O}_u]$ and $[\mathcal{O}_u(\lambda)]$ do not depend on the choice of the fixed point u, but only on σ.
Hint: Use the repetitivity of $\mathcal{E}_u$, or equivalently, the uniform recurrence of the fixed point u.

Exercise 5.6 Prove that the set of irreducible balanced pairs occurring by reducing $(u, S^{|w|}(u))$ (here S denotes the shift) is a finite set.
Hint: Use the uniform recurrence of the fixed point u.

Exercise 5.7 List among the graphs introduced in the present chapter the ones that contain the prefix-suffix graph (or the graph obtained by reversing the direction of its edges) as a subgraph.

5.11 Notes

Section 5.1

As introduced for instance in (Thurston 1989) and in (Fabre 1995), one can associate in a natural way with the β-shift (see Section 2.3.2.1) a substitution σ_β called *β-substitution*, in the case where β is a Parry number. For more details, see Exercise 5.1. Compare also with the ideas underlying Definition 3.4.10. If β is a Pisot number, the associated substitution can be Pisot reducible as well as Pisot irreducible. An example of a Pisot reducible β-substitution is given by the smallest Pisot number β which is the positive root of $X^3 - X - 1$ (see Example 2.3.54).

Fractal geometry is deeply related to the study of numeration systems. One of the most famous examples of fractal tiles that come from numeration systems is the twin dragon fractal related to expansions of Gaussian integers in base $-1 + i$ (see (Knuth 1998, p. 206)). More generally, tilings can be introduced in the framework of canonical numeration systems (see Section 2.4 and the references in the survey (Akiyama and Thuswaldner 2004)), of shift radix systems (see Section 2.4.4 and (Berthé, Siegel, Steiner, et al. 2009)), or of abstract numeration systems (see Chapter 3 and (Berthé and Rigo 2007a)). In particular, the study of the boundary of central tiles has proved to be particularly efficient in order to derive properties of numeration systems. Under the tiling condition and in the cubic case $n = 3$, points lying at the intersection of tiles in the self-replicating tiling have been described as complex numbers with multiple expansions in some numeration system (see *e.g.* (Messaoudi 1998, Messaoudi 2000, Sadahiro 2006)). For more on the relations between central tiles and numeration systems, see the survey (Barat, Berthé, Liardet, et al. 2006).

The construction of central tiles also has consequences for the effective construction of Markov partitions for toral automorphisms, the main eigenvalue of which is a Pisot number. See, for instance, (Kenyon and Vershik 1998), (Praggastis 1999), (Schmidt 2000), and (Lindenstrauss and Schmidt 2005). For more on connections between beta-numeration, Vershik's adic transformation (see (Vershik and Livshits 1992)) and codings of hyperbolic automorphisms, see the survey (Sidorov 2003), and in the same vein, (Einsiedler and Schmidt 2002).

The study of central tiles has also led to particularly interesting applications in number theory. This was one of the motivations of (Rauzy 1982). Central tiles and their associated tilings are indeed efficient tools to compute best simultaneous Diophantine approximations (see (Chekhova, Hubert, and Messaoudi 2001), (Hubert and Messaoudi 2006) and (Ito, Fujii, Higashino, et al. 2003)), or to characterise points with purely periodic beta-expansions (see (Hama and Imahashi 1997), (Akiyama, Barat, Berthé, et al. 2008) or (Adamczewski, Frougny, Siegel, et al. 2010)).

Section 5.2

In the case of a unit Pisot reducible substitution, besides $\mathbb{H}_c$ and $\mathbb{H}_e$ a third space plays a role. This is the space $\mathbb{H}_s$ generated by the eigenspaces corresponding to the eigenvalues of $\mathbf{M}_\sigma$ that are not conjugate to β. The projection of the broken line L_u on $\mathbb{H}_c$ along $\mathbb{H}_e \oplus \mathbb{H}_s$ still provides a bounded set in $\mathbb{H}_c$ which allows the definition of a central tile also in this case (for more details, see (Ei, Ito, and Rao 2006, Section 3.2) and (Berthé and Siegel 2005)).

The study of the spectrum of Pisot substitutive dynamical systems was one of the main motivations for the introduction of central tiles. Pisot irreducible substitutions are indeed conjectured to have discrete spectrum. For a detailed account of the spectral theory of substitutive dynamical systems, see (Queffélec 1987), (Pytheas Fogg 2002, Chapter 7) and (Barge and Kwapisz 2006). See also Section 6.9.

The notion of *coincidence* (in its various forms) has proved to be an efficient way for proving discrete spectrum. The coincidence condition was first introduced by Dekking (Dekking 1978) for substitutions with constant length. Using this notion, Dekking completely characterised the substitutions with constant length whose associated symbolic dynamical system has discrete spectrum. Later, Arnoux and Ito introduced the notion of combinatorial strong coincidence (under the name *strong coincidence*) in

(Arnoux and Ito 2001), which lead Ito and Rao to define the super coincidence condition (under the name *super coincidence*) in (Ito and Rao 2006). The super coincidence condition has been also introduced independently in (Barge and Kwapisz 2006) (under the name *geometric coincidence condition*). For a complete proof of the equivalence between discrete spectrum and the super coincidence condition for unit Pisot irreducible substitutions, see (Barge and Kwapisz 2006). The notion of coincidence has also been exploited in the framework of substitution tiling spaces and substitution Delone multisets. Lee (see (Lee 2007)) introduced the notion of *algebraic coincidence* in order to characterise substitution Delone multisets that have a pure point diffraction spectrum. Using the notation of Section 5.7 algebraic coincidence can be stated as follows. Let $\Lambda_i := \bigcup_{n\in\mathbb{N},\ u_n=i} T_n$. We say that the substitution σ satisfies the *algebraic coincidence* condition if there exist a positive integer M and $\xi \in \Lambda_i$ for each $i \in A$ such that $\xi + \beta^M \Xi(\mathcal{E}_u) \subseteq \Lambda_i$. For a review of various notions of coincidences that are related to substitutions and substitution Delone multisets, see (Sing 2006) and the discussion in (Lee 2007). See also (Lee, Moody, and Solomyak 2003) and (Fretlöh and Sing 2007) for the related notion of *modular coincidence*.

The numeration system based on words alluded to in the proof of Theorem 5.2.3 is known as the *Dumont–Thomas numeration system* (see *e.g.* (Dumont and Thomas 1989, Rauzy 1990, Dumont and Thomas 1993) and Section 9.4.2). More precisely one checks that every finite prefix of u can be uniquely expanded as $\sigma^n(p_n)\sigma^{n-1}(p_{n-1})\cdots p_0$, where $p_n \neq \varepsilon$, and $(p_0, a_0, s_0)\cdots(p_n, a_n, s_n)$ is the sequence of labels of a path in the prefix-suffix automaton $\mathcal{G}_\sigma$ (see (Dumont and Thomas 1989, Theorem 1.5)). Hence, we can expand the non-negative integer N as $N = |\sigma^n(p_n)| + \cdots + |p_0|$, where $u_0 \cdots u_{N-1} = \sigma^n(p_n)\sigma^{n-1}(p_{n-1})\cdots p_0$. Let β be the Perron–Frobenius eigenvalue of σ. This numeration system also provides generalised radix expansions of positive real numbers, with digits belonging to a finite subset of the number field $\mathbb{Q}(\beta)$. We first define the mapping $\delta_\sigma\colon\ A^* \to \mathbb{Q}(\beta),\ p \mapsto \langle \mathbf{P}(p), \mathbf{w}_\beta\rangle$, where $\mathbf{w}_\beta$ is a left eigenvector of $\mathbf{M}_\sigma$ with positive entries associated with the Perron–Frobenius eigenvalue β. One has $\delta_\sigma(\sigma^n(p)) = \beta^n\delta_\sigma(p)$, for every n and $p \in A^*$. We then associate with the combinatorial expansion $(p_n, a_n, s_n)\ldots(p_0, a_0, a_0)$ the real number $\delta_\sigma(p_n)\beta^n + \cdots + \delta_\sigma(p_0) \in \mathbb{Q}(\beta)$. To recover the β-numeration in the particular case where σ is a β-substitution, $\mathbf{w}_\beta$ has to be normalised so that its first coordinate is equal to 1: the coordinates of $\mathbf{w}_\beta$ are then of the form $T_\beta^i(1)$, for $0 \le i \le n-1$, with $T_\beta\colon x \mapsto \{\beta x\}$. We have chosen to work in the present chapter with an eigenvector $\mathbf{v}_\beta$ normalised so that its coordinates belong to $\mathbb{Z}[\beta]$. This choice of normalisation plays a role in particular in the proof of Lemma 5.7.2. One checks that if σ is a unit Pisot irreducible β-substitution, the theory and the results of this chapter also hold by working with $\mathbf{w}_\beta$ normalised in a way that its first coordinate is equal to 1, instead of working with $\mathbf{v}_\beta$. In particular, the proof of Lemma 5.7.2 can easily be adapted, by noticing that there exists a positive integer $D > 0$ such that the coordinates of $\mathbf{w}_\beta$ all belong to $\frac{1}{D}\mathbb{Z}[\beta]$.

Section 5.3

GIFS substitutions, introduced in (Arnoux and Ito 2001), were inspired by the geometric formalism of (Ito and Ohtsuki 1993), whose aim was to provide explicit Markov partitions for hyperbolic automorphisms of the torus associated with particular substitutions produced by Brun's continued fraction algorithm. GIFS substitutions have already proved their efficiency for Diophantine approximation (Ito, Fujii, Higashino, et al. 2003), in word combinatorics (Arnoux, Berthé, and Siegel 2004), and in discrete geometry (Arnoux, Berthé, and Ito 2002), (Fernique 2006) and (Fernique 2009).

There is a second multiple tiling defined on $\mathbb{H}_c$ that plays an important role in the study of the substitutive symbolic dynamical system (X_σ, S). It is obtained by projecting points in $\mathbb{Z}^n$ that lie on the hyperplane with equation $\langle \mathbf{x}, (1, \dots, 1)\rangle = 0$ by π_c. The corresponding translation set, called the *lattice translation set*, is thus defined as $\{[\gamma, i]^* \in \pi_c(\mathbb{Z}^n) \times A \mid \gamma \in \sum_{k=2}^{n} \mathbb{Z}(\pi_c(\mathbf{e}_k) - \pi_c(\mathbf{e}_1))\}$. It is clearly periodic. According to (Canterini and Siegel 2001b), if σ is a unit Pisot irreducible substitution that satisfies the combinatorial strong coincidence condition, the lattice translation set is a Delone set that also provides a multiple tiling for the subtiles of the central tile. This multiple tiling is called the *lattice multiple tiling*. Rauzy introduced in the seminal paper (Rauzy 1982) the notion of central tile with respect to this tiling. According to (Ito and Rao 2006) (see also (Barge and Kwapisz 2006, Remark 18.5)), we know that the lattice multiple tiling is a tiling if, and only if, the self-replicating multiple tiling is a tiling, if σ is assumed to be a unit Pisot irreducible substitution.

There are two dynamical systems that can be associated in a natural way with a unit Pisot substitution, namely the substitutive dynamical system (X_σ, S) (with its natural $\mathbb{Z}$-action by the shift), and the one-dimensional tiling space associated with the self-similar tiling of the expanding line (described in terms of an $\mathbb{R}$-action by translations). The lattice multiple tiling is intimately connected to the spectral properties of the substitutive dynamical system (X_σ, S) (see (Queffélec 1987) and (Pytheas Fogg 2002, Chapter 7)), whereas the self-replicating multiple tiling is connected to the spectral properties of the one-dimensional tiling space associated with the tiling of the expanding line (see (Barge and Kwapisz 2006)). Note that there exist unit Pisot reducible substitutions for which (X_σ, S) does not have discrete spectrum, as shown by (Baker, Barge, and Kwapisz 2006, Example 5.3). See also (Clark and Sadun 2006) for the study of the spectral impact of deformations of the lengths of tiles for the tiling spaces associated with a substitution. More generally, see (Sadun 2008) for a topological study of tiling spaces with aperiodic order.

Section 5.4

The geometric finiteness property is intimately related to the so-called (F) property (introduced in Section 2.3.2.2 in the beta-numeration framework). It is expressed in (Berthé and Siegel 2005) in terms of the Dumont–Thomas numeration. It also appears in (Fuchs and Tijdeman 2006) in a related context. Note that we can use the vast literature on the (F) property in the beta-numeration framework to exhibit classes of beta-substitutions that satisfy the geometric finiteness property (see *e.g.* (Baker, Barge, and Kwapisz 2006) and (Barge and Kwapisz 2005)).

The so-called (W) or *weak finiteness* property was first introduced in (Hollander 1996). He has proved that the (W) property implies the pure discreteness of the spectrum of the irreducible beta-shift. The (W) property can be stated for a Pisot number β as follows:

$$\forall\, z \in \mathbb{Z}[\beta^{-1}] \cap [0,1),\ \forall\, \varepsilon > 0,\ \exists x, y \in \mathrm{Fin}(\beta) \text{ such that } z = x - y \text{ and } y < \varepsilon.$$

The (W) property has been proved in (Akiyama 2002) to be equivalent with the tiling property. An algorithm which can tell whether a given Pisot number β has (F) or (W) property is described in (Akiyama, Rao, and Steiner 2004). The condition of Theorem 5.4.14 is related to the (W) property.

Section 5.5

Similar graphs have appeared in several restricted contexts with different names (see *e.g.* the references in (Akiyama and Thuswaldner 2004) for contact graphs for tiles related to matrix number systems). They are used either to describe beta-expansions for 0 (Akiyama 2002), to describe multiple expansions (Messaoudi 1998, Messaoudi 2000, Durand and Messaoudi 2009), to compute the Hausdorff dimension of the boundary of central tiles (Messaoudi 2000, Feng, Furukado, Ito, et al. 2006, Thuswaldner 2006), or to obtain pure discrete spectrum conditions for substitutive dynamical systems (by referring to the lattice translation set) (Siegel 2004). The knowledge on intersections between tiles also yields criteria for topological properties of central tiles (connectivity, disklikeness, non-trivial fundamental group) (see (Messaoudi 2000), (Siegel 2004), (Siegel and Thuswaldner 2010)).

Contact graphs are inspired by the contact matrix defined in (Gröchenig and Haas 1994) for self-similar lattice tilings. They have been defined in (Thuswaldner 2006) in the framework of substitutions. The properties of the contact graph are based on the polyhedral tiling generated by the geometric tips. This polyhedral tiling property is very specific to the Pisot irreducible case. This is the main reason why the contact graph can only be defined in the Pisot irreducible case, whereas the two-piece ancestor and the boundary graphs can be defined in the Pisot reducible case with slight modifications (see (Siegel and Thuswaldner 2010)). In the Pisot reducible case, some examples of substitutions have been studied in (Ei and Ito 2005) by using the mth polyhedral approximations of Definition 5.5.9. Unfortunately, no generic algorithm based on this approach exists so far.

Section 5.6

The notion of strand, introduced in (Barge and Diamond 2002), has been very fruitfully developed in the form of the strand space model for one-dimensional substitution tiling spaces in (Barge and Kwapisz 2006), see also (Barge and Kwapisz 2005), (Barge and Diamond 2007) and (Barge, Diamond, and Swanson 2009).

Section 5.7

Lemma 5.7.10 is strongly related to the notion of algebraic coincidence (see (Lee 2007)).

Much more than Lemma 5.7.11 can be said. In fact, the $\mathbb{Z}$-module generated by the lengths $\pi_e \circ \mathbf{P}(i)$, for $i \in A$, is equal to $\Xi(\mathcal{E}_u)$. For a proof, see (Sing 2006, Lemma 6.34) and (Barge and Kwapisz 2006, Section 12). Furthermore, for examples of substitutions for which we have the strict inclusion $\Xi(\mathcal{E}_u) \subset \mathbb{Z}[\beta]$, see (Sing 2006, Remark 6.36).

In (Sirvent and Solomyak 2002) the spectrum of the two dynamical systems associated with a substitution of Pisot type (*i.e.*, the substitutive symbolic space (X_σ, S) and the one-dimensional tiling space), is studied by comparing the balanced pair algorithm (for the $\mathbb{Z}$-action) and the overlap algorithm (for the $\mathbb{R}$-action). It is proved in (Clark and Sadun 2006) and in (Barge and Kwapisz 2006) that for a unit Pisot irreducible substitution, the tiling space has discrete spectrum if, and only if, the substitutive symbolic dynamical system has discrete spectrum.

Section 5.8

In Section 5.8 we have dealt exclusively with Pisot irreducible substitutions. Recently, (Martensen 2004) has generalised the balanced pair algorithm to the Pisot reducible case. In this case one has to identify certain patterns in order to get a proper behaviour of the algorithm, *i.e.*, to show its termination to be equivalent to the fact that the dynamical system associated with the substitution in question is purely discrete.

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