RANDOM WALKS ON INFINITE GRAPHS AND GROUPS
(Cambridge Tracts in Mathematics 138)

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The simple random walk on the integers is one of the simplest random processes
that one can imagine. It generalizes to any finitely generated group Γ equipped

with a finite set S of generators. If $X_n \in \Gamma$ denotes the position at time n, then $X_{n+1} = X_n \xi_{n+1}$, where ξ_n is chosen uniformly at random in S. More generally, ξ_n can be picked according to a given probability measure μ. Then the probability that the walk started at the neutral element $X_0 = e$ reaches x at time n is $\mathbf{P}_e(X_n = x) = \mu^{(n)}(x)$, where $\mu^{(n)}$ denotes the nth convolution power of μ. We always assume below that the generating set S is symmetric, that is, $S = S^{-1}$.

Will a random walk return infinitely many times to its starting point? This is the question discussed in Pólya's seminal article [6], in the case of integer lattices. If the answer is yes, then the walk is called recurrent; otherwise, it is called transient. Pólya's well-known finding is that the simple random walk on the integer lattice in Euclidean space is recurrent in one or two dimensions, and transient in dimension three or higher. Indeed, transience/recurrence is equivalent to the convergence/divergence of the series $\sum \mathbf{P}_e(X_n = e)$ and, in dimension d,

$$\mathbf{P}_e(X_{2n} = e) \sim c_d n^{-d/2} \quad \text{as } n \text{ tends to infinity.} \tag{1}$$

(For parity reasons, one cannot return to the starting point at odd time.) Spitzer's famous book [7] gives a thorough and beautiful treatment of random walks on integer lattices.

For general groups, one of the most basic and natural questions about random walks concerns the asymptotic behaviour of the probability of return to the starting point. How does (1) generalize to non-Abelian groups? For instance, can one characterize those groups which carry a recurrent simple random walk? The first work on random walks on general finitely generated groups is Kesten's thesis [4]. In the sequel [5], he proves the fundamental result that $\mathbf{P}_e(X_n = e)$ decays exponentially fast if and only if the group is non-amenable. The next crucial development concerning the behaviour of $\mathbf{P}_e(X_n = e)$ came more than twenty years later. To describe this, let $V(n)$ be the number of elements in the group that can be written as words of length at most n in the generators $s \in S$. Write $f(n) \approx g(n)$ if there are constants such that $c_1 f(c_2 n) \leqslant g(n) \leqslant c_3 f(c_4 n)$ for all n. During the 1980s, Varopoulos [8] proved that if $V(n) \geqslant cn^d$, then $\mathbf{P}_e(X_n = e) \leqslant Cn^{-d/2}$. This is remarkable because no further assumption on the group Γ is made. One celebrated consequence is that the only recurrent groups are the finite extensions of $\{0\}$, $\mathbb{Z}$ and $\mathbb{Z}^2$. Together with deep theorems concerning the algebraic structure of groups (theorems due to Malcev, Gromov, Tits, Wolf, and others), this leads to the following result. For a simple random walk on a discrete subgroup Γ of a connected Lie group, three and only three behaviours may occur: (i) $\mathbf{P}_e(X_{2n} = e) \approx n^{-d/2}$ for some integer d; (ii) $\mathbf{P}_e(X_{2n} = e) \approx \exp -n^{1/3}$; (iii) $\mathbf{P}_e(X_{2n} = e) \approx \exp(-n)$. Moreover: case (i) happens if and only if Γ is virtually nilpotent and $V(n) \approx n^d$; case (ii) happens if and only if Γ is virtually polycyclic and $V(n) \approx \exp(n)$; case (iii) happens if and only if Γ is non-amenable. Still, today, there are many finitely generated groups for which the behaviour of $\mathbf{P}_e(X_n = e)$ is not well understood, for instance, metabelian (that is, two-steps solvable) non-polycyclic groups.

Another fundamental aspect of the theory of random walks concerns the existence and behaviour of harmonic functions and the related boundary theories. (A function u is (μ)-harmonic if it satisfies the convolution equation $u * \mu = u$.) Indeed, this aspect played an important role at an early stage of the development of the theory, and is still an active area of study. A celebrated problem in this direction concerns the existence of bounded or positive harmonic functions: a measure μ has the Liouville property (respectively, the strong Liouville property) if any bounded

(respectively, positive) harmonic function is constant. It is still an open problem today whether or not these Liouville properties for simple random walks on a group are, in general, independent of the generating set.

Woess' book gives a well-documented, informative and personal treatment of the theory of random walks as it has evolved since Spitzer's book [7]. Although it is not meant to be self-contained, it gives careful proofs of most of the results that are discussed. The book actually treats the more general theory of random walks on graphs, but manages always to stay close to the heart of the matter. It includes some beautiful results on random walks on planar graphs. Random walks on Cayley graphs (that is, simple random walks on groups) are treated as a special case of random walks on graphs having a vertex-transitive group of automorphisms. The first chapter studies the 'type problem', that is, whether or not a given walk is recurrent. It gives a thorough and detailed treatment, including many interesting specific examples of recurrent graphs. The second chapter concerns the amenable/non-amenable dichotomy, and the problem of computing the so-called spectral radius $\rho = \limsup_{n \to \infty} \sqrt[n]{\mathbf{P}_e(X_n = e)}$ of some walks. The third chapter treats the asymptotic behaviour of $\mathbf{P}_e(X_n = e)$. Although many satisfactory results concerning the rough asymptotic behaviour of $\mathbf{P}_e(X_n = e)$ (in the sense of the relation $\approx$) are known, obtaining precise asymptotic results such as (1) is, in many cases, an open problem. In this direction, Woess describes a rich collection of results concerning specific groups. A recent development in this direction that is not included in the book is [1]. The fourth and last chapter gives a nice treatment of certain aspects of boundary theory. (I was surprised not to find [2] in the bibliography, and I wish that the results of [3] had been included, at least in the further results section.) The book focuses chiefly on positive harmonic functions, leaving the task of giving a complete treatment of Poisson boundary theory to another author.

Random walk theory is connected with many other areas of mathematics. Without distracting the author from its main theme, these links appear all through the text. Some readers will find that certain connections (for example, to volume growth, isoperimetry, geometric group theory, algebraic structure, covering of compact manifolds) could have been developed more, but this would have led to a voluminous and very different book.

This is an excellent book, where beginners and specialists alike will find useful information. It will become one of the major references for all those interested directly or indirectly in random walks. I highly recommend it.

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