(43) $\sum_{k=1}^{\infty} \frac{k}{2^k}$, Konvergenz untersuchen und ggf. Grenzwert

Konvergenz: Quotientenkriterium

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(k+1)}{2^{k+1}} \cdot \frac{2^k}{k}\right| = \frac{k+1}{2k} \xrightarrow[k\to\infty]{} \frac{1}{2} < 1 \Rightarrow \text{konvergent!}$$

Summe mit Abelscher Summation aus Bsp 22

Bsp 22: $\sum_{k=1}^{n} a_k b_k = s_n b_{n+1} - \sum_{k=1}^{n} s_k (b_{n+1} - b_n)$; $s_n = \sum_{k=1}^{n} a_k$

Wähle $a_k = \frac{1}{2^k}$, $b_k = k$ dann $b_{k+1} - b_k = k+1-k = 1$ und

$$s_k = \sum_{i=1}^{k} \frac{1}{2^i} = \frac{1-\frac{1}{2^{k+1}}}{1-\frac{1}{2}} - 1 = 2 \cdot \frac{2^{k+1}-1}{2^{k+1}} - 1 = \frac{2^{k+1}-2}{2^{k+1}} = \underline{\underline{1-\frac{1}{2^k}}}$$

Damit nach Abelscher Summation:

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} \frac{k}{2^k} = \left(1-\frac{1}{2^n}\right)(n+1) - \sum_{k=1}^{n}\left(1-\frac{1}{2^k}\right)\cdot 1 = n - \frac{n}{2^n} + 1 - \frac{1}{2^n} - \underbrace{\sum_{k=1}^{n} 1}_{n} + \sum_{k=1}^{n} \frac{1}{2^k} =$$

$$= -\frac{n}{2^n} - \frac{1}{2^n} + \sum_{k=0}^{n} \frac{1}{2^k} = -\frac{n}{2^n} - \frac{1}{2^n} + \frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}} = -\frac{n}{2^n} - \frac{1}{2^n} + 2\left(1-\frac{1}{2^{n+1}}\right)$$

$\left[\frac{n}{2^n} \text{ ist Nullfolge: } \frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{n} \cdot \frac{1}{2} \overset{n \geq 2}{\leq} \frac{3}{2} \cdot \frac{1}{2} \Rightarrow a_{n+1} \leq \frac{3}{4} a_n \Rightarrow a_{n+1} \leq \left(\frac{3}{4}\right)^{n} a_i \longrightarrow 0 \right]$

$\sum_{k=1}^{\infty} \frac{k}{2^k} = \lim_{n\to\infty} -\frac{n}{2^n} - \frac{1}{2^n} + 2\left(1-\frac{1}{2^{n+1}}\right) = \underline{\underline{2}}$