Fourier series on fractals

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Spectral sets

Definition

A set Ω of positive finite Lebesgue measure is called spectral if there exists a set $\Lambda \subset \mathbb{R}^d$, such that $\{\exp(2\pi i\lambda \cdot x) \mid \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\Omega)$.

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Tao, Matolcsi et.al.: The Fuglede Conjecture fails in dimension $d \geq 3$.

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Definition

Let $e_\lambda(x) := e^{2\pi i\lambda\cdot x}$. A Borel probability measure μ on $\mathbb{R}^d$ is called *spectral* if there exists a set $\Lambda \subset \mathbb{R}^d$ such that $\{e_\lambda \,|\, \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. Then Λ is called a *spectrum* for the measure μ.

The Jorgensen-Pedersen example

Example: Cantor set, using division by 4, keep the first and the third quarter. The Hausdorff measure μ_4 on this Cantor set, with dimension $\ln 2/\ln 4$, is a spectral measure with spectrum

$$\Lambda := \left\{ \sum_{k=0}^{n} 4^k a_k \,|, a_k \in \{0, 1\} \right\}.$$

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The Middle Third Cantor measure is far from spectral: there are no three mutually orthogonal exponential functions.

Affine iterated function systems

Let R be a $d \times d$ expansive integer matrix, let B be a finite subset of $\mathbb{Z}^d$, $0 \in B$, and let $N := \#B$. Define the affine maps

$$\tau_b(x) = R^{-1}(x + b), \quad (x \in \mathbb{R}^d, b \in B)$$

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There exists a unique compact set such that

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There is a unique Borel probability measure $\mu = \mu_B$ on $\mathbb{R}^d$ such that

$$\int f \, d\mu = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b \, d\mu, \quad (f \in C_c(\mathbb{R}^d))$$

The measure μ is supported on X_B.

Convergence of mock Fourier series

Theorem (Strichartz)

For the Jorgensen-Pedersen Cantor set, the Fourier series of continuous functions converge uniformly, Fourier series of L^p-functions converge in L^p.

Connections to wavelet theory

The Fourier transform of μ:

$$\hat{\mu}(x) = \prod_{n=1}^{\infty} \hat{\delta}_B\left((R^*)^{-n} x \right), \quad \hat{\delta}_B(x) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot x}$$

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Orthogonality:

$$\sum_{\lambda\in\Lambda} |\hat{\mu}(x+\lambda)|^2 = 1$$

Hadamard pairs

Let L be a subset of $\mathbb{Z}^d$ of the same cardinality as B, $0 \in L$. We say that (B, L) form a Hadamard pair if one of the following equivalent conditions is satisfied

1. The matrix

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3. The measure $\delta_B = \frac{1}{N}\sum_{b\in B}\delta_b$ is spectral with spectrum $(R^*)^{-1}L$.

δ-cycles

Suppose (B, L) form a Hadamard pair.
Want to get the following spectrum for μ.

$$\Lambda := \left\{ \sum_{n=0}^{\infty} R^k l_k \mid l_k \in L \right\}$$

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Definition

A set $\{x_0, \dots, x_{p-1}\}$ is called a δ-cycle, if there exist $l_0, \dots, l_{p-1} \in L$ such that $(R^*)^{-1}(x_i + l_i) = x_{i+1}$, where $x_p := x_0$, and $|\hat{\delta}_B(x_i)| = 1$, for all $i \in \{0, \dots, p-1\}$

The Łaba-Wang theorem

Theorem (Łaba-Wang)

In dimension $d = 1$, suppose $R \in \mathbb{Z}$ and $0 \in B, L \subset \mathbb{Z}$ form a Hadamard pair. Let μ_B be the invariant measure for the IFS $(\tau_b)_{b\in B}$. Then the set

$$\Lambda := \left\{ \sum_{n=0}^{\infty} R^k l_k \mid l_k \in L \right\}$$

is a spectrum for the measure μ_B if and only if the only δ-cycle is $\{0\}$.

An improvement on the Ł-W theorem

Theorem (D, Jorgensen)

In dimension $d = 1$, suppose $0 \in B, L \subset \mathbb{Z}$, (B, L) form a Hadamard pair, and let μ_B be the invariant measure of the IFS $(\tau_b)_{b \in B}$. Then μ_B is a spectral measure.

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A spectrum for μ_B is the smallest set that contains $-C$ for all δ-cycles C, and such that

$$R^*\Lambda + L \subset \Lambda.$$

Examples

For the Jorgensen-Pedersen example $R = 4$, $B = \{0, 2\}$. We can take $L = \{0, 3\}$. Then $\hat{\delta}_B(x) = \frac{1}{2}(1 + e^{2\pi i \cdot 2x})$.

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Other than the trivial δ-cycle $\{0\}$, there is an additional one $\{1\}$. $1 = \frac{1}{4}(1 + 3)$, and $|\hat{\delta}_B(1)| = 1$.

$$\Lambda(0) = \left\{ \sum_{k=0}^{n} 4^k l_k \mid l_k \in \{0, 3\} \right\}$$

$$\Lambda(1) = \left\{ -1 - \sum_{k=0}^{n} 4^k l_k \mid l_k \in \{0, 3\} \right\}$$

Then $\Lambda(0) \cup \Lambda(1)$ is a spectrum for μ_B.

Higher dimensions

Conjecture (D-Jorgensen)

Let $0 \in B, L \subset \mathbb{Z}^d$, and suppose (B, L) form a Hadamard pair. The invariant measure μ_B for the IFS $(\tau_b)_{b \in B}$ is a spectral measure.

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1. True for dimension $d = 1$.

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1. True for dimension $d = 1$.
2. True for higher dimensions under the assumption that (B, L) is "reducible".

Examples

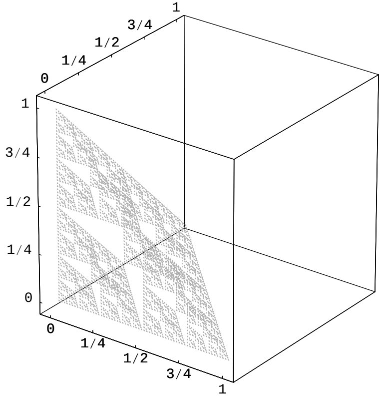

Figure: The Eiffel Tower. $R = 2I_3$, $B = \{0, e_1, e_2, e_3\}$

The Fourier transform

Let μ be a spectral measure with spectrum Λ. The Fourier transform $\mathcal{F} : L^2(\mu) \to l^2(\Lambda)$ is defined by

$$(\mathcal{F}f)(\lambda) = \langle f \, , \, e_\lambda \rangle \, , \quad (f \in L^2(\mu), \lambda \in \Lambda).$$

The group of local translations

Define the multiplication operator M_{e_t} on $l^2(\Lambda)$

$$M_{e_t}(a_\lambda)_\lambda = (e^{2\pi i t\cdot\lambda}a_\lambda)_\lambda.$$

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$$U_\Lambda(t) = \mathcal{F}^{-1} M_{e_t} \mathcal{F}, \quad (t \in \mathbb{R}^d).$$

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Theorem

Suppose $O, O+t$ is contained in $\mathrm{supp}(\mu)$. *Then*

$$(U_\Lambda(t)f(x) = f(x+t), \quad (x \in O)$$

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Corollary

If μ is a spectral measure and $O, O + t \subset \operatorname{supp}(\mu)$ then $\mu(O) = \mu(O + t)$.

Finite spectral sets

Theorem

Let A be a finite subset of $\mathbb{R}^n$. The following affirmations are equivalent:

1. *The set A is spectral.*
2. *There exists a continuous group of unitary operators $(U(t))_{t\in\mathbb{R}^n}$ on $L^2(A)$, i.e., $U(t+s) = U(t)U(s)$, $t,s \in \mathbb{R}^n$ such that*
$$U(a-a')\chi_a = \chi_{a'} \quad (a,a' \in A), \tag{3.1}$$
where
$$\chi_a(x) = \begin{cases} 1, & x = a \\ 0, & x \in A \setminus \{a\}. \end{cases}$$

Frames

Definition

A family of vectors $(v_i)_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f\,,\,v_i\rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}).$$

Fourier frames for the Cantor set

Consider the Middle Third Cantor set with its invariant measure μ_3, i.e., $R = 3$, $B = \{0, 2\}$. Jorgensen and Pedersen proved that there are not more than two orthogonal exponentials in $L^2(\mu_3)$.

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Let μ be a finite Borel measure on $\mathbb{R}^d$. A set Λ in $\mathbb{R}^d$ is called a *frame spectrum* if $\{e_\lambda \mid \lambda \in \Lambda\}$ is a frame for $L^2(\mu)$.

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Question

Construct a frame spectrum for the Middle Third Cantor set.

Frame spectrum and geometry

Question (Mark Kac)

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Question

What geometric properties of the measure μ can be deduced if we know a spectrum/ frame spectrum of μ?

Beurling dimension

Definition

Let $Q = [0,1]^d$ be the unit cube. Let Λ be a discrete subset of $\mathbb{R}^d$, and let $\alpha > 0$. Then the α-upper Beurling density is

$$\mathcal{D}_\alpha(\Lambda) := \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap (x + hQ))}{h^\alpha}.$$

Then $\mathcal{D}_\alpha(\Lambda)$ is constant ∞ then 0 , with discontinuity at exactly one point. This point is called the upper Beurling dimension of Λ.

Hausdorff meets Beurling

Theorem

Let μ_B be the invariant measure for an affine IFS, with no overlap. Suppose Λ is a frame spectrum for μ_B, and Λ is "not too sparse". Then the Beurling dimension of Λ is equal to the Hausdorff dimension of the attractor $X_B (= \mathrm{supp}(\mu))$.