Algorithmic problems in the research of number expansions

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Notations I.

- Lattice Λ in $\mathbb{R}^n$
- $M : \Lambda \to \Lambda$ such that $\det(M) \neq 0$
- $0 \in D \subseteq \Lambda$ a finite subset

Definition The triple (Λ, M, D) is called a *number system* (GNS) if every element x of Λ has a unique, finite representation of the form $x = \sum_{i=0}^{l} M^i d_i$, where $d_i \in D$ and $l \in \mathbb{N}$.

Notations II.

- Similarity preserves the number system property, i.e, if M_1 and M_2 are similar via the matrix Q and (Λ, M_1, D) is a number system then $(Q\Lambda, M_2, QD)$ is a number system as well.
- No loss of generality in assuming that M is integral acting on the lattice $\mathbb{Z}^n$.
- If two elements of Λ are in the same coset of the factor group $\Lambda / M\Lambda$ then they are said to be congruent modulo M.

Notations III.

Theorem 1[1] If (Λ, M, D) is a number system then

1. D must be a full residue system modulo M,
2. M must be expansive,
3. $\det(I - M) \neq \pm 1$.

If a system fulfills these conditions it is called a *radix system*.

Notations IV.

- Let $\phi : \Lambda \to \Lambda$, $x \stackrel{\phi}{\mapsto} M^{-1}(x - d)$ for the unique $d \in D$ satisfying $x \equiv d \pmod{M}$.
- Since M^{-1} is contractive and D is finite, there exists a norm on Λ and a constant C such that the orbit of every $x \in \Lambda$ eventually enters the finite set $S = \{p \in \Lambda \mid \|x\| < C\}$ for the repeated application of ϕ.
- This means that the sequence $x, \phi(x), \phi^2(x), \ldots$ is eventually periodic for all $x \in \Lambda$.

Notations V.

- (Λ, M, D) is a GNS iff for every $x \in \Lambda$ the orbit of x eventually reaches 0.
- A point x is called periodic if $\phi^k(x) = x$ for some $k > 0$.
- The orbit of a periodic point is called a *cycle*.
- The decision problem for (Λ, M, D) asks if they form a GNS or not.
- The classification problem means finding all cycles.

Content

- How to decide expansivity?
- How to generate expansive operators?
- How to decide the number system property?
- Case study: generalized binary number systems.
- How to classify the expansions?
- How to construct number systems?

Expansivity I.

$\Lambda = \mathbb{Z}^n$. Given operator M examine $P =$charpoly(M).

- A polynomial is said to be *stable* if
  1. all its roots lie in the open left half-plane, or
  2. all its roots lie in the open unit disk.

  The first condition defines Hurwitz stability and the second one Schur stability.
- There is a bilinear mapping between these criterions (Möbius map).

Expansivity II.

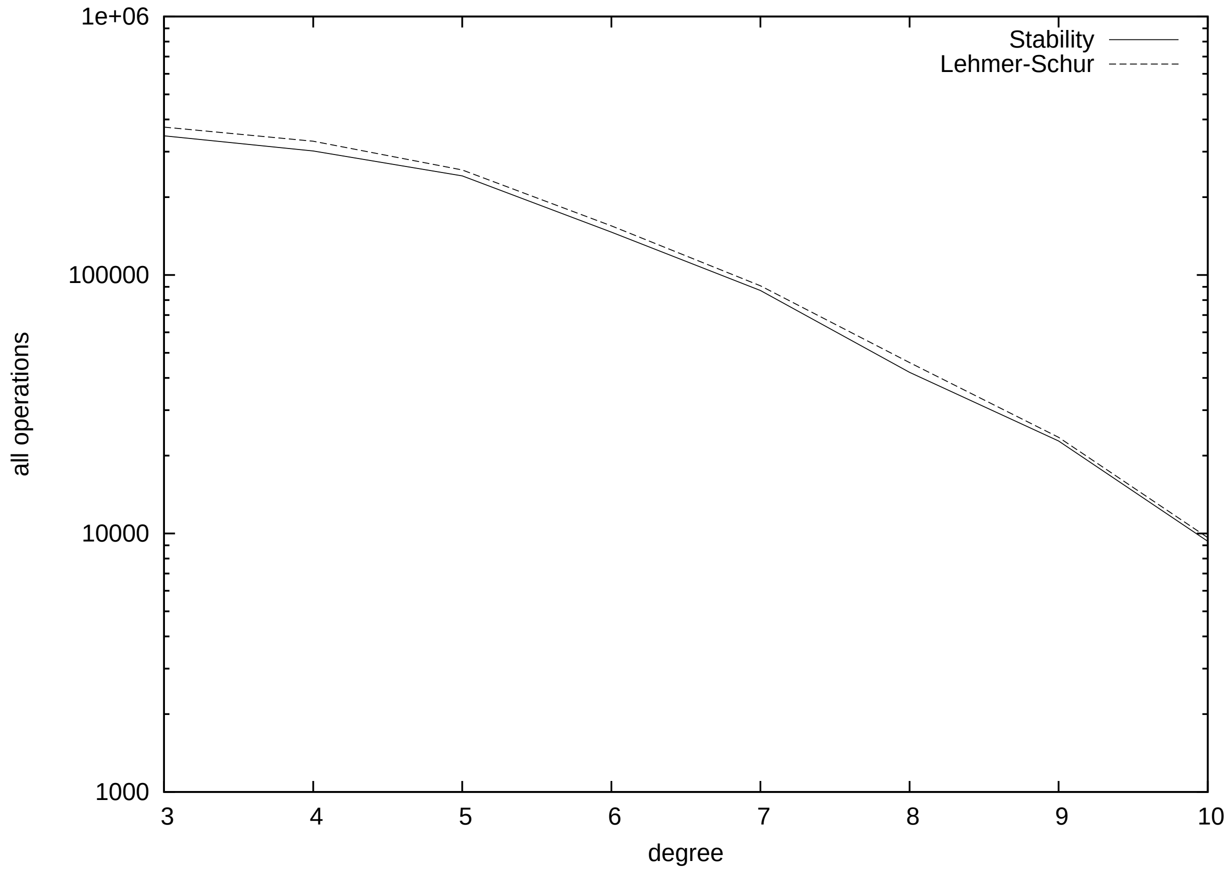
- Schur stability: Algorithm of Lehmer-Schur.
- Hurwitz stability: An n-terminating continued fraction algorithm of Hurwitz.

Results:

- For arbitrary polinomials Lehmer-Schur is faster.
- For stable polynomials Hurwitz-method is faster.
- Caution: Intermediate expression swell may occur.

Expansivity III.

Comparision of the methods for stable polynomials.



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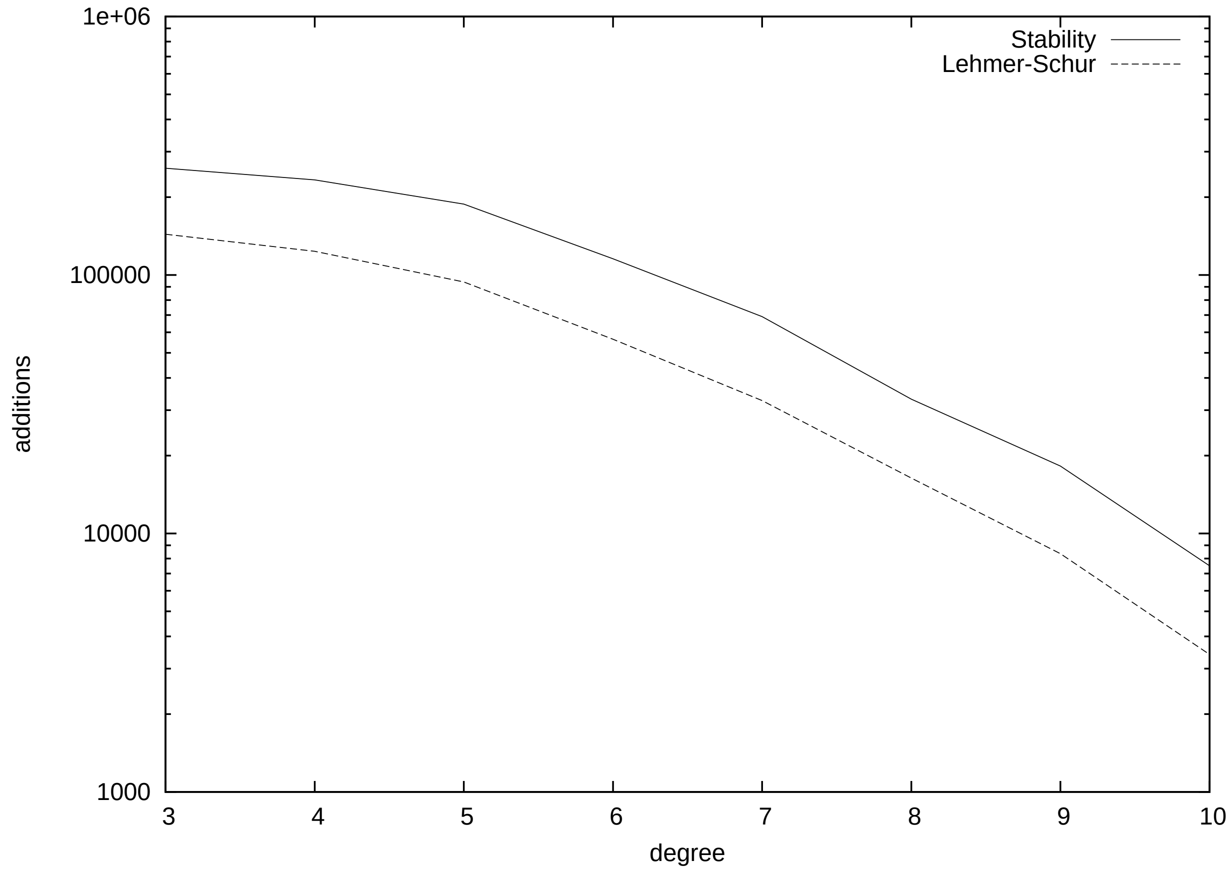

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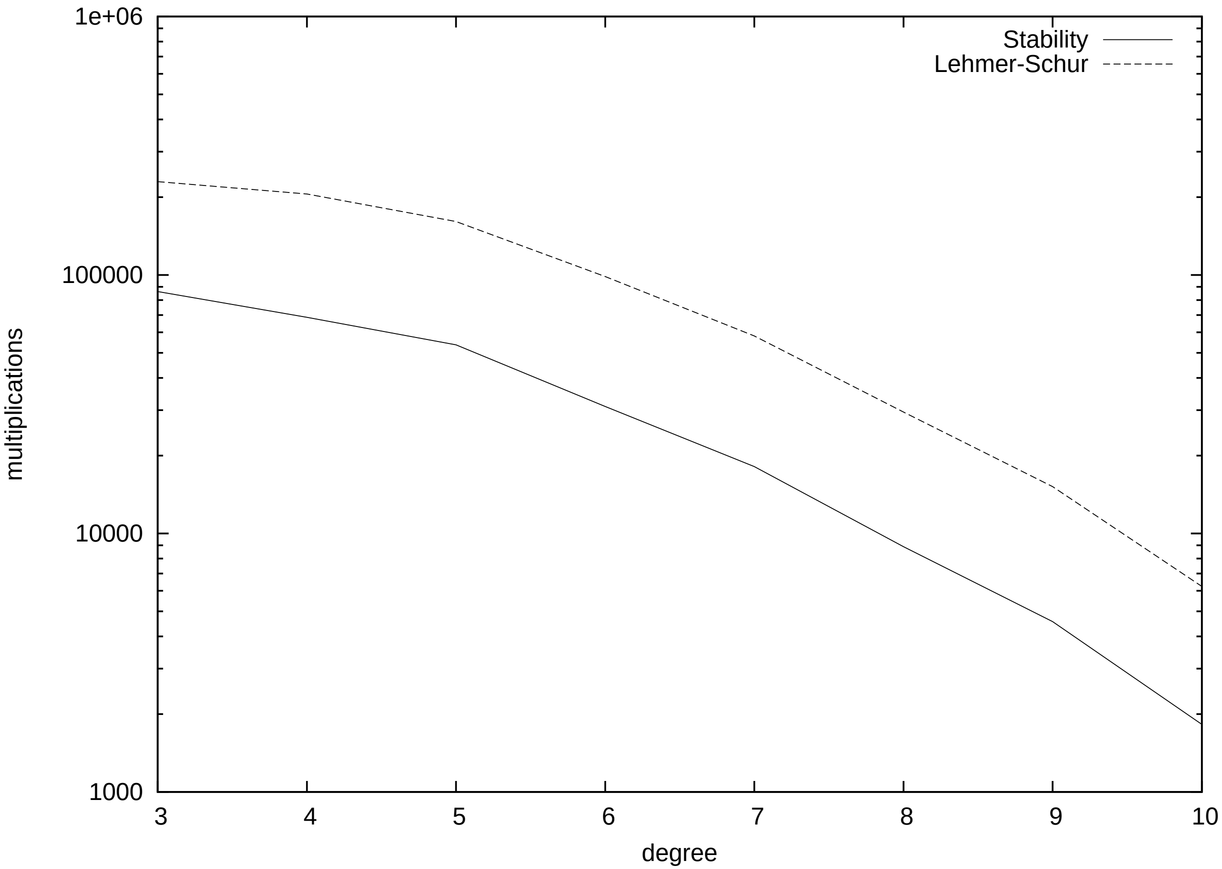


Expansivity IV.

Hurwitz-method works also for symbolic coeffs.
Let $a(x) = a_0 + a_1 x + a_2 x^2 + x^3 \in \mathbb{Z}[x]$.
Hurwitz-method gives that $a(x)$ is expansive if

$$\frac{3a_0 - a_1 - a_2 + 3}{a_0 - a_1 + a_2 - 1}, \frac{a_0 + a_1 + a_2 + 1}{3a_0 - a_1 - a_2 + 3}$$

$$\frac{8(a_0^2 - a_0 a_2 + a_1 - 1)}{(a_0 - a_1 + a_2 - 1)(3a_0 - a_1 - a_2 + 3)},$$

are all positive.

For the details (with Maple code) see [2].

Expansivity V.

How to generate expansive integer polynomials with given degree and constant term?

- Using Las Vegas type randomized algorithm, which produces an expansive polynomial in $\mathbb{R}[x]$, then makes round.
- Using the algorithm of Dufresnoy and Pisot [3], which works well for small constant term.

Expansivity VI.

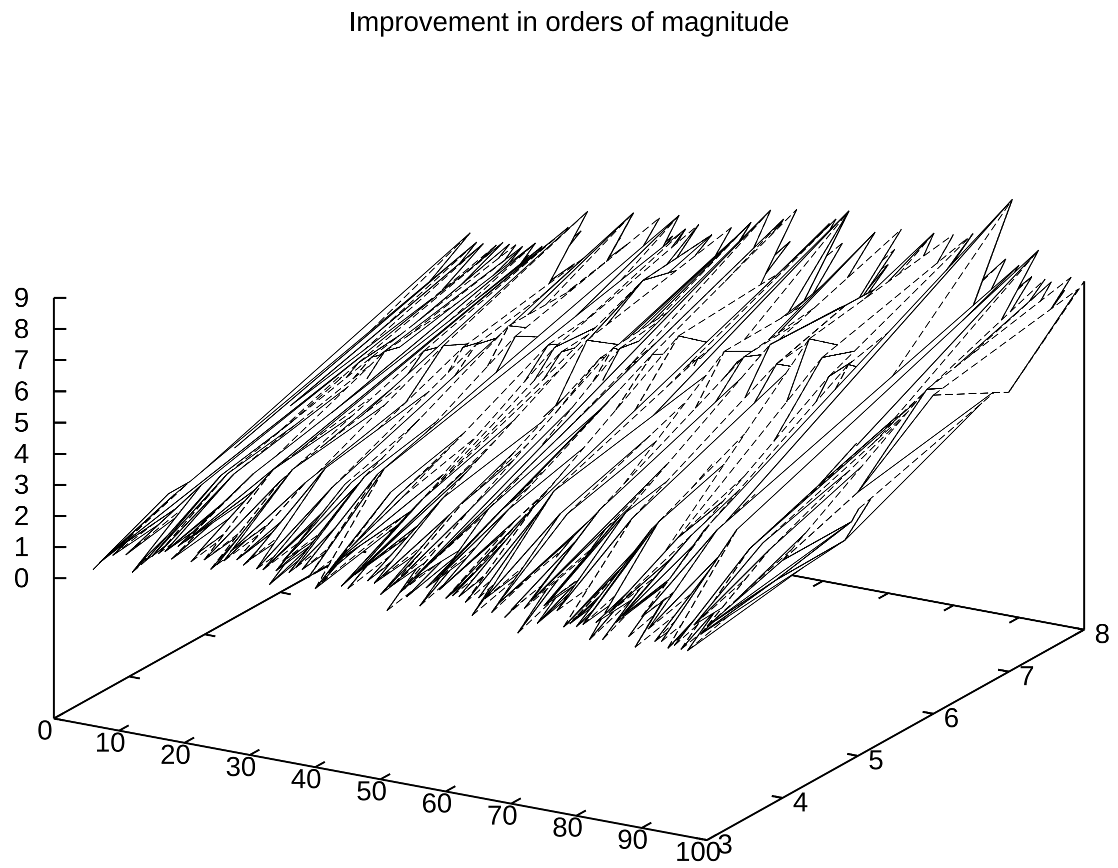
- Generating random expansive matrices seems difficult.
- One can apply an integer basis transformation to the companion matrix of a polynomial.
- This method generates all expansive matrices only if the class number of the order corresponding to the polynomial is 1.

GNS Decision I.

- The original method uses a covering of the set of fractions H (all periodic points lie in the set $-H$). Since H is compact, it gives lower and upper bounds on the coordinates of periodic points [4].
- It can be combined with a basis transformation using a simulated annealing type randomized algorithm in order to improve the bounds [5].

GNS Decision II.

The average improvement in the volume of the covering set expressed in orders of magnitude.



GNS Decision III.

- Brunotte's canonical number system decision algorithm [6] can be extended (M is the companion of the monic, integer polynomial, $D = \{(i, 0, 0, \ldots 0)^T \mid 0 \leq i < |\det M|\}$).

Function CONSTRUCT-SET-E(M, D)

```
E ← D , E′ ← ∅ ;
while E ≠ E′ do
    E′ ← E;
    forall e ∈ E and d ∈ D do
        put φ(e + d) into E;
    end
end
return E;
```

GNS Decision IV.

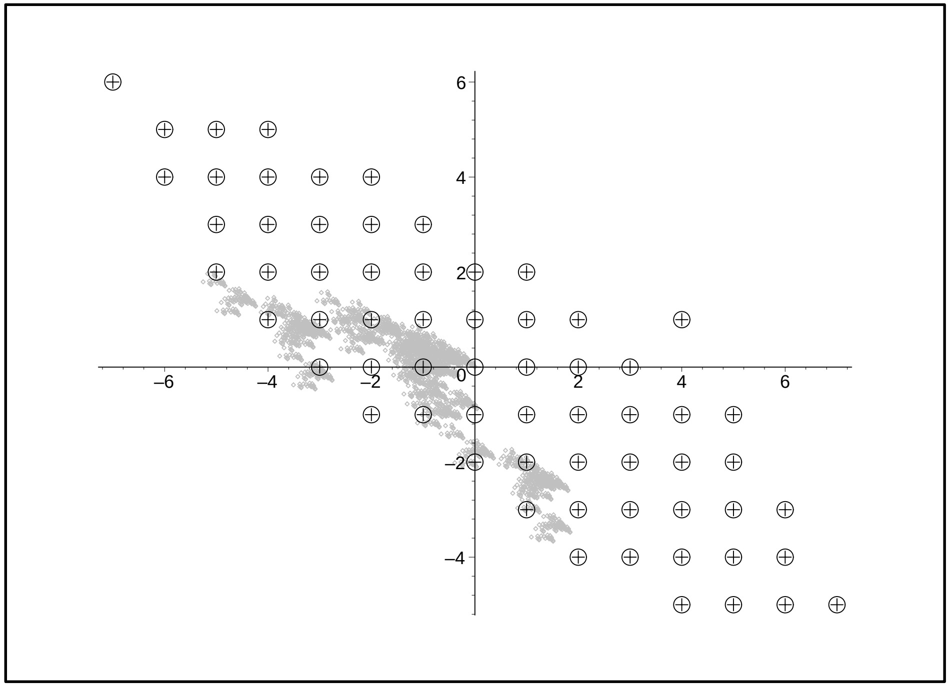
The previous algorithm terminates. Denote $B = \{(0, 0, \ldots, 0, \pm 1, 0, \ldots, 0\}$ the n basis vectors and their opposites.

Function SIMPLE-DECIDE(M, D)

$E \leftarrow$ CONSTRUCT-SET-E(M, D);
forall $p \in B \cup E$ **do**
    if p *has no finite expansion* **then**
        return false ;
end
return true;

GNS Decision V.

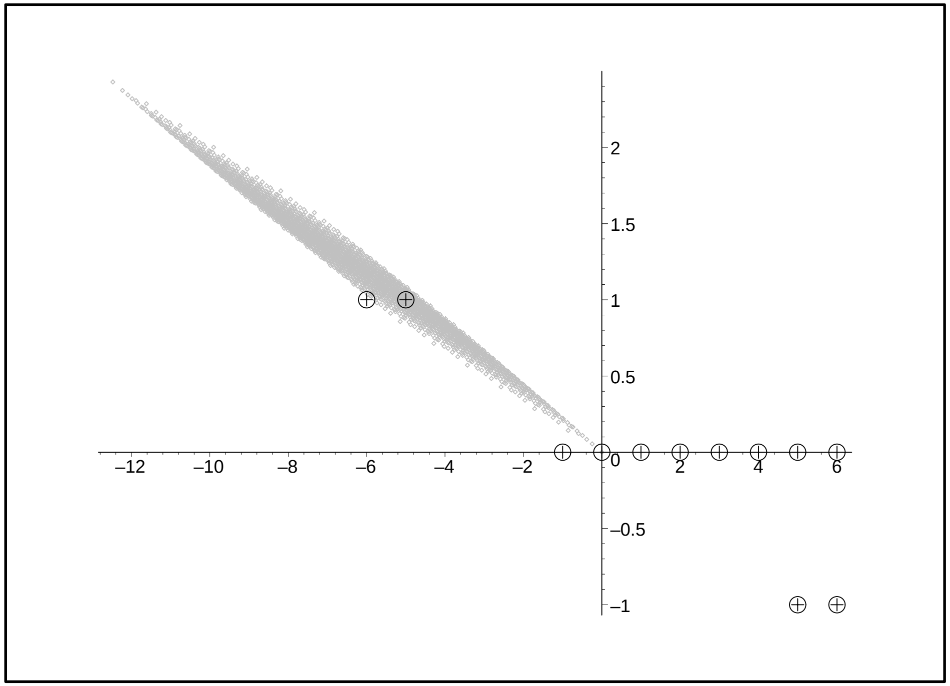
$M = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}, D = \{(0,0),(1,0),(0,1),(4,1),(-7,6)\}.$



Changing the basis to $\{(1,0),(-1,1)\}$ decreases the volume from 42 to 24. $|E| = 65$.

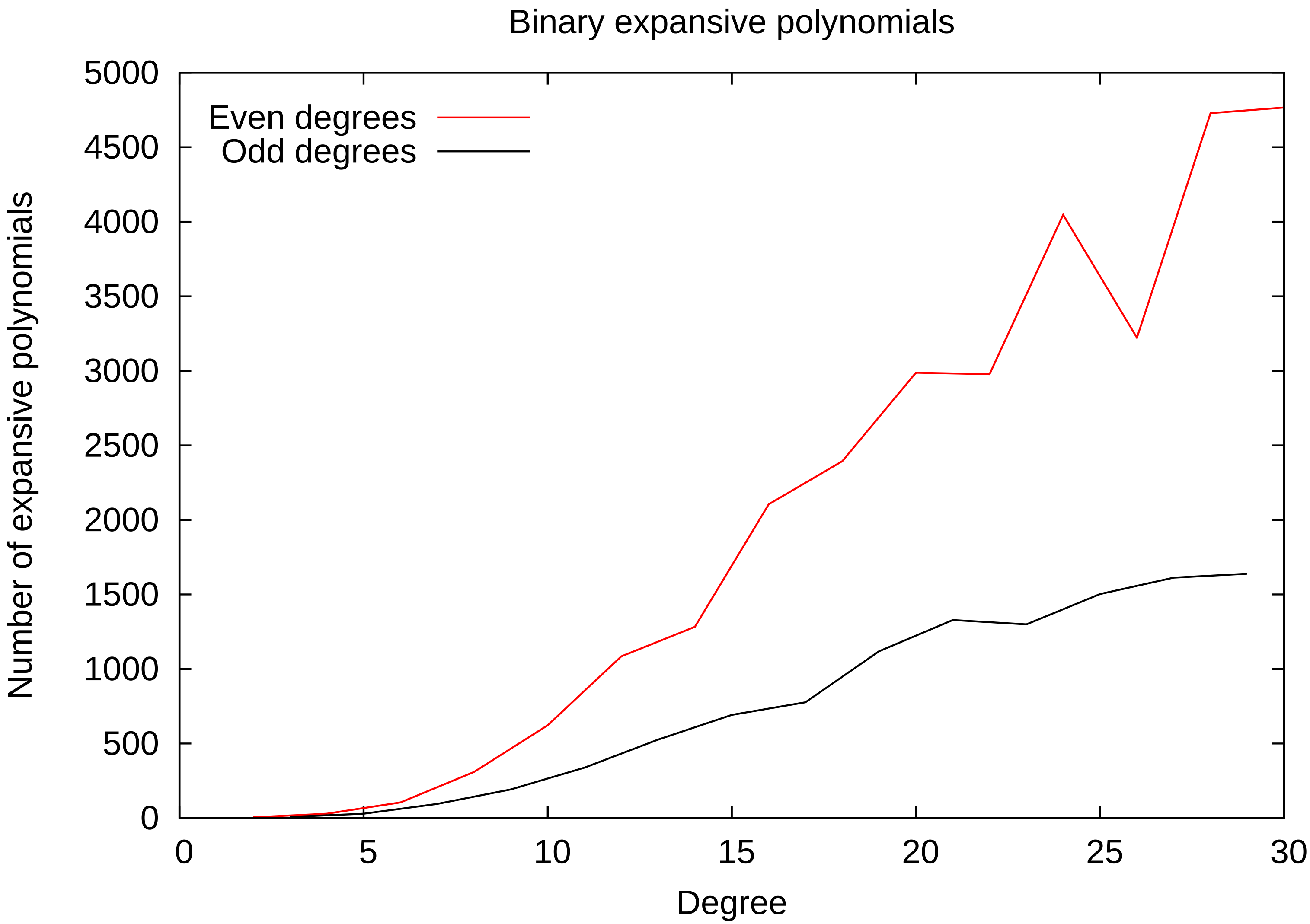
GNS Decision VI.

$M = \begin{pmatrix} 0 & -7 \\ 1 & 6 \end{pmatrix}$, D is canonical.



Replacing the basis vector $(0, 1)$ with $(-5, 1)$ gives volume 4 instead of 64. $|E| = 12$.

Binary Case I.



Binary Case II.

Degree	2	3	4	5	6	7	8	9	10	11
Expansive	5	7	29	29	105	95	309	192	623	339
CNS	4	4	12	7	25	12	20	12	42	11

Problems: in higher dimensions the volume of the covering set or the set E are sometimes too big. The largest E encountered is of size $21\,223\,091$, for $2+3x+3x^2+3x^3+3x^4+3x^5+3x^6+3x^7+3x^8+2x^9+x^{10}$. The number of points in the covering set of this sapmle is $226\,508\,480\,352\,000$.

GNS Classification I.

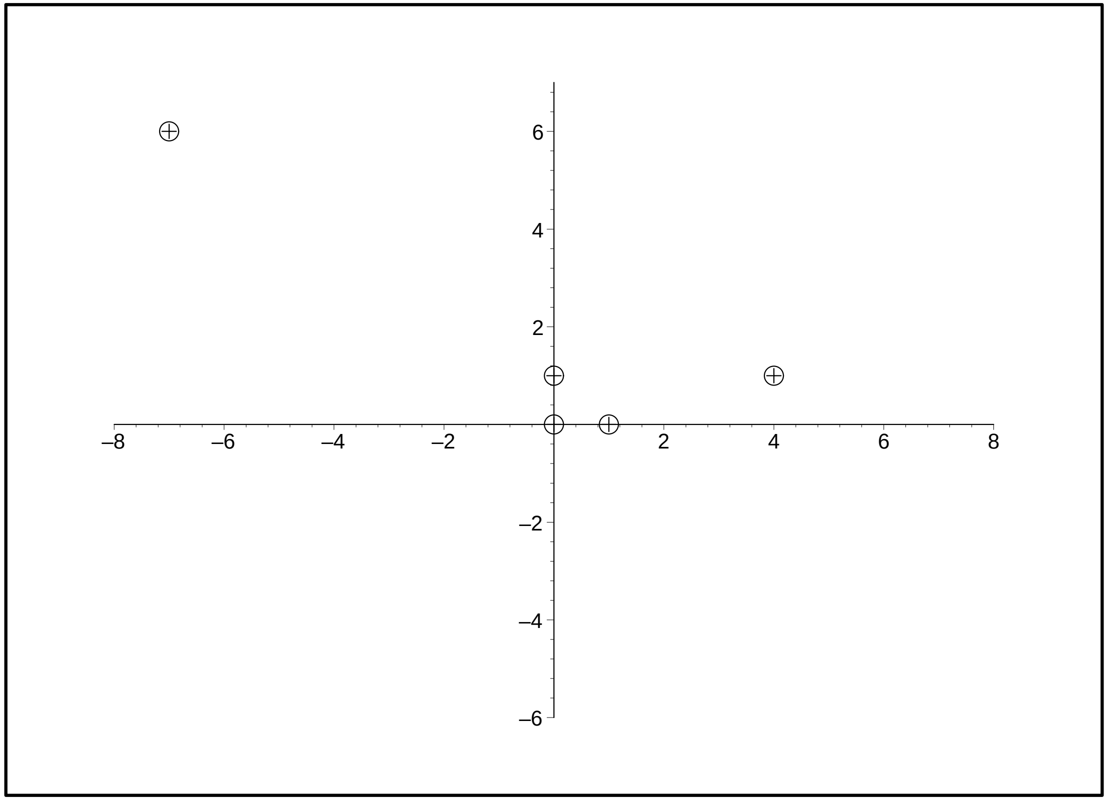
Two methods: covering and simple classify.

Function SIMPLE-CLASSIFY(M, D)

$\mathcal{D} \leftarrow D$;
finished $\leftarrow$ false;
while *not finished* **do**
  $\mathcal{E} \leftarrow$ CONSTRUCT-SET-E$(M, \mathcal{D})$;
  finished $\leftarrow$ true;
  forall $p \in \mathcal{E} \cup B$ **do**
    if p *does not run eventually into* $\mathcal{D}$ **then**
      put newly found periodic points into $\mathcal{D}$;
      finished $\leftarrow$ false;
  end
end
return $\mathcal{D} \setminus D$ (the set of non-zero periodic points);

SIMPLE-CLASSIFY

$M = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}, D = \{(0,0),(1,0),(0,1),(4,1),(-7,6)\}.$



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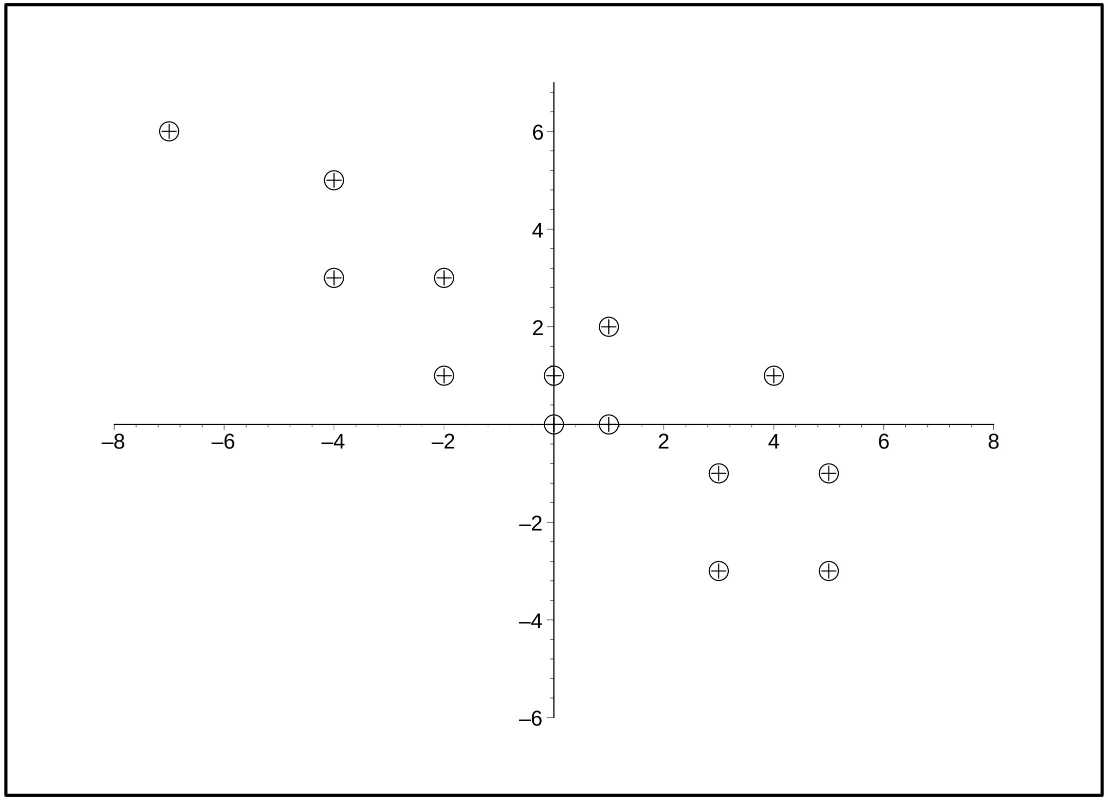

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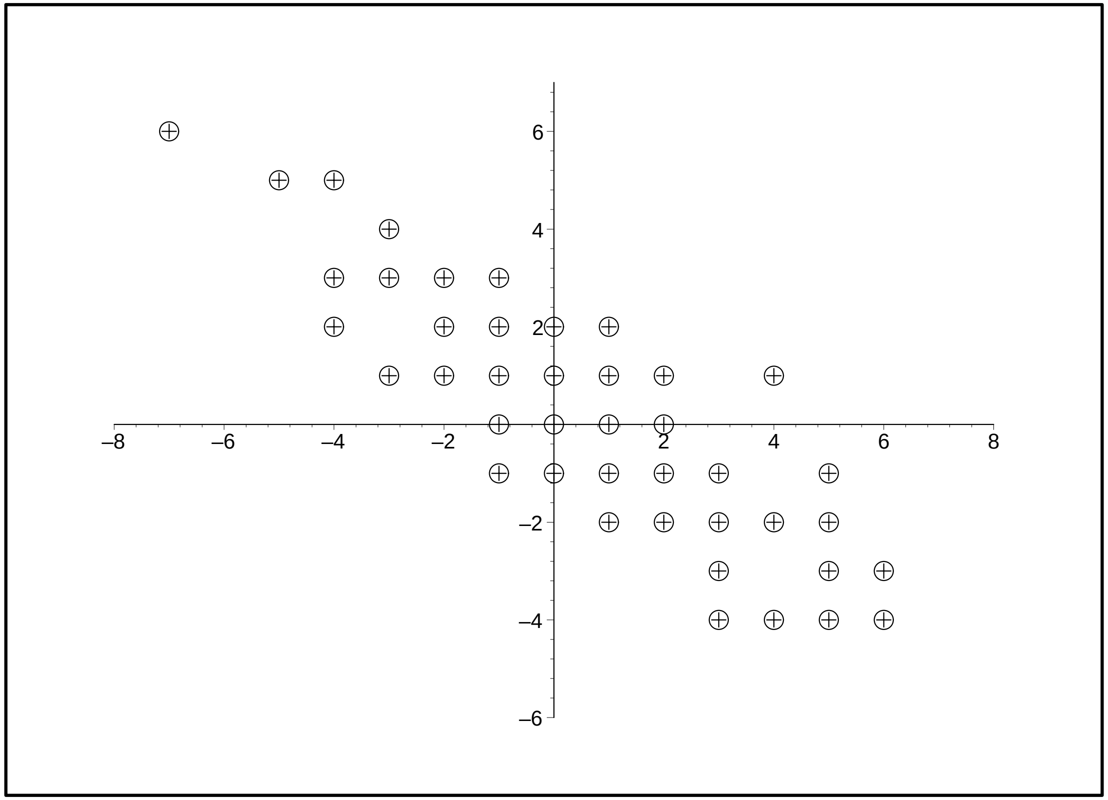

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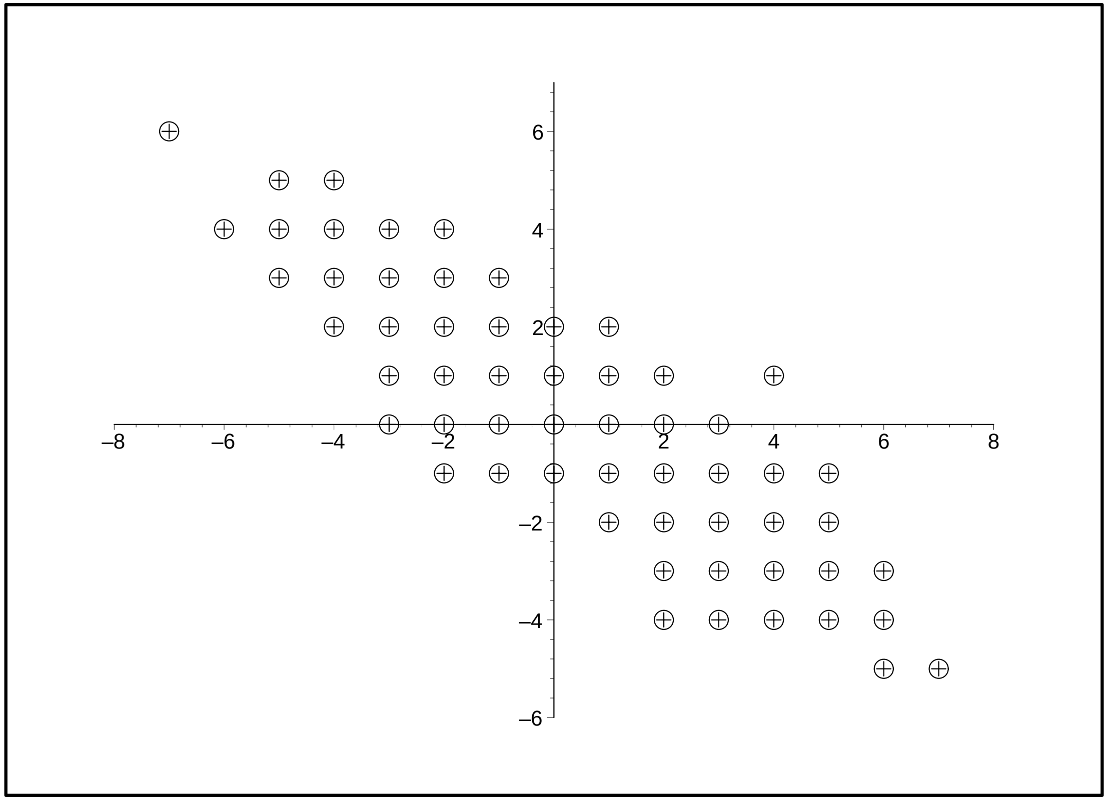

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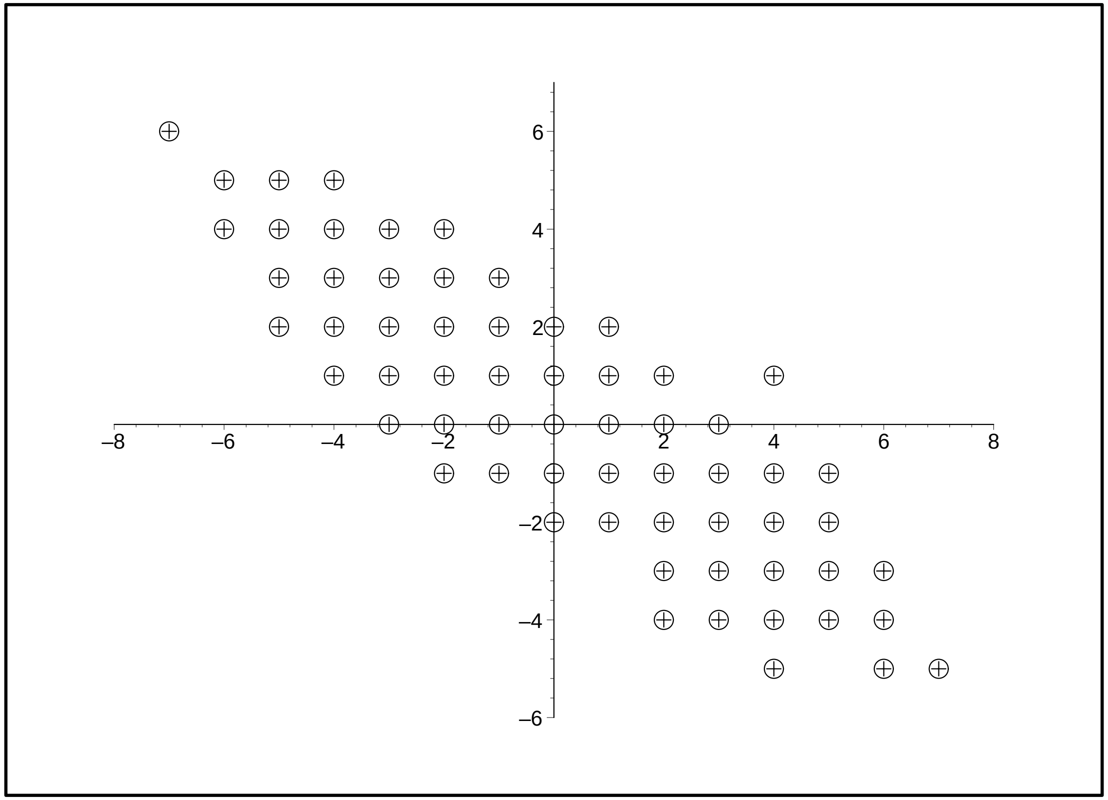

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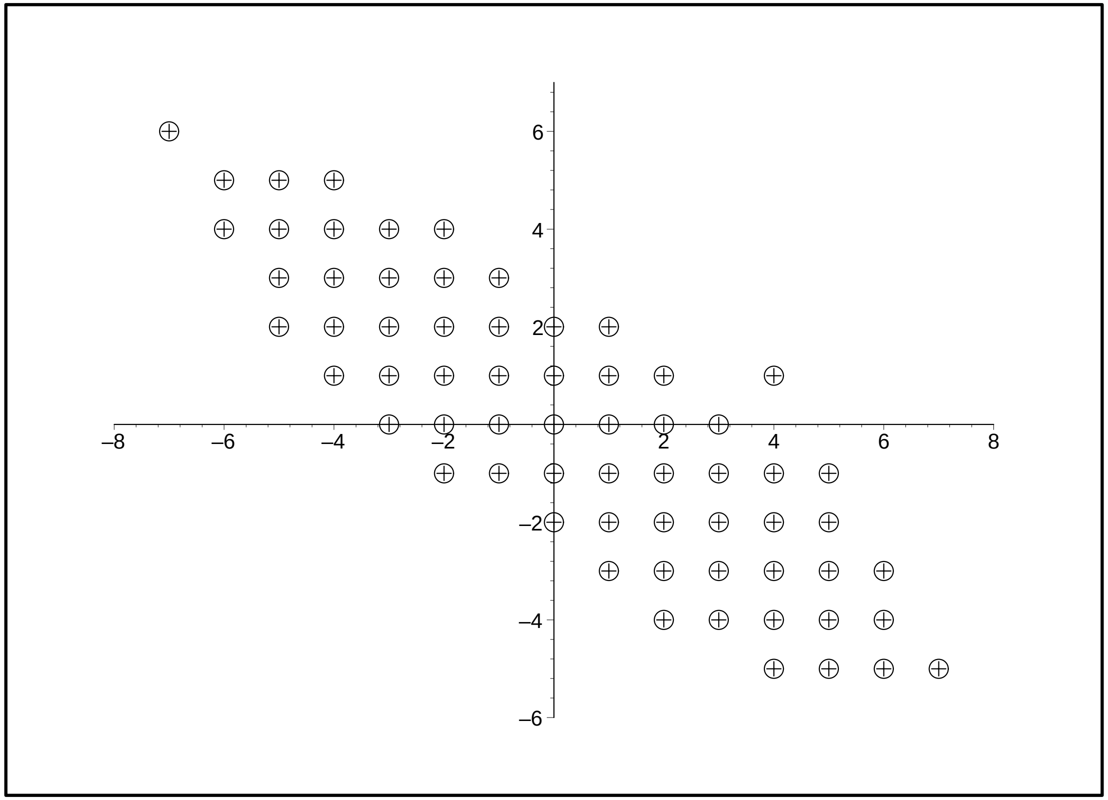

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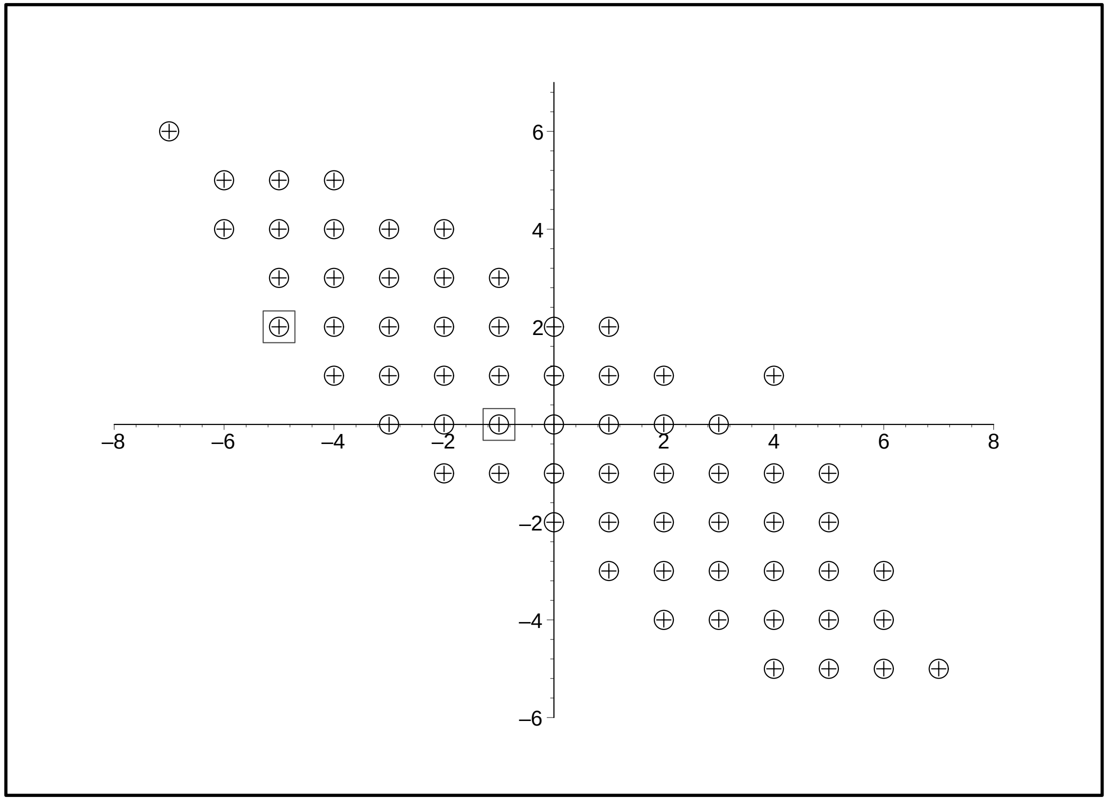

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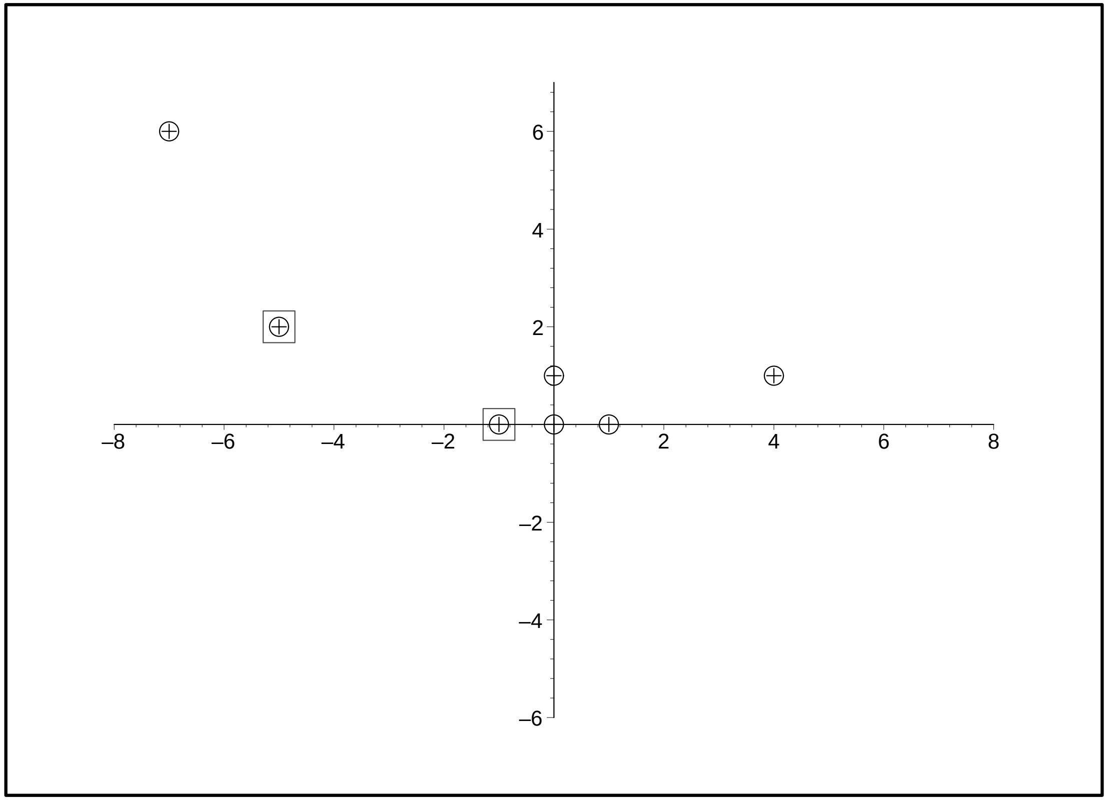

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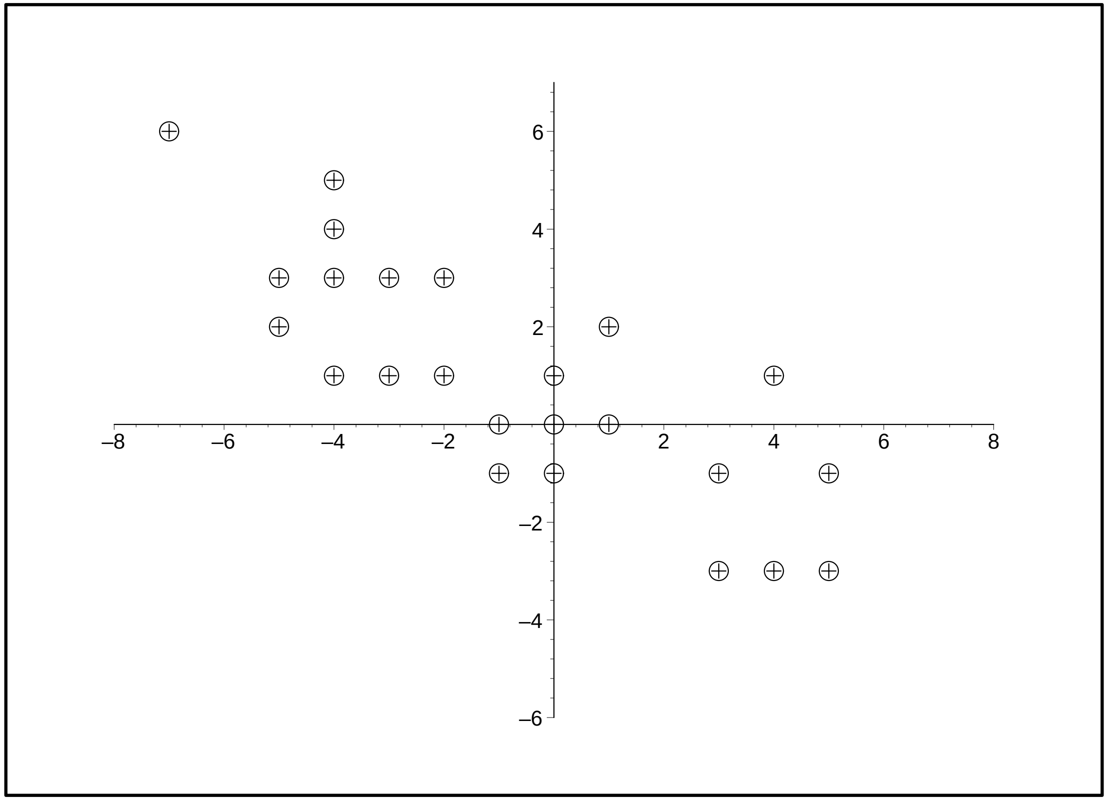

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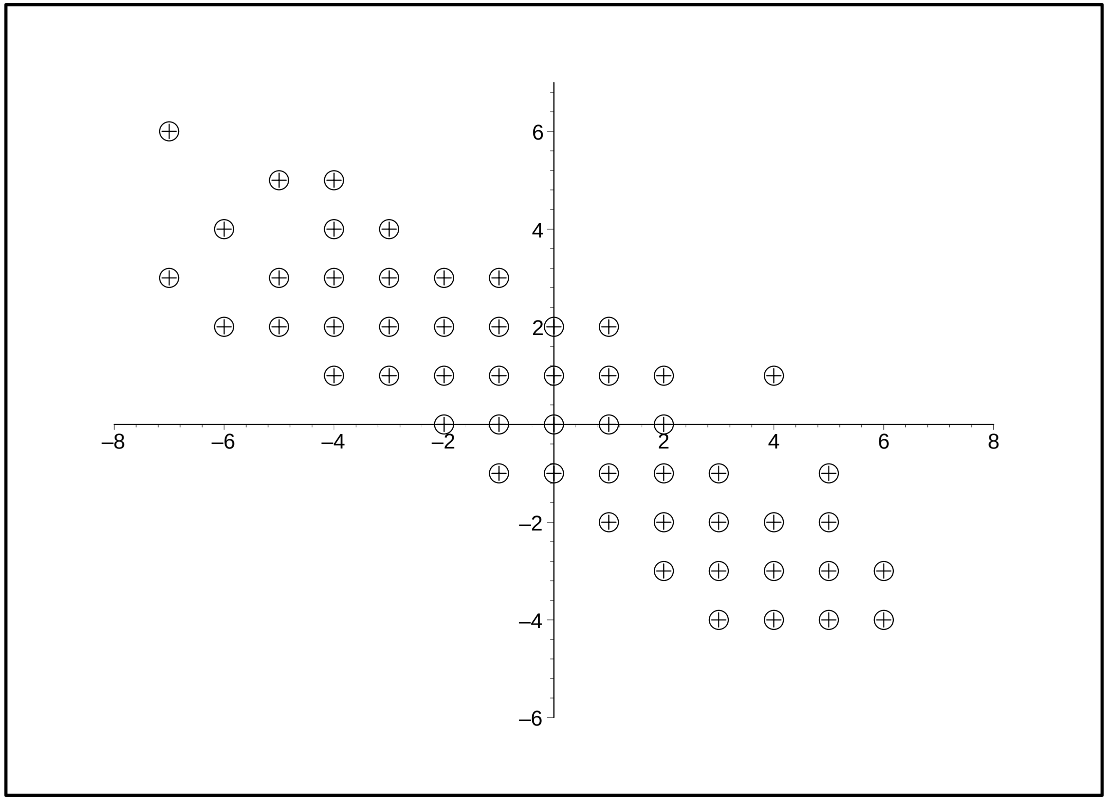

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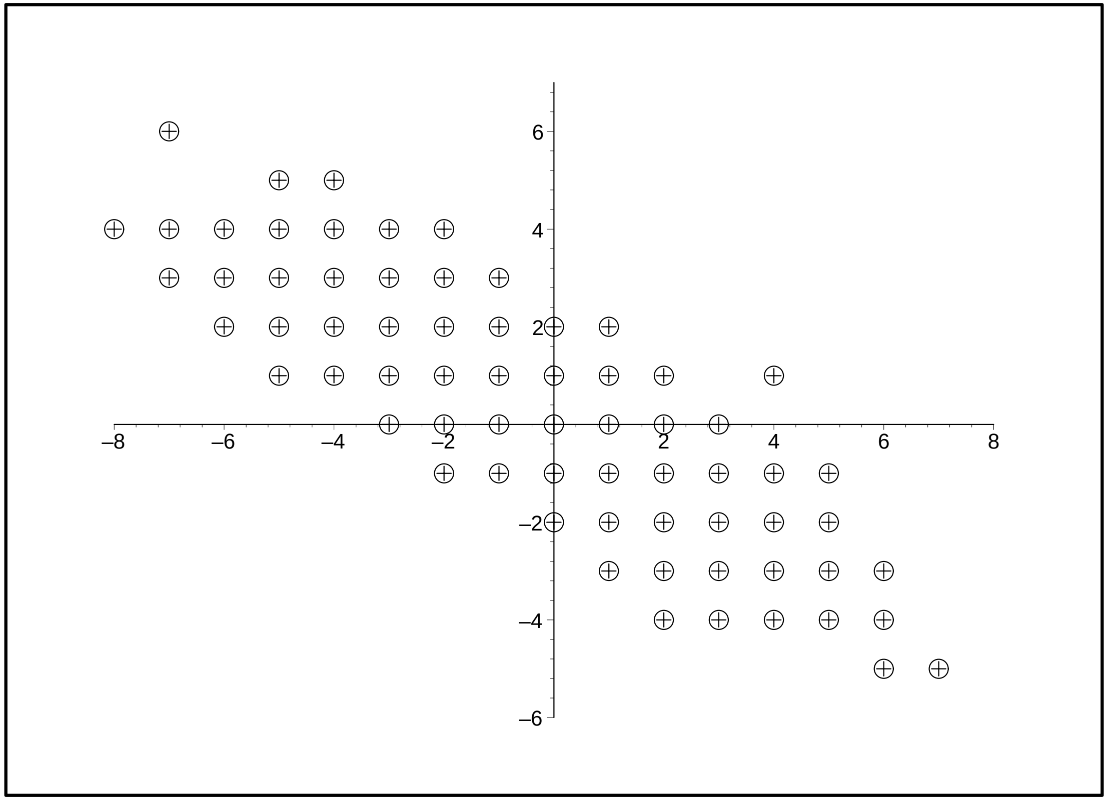

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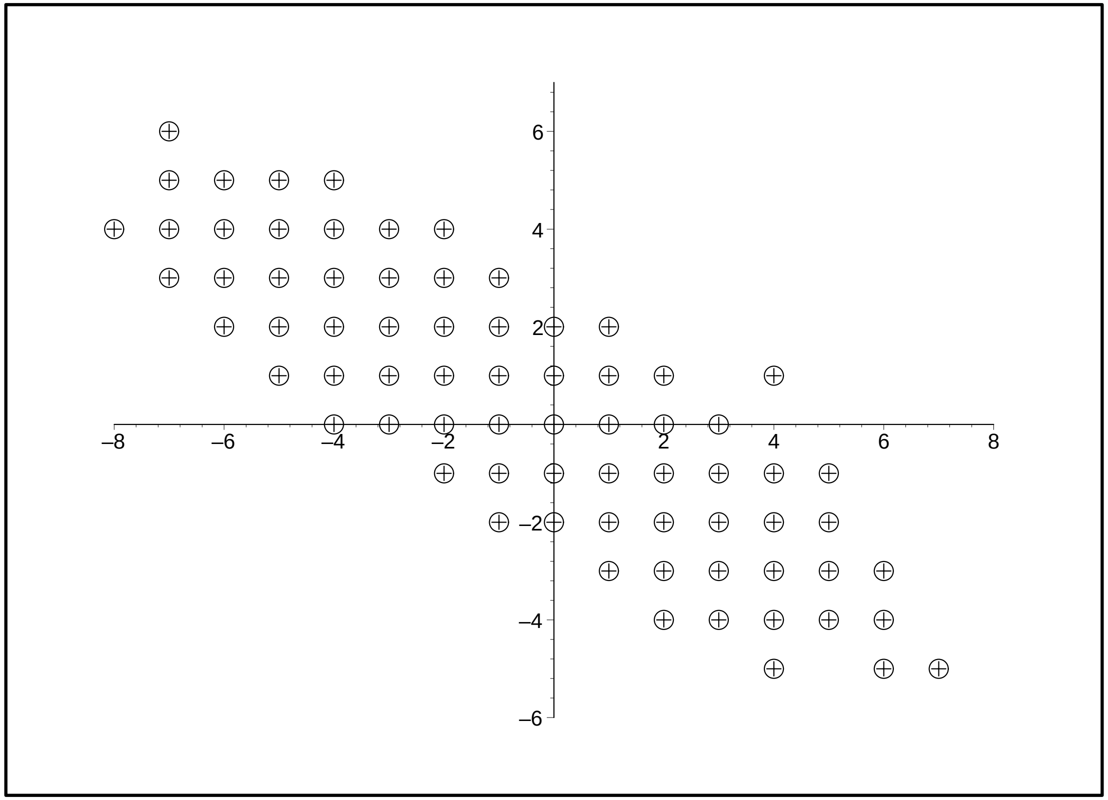

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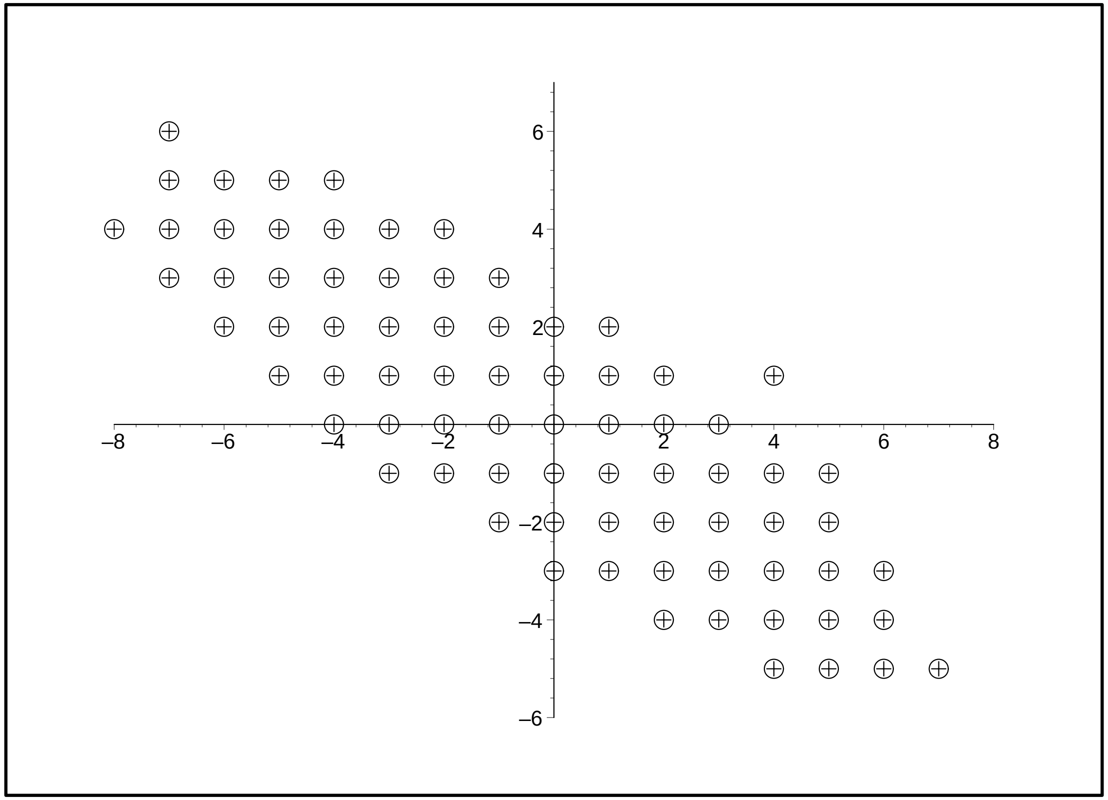

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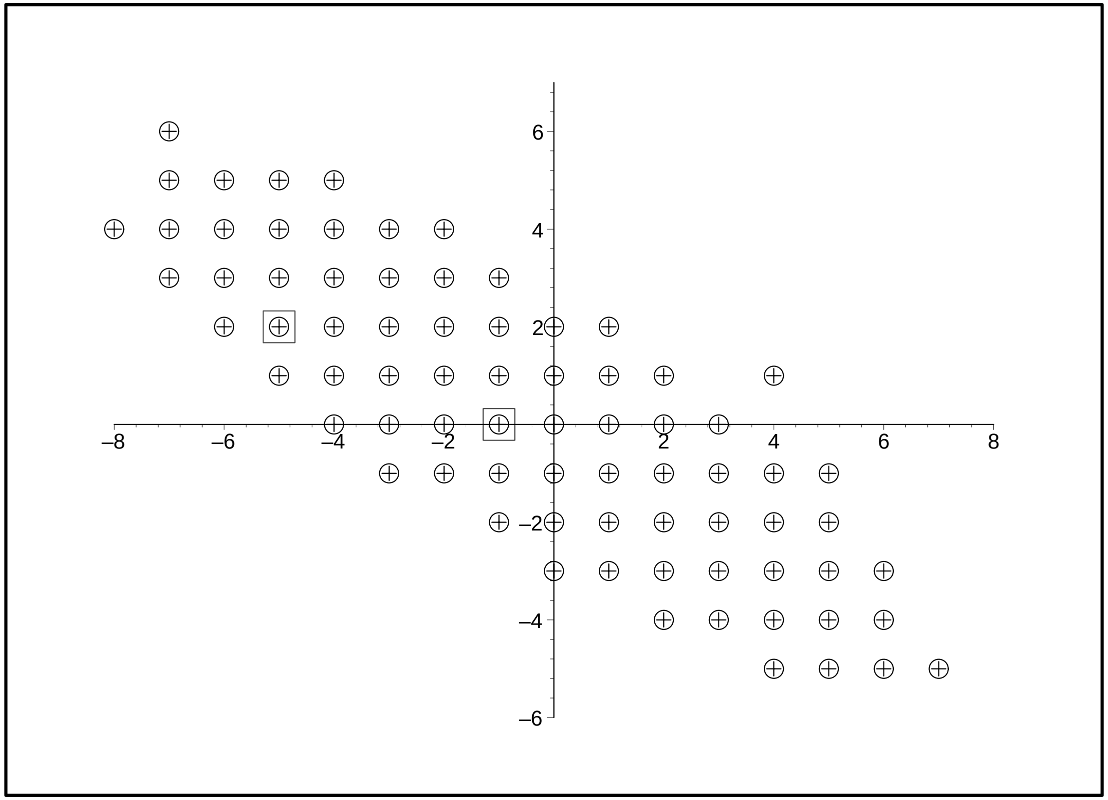

GNS Classification II.

Comparing covering and simple classify:

- Covering is parallelizable.
- Both give negative answers fast.
- Either can beat the other in some cases.
- Experiments show that the algorithmic complexity of the worst case is exponential.

GNS Construction I.

- Given lattice Λ and operator M satisfying criteria 2) and 3) in Theorem 1 is there any suitable digit set D for which (Λ, M, D) is a number system?
- If yes, how many and how to construct them?

GNS Construction II.

Theorem (Kátai) Let Λ be the set of algebraic integers in an imaginary quadratic field and let $\alpha \in \Lambda$. Then there exists a suitable digit set D by which (Λ, α, D) is a number system if and only if $|\alpha| > 1$, $|1 - \alpha| > 1$ hold.

Theorem [8] Let Λ be the set of algebraic integers in the real quadratic field $\mathbb{Q}(\sqrt{2})$ and let $0 \neq \alpha \in \Lambda$. If $\alpha, 1 \pm \alpha$ are not units and $|\alpha|, |\overline{\alpha}| > \sqrt{2}$ then there exists a suitable digit set D by which (Λ, α, D) is a number system.

GNS Construction III.

Theorem [9] For a given matrix M if $\rho(M^{-1}) < 1/2$ then there exists a digit set D for which (Λ, M, D) is a number system.

Theorem [9] Let the polynomial $c_0 + c_1 x + \cdots + x^n \in \mathbb{Z}[x]$ be given and let us denote its companion matrix by M. If the condition $|c_0| > 2\sum_{i=1}^{n} |c_i|$ holds then there exists a suitable digit set D for which $(\mathbb{Z}^n, M, D)$ is a number system.

References

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Thank you!