Mini-Workshop

Graz University of Technology

20th September 2013

08:50-09:00	Opening	
09:00-10:00	Oliver Riordan (University of Oxford)	The evolution of Achlioptas processes
	Coffee break	
10:20-10:50	Christoph Koch (Graz University of Techology)	Size of the largest component in a multi-type generalization of Erdős-Rényi random graphs
11:00-12:00	Oliver Cooley (Graz University of Technology)	The Giant Component in Random Hypergraphs
	Lunch	
13:30-14:00	Matan Harel (Courant Institute)	Localization in Random Geometric Graphs with Too Many Edges
14:10-14:40	Marcin Witkowski (Adam Mickiewicz University)	Random Lifts of Graphs
	Coffee break	
15:00-16:00	Charilaos Efthymiou (Goethe University Frankfurt)	MCMC sampling colourings and independent sets of $G(n, d/n)$ near the uniqueness threshold.

Abstracts

The evolution of Achlioptas processes

Oliver Riordan (University of Oxford)

In the *Erdős–Rényi random graph process*, starting from an empty graph, in each step a new random edge is added to the evolving graph. One of its most interesting features, both mathematically and in terms of applications, is the 'percolation phase transition': as the ratio of the number of edges to vertices increases past a certain critical point, the global structure changes radically, from only small components to a single macroscopic ('giant') component plus small ones.

An *Achlioptas process* is any one of a family of variants of the classical Erdős–Rényi process: starting from an empty graph, in each step *two* potential edges are chosen uniformly at random, and using some rule *one* of them is selected and added to the evolving graph. One might expect these processes to behave not too differently from the Erdős–Rényi one. However, simulations of Achlioptas, D'Souza and Spencer suggested that for certain rules (in particular the 'product rule' suggested by Bollobás) the percolation phase transition has a radically different form: more or less as soon as the macroscopic component appears, it is already extremely large. This phenomenon is known as 'explosive percolation'. I will briefly explain this striking and unusual phenomenon, and discuss results for general classes of Achlioptas processes obtained in joint work with Lutz Warnke.

Size of the largest component in a multi-type generalization of Erdős-Rényi random graphs

Christoph Koch (Graz University of Technology)

Galton-Watson branching processes are a very efficient tool to establish the size of the giant component in the supercritical phase of Erdős-Rényi random graphs. Recently Béla Bollobás and Oliver Riordan showed that a similar approach works also in the weakly supercritical region providing a new short proof of the size of the largest component. In the talk we will discuss how this approach can be adapted to a generalized random graph model containing different types of vertices. This involves a multi-type branching process, a notion of a dual branching process, and the width of a rooted tree associated with a branching process as well as the second moment method. This is joint work with Mihyun Kang and Angélica Pachón.

The Giant Component in Random Hypergraphs

Oliver Cooley (Graz University of Technology)

The fact that the existence of a giant component in the random graph $G(n, p)$ exhibits a phase transition at $p \sim 1/n$ was proved (with slightly different terminology) in the seminal paper of Erdős and Rényi in 1960, the paper in which the concept of random graphs was first introduced. And yet the analogous result for hypergraphs was until now known only for a special case.

More precisely for $1 \leq j \leq k-1$, two j-sets J_1, J_2 of vertices in a k-uniform hypergraph are j-connected if there is a sequence $E_1, \ldots, E_\ell$ of edges such that $J_1 \subset E_1$, $J_2 \subset E_\ell$ and $E_i \cap E_{i+1} \geq j$. A *giant* j-connected component consists of $\Theta(n^j)$ j-sets. The phase transition threshold for $j = 1$ and general k was determined by Schmidt-Pruzan and Shamir in 1985.

Recently Krivelevich and Sudakov gave a new and simple proof of the classical Erdős-Rényi result ($j = 1, k = 2$). In this talk we show how this new proof can, with some additional work, be generalised to determine the phase transition thresholds for all $1 \leq j \leq k-1$.

This is joint work with Mihyun Kang and Yury Person.

Localization in Random Geometric Graphs with Too Many Edges

Matan Harel (Courant Institute)

Consider a random geometric graph $G(\chi_n, r(n))$, given by a connecting two vertices of a Poisson Point Process of intensity n on the unit torus whenever their distance is smaller than the parameter $r(n)$. This model is conditioned on the rare event that the number of edges observed, $|E|$, is greater than the $[1+\delta(n)]\mathbb{E}[|E|]$, producing a three parameter model. We show that, for δ constant or vanishing sufficiently slowly in n, with high probability, there exists a "giant clique" with almost all of the $\delta\mathbb{E}[|E|]$ excess edges. Furthermore, if δ vanishes sufficiently quickly, the largest clique will be, at most, a constant multiple of the usual clique number of the unconditioned random geometric graph; roughly, all excess edges will result from the entropy-like effects of vanishing Radon-Nikodyn derivatives. Finally, we discuss progress in finding a phase transition function $\delta_0(n)$ so that when $\delta \gg \delta_0$, the giant clique scenario holds, while $\delta \ll \delta_0$ implies the entropy scenario.

Random Lifts of Graphs

Marcin Witkowski (Adam Mickiewicz University)

In the talk we present several results on random lifts of graphs, a random graph model defined by Amit and Linial [1].

For graphs G and H, a map $\pi : V(H) \to V(G)$ is a covering map from H to G if for every $v \in V(H)$ the restriction of π to the neighborhood of v is a bijection onto the neighborhood of $\pi(v) \in V(G)$. If such a mapping exists, we say that H is a lift of G. Moreover, if the number of vertices of H mapped to a vertex v of the base graph G is n for all vertices $v \in G$, we say that H is an n-lift of G. The set of all n-lifts of G we denote as $L_n(G)$.

The random n-lift of a graph G is obtained by choosing uniformly at random one graph from the set $L_n(G)$. Equivalently, the random n-lift of G can be generated by choosing independently and uniformly at random for every edge $\{u, v\} \in E(G)$ a perfect matching between the set of n vertices that are mapped to u and the set of n vertices mapped to v.

In this talk we briefly survey some basic properties of random lifts and its applications from [1, 2, 3, 4, 5, 6]. In particular, we show how lifts can be used to generate expanders and present a closer insight into connectivity and Hamiltonicity of random lifts.

References

[1] A. Amit and N. Linial, *Random graph coverings I: General theory and graph connectivity*, Combinatorica, **22** (2002), 1–18.

[2] A. Amit and N. Linial, *Random lifts of graphs: edge expansion*, Combinatorics, Probability & Computing, **15** (2006), 317–332.

[3] K. Burgin, P. Chebolu, C. Cooper, and A.M. Frieze, *Hamilton cycles in random lifts of graphs*, European Journal of Combinatorics, **27** (2006), 1282–1293.

[4] A. Marcus and D. Spielman and N.Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all Degrees*, ArXiv e-prints, (2013).

[5] M. Witkowski, *Random lifts of graphs are highly connected*, Electronic Journal of Combinatorics **20(2)** (2013).

[6] T. Łuczak and Ł. Witkowski and M. Witkowski, *Hamilton cycles in random lifts of graphs*, ArXiv e-prints, (2013).

Title: MCMC sampling colourings and independent sets of G(n,d/n) near the uniqueness threshold

Charilaos Efthymiou (Goethe University Frankfurt)

Sampling from Gibbs distribution is a central problem in computer science as well as in statistical physics. In this work we focus on the k-colouring model and the hard-core model with fugacity λ when the underlying graph is an instance of Erdos-Renyi random graph $G(n,p)$, where $p = d/n$ and d is fixed. We use the Markov Chain Monte Carlo method for sampling from the aforementioned distributions. In particular, we consider Glauber (block) dynamics. We show a dramatic improvement on the bounds for rapid mixing in terms of the number of colours and the fugacity for the corresponding models. For both models the bounds we get are only within small constant factors from the conjectured ones by the statistical physicists.

We use Path Coupling to show rapid mixing. For k and λ in the range of our interest the technical challenge is to cope with the high degree vertices, i.e. vertices of degree much larger than the expected degree d. The usual approach to this problem is to consider block updates rather than single vertex updates for the Markov chain. Taking appropriately defined blocks the effect of high degree vertices somehow diminishes. However, devising such a construction of blocks is a highly non trivial task.

We develop for a first time a weighting schema for the paths of the underlying graph. Vertices which belong to "light" paths, only, can be placed at the boundaries of the blocks. Then the tree-like local structure of $G(n, d/n)$ allows the construction of simple structured blocks.