On random subsequences of \{n\alpha\}

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Abstract

By a classical result of Philipp (1977), for any positive sequence \((n_k)_{k \geq 1}\) satisfying the Hadamard gap condition, the discrepancy of \((n_kx)_{1 \leq k \leq N} \mod 1\) satisfies the law of the iterated logarithm. For sequences \((n_k)_{k \geq 1}\) growing subexponentially this result becomes generally false and the asymptotic behavior of the discrepancy remains unknown. In this paper we show that for randomly sampled subsequences \((n_k)_{k \geq 1}\) the discrepancy \(D_N\) of \((n_kx)_{1 \leq k \leq N} \mod 1\) and its \(L^p\) version \(D_N^{(p)}\) not only satisfy a sharp form of the law of the iterated logarithm, but we also describe the precise asymptotic behavior of the empirical process of the sequence \((n_kx)_{1 \leq k \leq N}\), leading to substantially stronger consequences.

1 Introduction

An infinite sequence \((x_k)_{k \geq 1}\) of real numbers is said to be uniformly distributed mod 1 if for every pair \(a, b\) of real numbers with \(0 \leq a < b \leq 1\) we have

\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N} I_{[a,b)}(x_k)}{N} = b - a.
\]

Here \(I_{[a,b)}\) denotes the indicator function of the interval \([a, b)\), extended with period 1. It is easily seen that \((x_k)_{k \geq 1}\) is uniformly distributed mod 1 iff \(D_N(x_k) \to 0\) (or equivalently \(D_N^*(x_k) \to 0\)) where \(D_N(x_k)\) and \(D_N^*(x_k)\) are the the discrepancy, resp. star discrepancy of the finite sequence \((x_k)_{1 \leq k \leq N}\) defined by

\[
D_N(x_k) := \sup_{0 \leq a < b \leq 1} \left| \frac{\sum_{k=1}^{N} I_{[a,b)}(x_k)}{N} - (b - a) \right|,
\]

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\[ D_N^*(x_k) := \sup_{0 < a \leq 1} \left| \frac{\sum_{k=1}^N I_{(0,a)}(x_k)}{N} - a \right|, \]

respectively. Clearly, \( D_N^* \leq D_N \leq 2D_N^* \). By a classical result of Weyl [23], for any increasing sequence \( (n_k)_{k \geq 1} \) of positive integers, \( (n_kx)_{k \geq 1} \) is uniformly distributed mod 1 for almost all \( x \) and consequently, \( D_N(n_kx) \rightarrow 0 \) a.e. Computing the precise order of magnitude of \( D_N(n_kx) \) is a difficult problem and has been solved only in a few special cases. In the case \( n_k = k \), Kesten [12] proved that

\[ ND_N(kx) \sim \frac{2}{\pi^2} \log N \log \log N \quad \text{in measure.} \tag{1.1} \]

For additional results for the discrepancy of \( \{na\} \) see Drmota and Tichy [6] and Kuipers and Niederreiter [13]. Philipp [15], [16] proved that if \( (n_k)_{k \geq 1} \) satisfies the Hadamard gap condition

\[ n_{k+1}/n_k \geq q > 1 \quad k = 1, 2, \ldots \tag{1.2} \]

then

\[ \frac{1}{4\sqrt{2}} \leq \limsup_{N \rightarrow \infty} \frac{ND_N(n_kx)}{\sqrt{2N \log \log N}} \leq C_q \quad \text{a.e.} \tag{1.3} \]

where \( C_q \) is a constant depending only on \( q \). Note that by the Chung-Smirnov law of the iterated logarithm (see e.g. [20], p. 504) we have

\[ \limsup_{N \rightarrow \infty} \frac{ND_N(\xi_k)}{\sqrt{2N \log \log N}} = \frac{1}{2} \quad \text{a.s.} \tag{1.4} \]

if \( (\xi_k)_{k \geq 1} \) is a sequence of independent, identically distributed, nondegenerate random variables in \( \mathbb{R} \). A comparison of (1.3) and (1.4) shows that under (1.2) the sequence \( \{n_kx\}_{k \geq 1} \) behaves like an i.i.d. sequence of random variables; here \( \{\cdot\} \) denotes fractional part. However, the analogy is not complete: the value of the limsup in (1.3) can be different from 1/2 and can also be a nonconstant function of \( x \). Denoting the value of the limsup by \( \Sigma_\theta \) for \( n_k = \theta^r \), \( \theta > 1 \), Fukuyama [8] showed that

\begin{align*}
\Sigma_\theta &= 1/2 \quad \text{if } \theta^r \text{ is irrational for all } r \in \mathbb{N}, \\
\Sigma_\theta &= \sqrt{42}/9 \quad \text{if } \theta = 2 \\
\Sigma_\theta &= \frac{\sqrt{(\theta + 1)\theta(\theta - 2)}}{2\sqrt{(\theta - 1)^3}} \quad \text{if } \theta \geq 4 \text{ is an even integer,} \tag{1.5} \\
\Sigma_\theta &= \frac{\sqrt{\theta + 1}}{2\sqrt{\theta - 1}} \quad \text{if } \theta \geq 3 \text{ is an odd integer.}
\end{align*}

Aistleitner [1] showed that the limsup equals 1/2 for a large class of integer sequences \( (n_k)_{k \geq 1} \) satisfying certain Diophantine conditions. For examples for a nonconstant
limsup function in (1.3) and its version for the star discrepancy $D^*_N$, see Aistleitner [2] and Fukuyama and Miyamoto [11].

Given an increasing sequence $(n_k)_{k \geq 1}$ of real numbers, let

$$F_N(t) = F_N(t, x) = \frac{1}{N} \sum_{k=1}^{N} I_{(-\infty, t)}(\{n_k x\})$$

denote the empirical distribution function of the sample $\{n_k x\}_{1 \leq k \leq N}$. Philipp [16] proved that under (1.2) the sequence

$$\alpha_N(t, x) = \sqrt{\frac{N}{2 \log \log N}} (F_N(t, x) - t), \quad 0 \leq t \leq 1, \quad N = 1, 2, \ldots$$

(1.6)
is relatively compact in the Skorohod space $D[0, 1]$ for almost all $x$ and under additional number theoretic assumptions on $n_k$ he determined the class of its limit functions in the $D[0, 1]$ metric. Since

$$D^*_N(n_k x) = \sup_t |F_N(t, x) - t|,$$

this leads immediately a precise LIL for $D_N(n_k x)$, but a functional LIL yields far deeper information on the distribution of $\{n_k x\}_{1 \leq k \leq N}$ than (1.3): it yields the precise asymptotics of various other functionals of the curve in (1.6) with constants obtained as extreme values of functionals on Hilbert space. Unlike the constants in (1.5) above, they can only be evaluated approximately.

For sequences $(n_k)_{k \geq 1}$ growing slower than exponentially, the LIL (1.3) becomes generally false (see Berkes and Philipp [3]), and the asymptotic behavior of $D_N(n_k x)$ remains open. It is then natural to ask about 'typical' behavior of the discrepancy, which requires to study $D_N(n_k x)$ for random sequences $(n_k)_{k \geq 1}$. A simple and natural model is when $(n_k)_{k \geq 1}$ is an increasing random walk, i.e. when $n_{k+1} - n_k$ are i.i.d. positive random variables. This model was investigated by Schatte [17], [19], Weber [22], Berkes and Weber [4]. For other randomization methods in the context of the LIL for the discrepancy of $(n_k x) \mod 1$ we refer to Fukuyama [9], [10]. The purpose of the present paper is to investigate the random walk model in more detail and to prove a functional LIL for the empirical process of $\{n_k x\}_{1 \leq k \leq N}$ in this model.

Let $X_1, X_2, \ldots$ be i.i.d. positive random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $X_1$ is supported on a finite interval $[a, b] \subset (0, \infty)$ and has a bounded density. Let $U$ be a random variable uniformly distributed on $(0, 1)$, independent of the sequence $(X_k)_{k \geq 1}$. Clearly, the existence of such a $U$ can be guaranteed by a suitable enlargement of the probability space. Put $S_n = \sum_{k=1}^{n} X_k$ and let

$$F^*_N(t, x) = \frac{1}{N} \sum_{k=1}^{N} I_{(-\infty, t)}(S_k x)$$
denote the empirical distribution function of the sample \( \{S_1x, \ldots, S_Nx\} \). We will prove the following result.

**Theorem.** With \( \mathbb{P} \)-probability one the sequence of functions

\[
\alpha_N(t, x) = \sqrt{\frac{N}{2 \log \log N}} (F_N^*(t, x) - t), \quad 0 \leq t \leq 1, \ N = 1, 2, \ldots
\]  

(1.7)
is relatively compact in the Skorohod space \( D[0, 1] \) for almost all \( x \in \mathbb{R} \) in the sense of Lebesgue measure and its class of limit functions is identical with the unit ball \( B_T \) of the reproducing kernel Hilbert space determined by the covariance function

\[
\Gamma(s, s') = \mathbb{E} g_s(U) g_{s'}(U) + \sum_{\varrho=1}^{\infty} \mathbb{E} g_s(U) g_{s'}(U + S_{\varrho}x) + \sum_{\varrho=1}^{\infty} \mathbb{E} g_{s'}(U) g_s(U + S_{\varrho}x), \quad (1.8)
\]

Here \( g_s = I(0, s) - s \) is the centered indicator function of the interval \((0, s)\), extended with period 1.

The absolute convergence of the series (1.8) will follow from the proof of the theorem. For background on reproducing kernel Hilbert spaces see e.g. Oodaira [14].

In the case when \( X_1 \) is uniformly distributed on \((0, 1)\), the \( \{S_kx\} \) are easily seen to be independent, uniform r.v.'s and \( \Gamma(s, s') \) reduces to the covariance function \( s(1-s') \) \((s < s')\) of the Brownian bridge. In this case the limit set in the theorem reduces to the set

\[
K = \{y(t) : y \text{ is absolutely continuous in } [0, 1], \ y(0) = y(1) = 0 \text{ and } \int_0^2 y'(t)^2 dt \leq 1\}
\]

obtained in the i.i.d. case by Finkelstein [7].

As noted above, \( \sup_{0 \leq t \leq 1} |F_N^*(t, x) - t| \) equals the star discrepancy of the sequence \( \{S_kx\}_{1 \leq k \leq N} \), while \( (\int_0^1 |F_N^*(t, x) - t|^p dt)^{1/p} \) is the \( L_p \) discrepancy \( D_N^{(p)}(S_kx) \) of the same sequence. Thus immediate consequences of our theorem are

\[
\limsup_{N \to \infty} \sqrt{\frac{N}{2 \log \log N}} D_N^*(S_kx) = \sup_{y \in B_T} \|y\|_{\infty}
\]

(1.9)

and

\[
\limsup_{N \to \infty} \sqrt{\frac{N}{2 \log \log N}} D_N^{(p)}(S_kx) = \sup_{y \in B_T} \|y\|_{p}, \quad (p \geq 1)
\]

(1.10)

\( \mathbb{P} \)-a.s. for almost all \( x \). With a different representation of the limit, relation (1.9) was obtained earlier in Schatte [19]. However, the theorem above provides much more precise information than (1.9), (1.10); for example, as a standard application of our theorem, one can characterize precisely how frequently \( (N/(2 \log \log N))^{1/2} D_N^*(S_kx) \) can get close to its limsup or get asymptotics for weighted versions of the discrepancy. We refer to Strassen [21] for applications of functional laws of the iterated logarithm.
2 Proof of the theorem

By Fubini’s theorem, it suffices to show that for any fixed \( x \neq 0 \) with \( \mathbb{P} \)-probability 1 the sequence (1.7) is relatively compact in \( D[0,1] \) and the set of its limit functions is \( B_\Gamma \). Since for any \( x \neq 0 \) the sequence \((X_k x)_{k \geq 1}\) also satisfies the condition of our theorem, without loss of generality we can assume \( x = 1 \).

Throughout this section, \( f : \mathbb{R} \to \mathbb{R} \) will denote bounded measurable functions satisfying

\[
f(t + 1) = f(t), \quad \int_0^1 f(t) dt = 0, \quad (2.1)
\]

and the \( L_2 \)-Lipschitz condition

\[
\left( \int_0^1 |f(t + h) - f(t)|^2 dt \right)^{1/2} \leq Ch \quad (2.2)
\]

for some constant \( C > 0 \). Note that if \( f = I_{(a,b)} - (b - a) \) for some \( 0 \leq a < b \leq 1 \), (extended as usual with period 1) then \( f \) satisfies (2.1), (2.2). Put

\[
A(f) = \|f\|^2 + 2 \sum_{k=1}^{\infty} \mathbb{E} f(U) f(U + S_k), \quad (2.3)
\]

where \( U \) is a uniform \((0,1)\) random variable, independent of \((X_j)_{j \geq 1}\). The absolute convergence of the series (2.3) will follow from the proof of Lemma 1. In the sequel, \( C \) and \( \lambda \) will denote positive constants, not always the same, which depend (at most) on the function \( f \) and the distribution of the random variable \( X_1 \).

**Lemma 1.** Let \( f \) satisfy (2.1), (2.2), fix an integer \( \ell \geq 1 \) and define a sequence of sets by

\[
\begin{align*}
I_1 &:= \{1, 2, \ldots, b\} \\
I_2 &:= \{p_1, p_1 + 1, \ldots, p_1 + b_1\} \quad \text{where} \quad p_1 \geq b + \ell + 2 \\
& \quad \vdots \\
I_n &:= \{p_{n-1}, p_{n-1} + 1, \ldots, p_{n-1} + b_{n-1}\} \quad \text{where} \quad p_{n-1} \geq p_{n-2} + b_{n-2} + \ell + 2 \\
& \quad \vdots
\end{align*}
\]

Then there exists a sequence \( \delta_1, \delta_2, \ldots \) of random variables, not depending on \( f \), satisfying the following properties:

(i) \( |\delta_n| \leq Ce^{-\lambda t} \) for all \( n \in \mathbb{N} \),
(ii) The random variables
\[ \sum_{i \in I_1} f(S_i), \sum_{i \in I_2} f(S_i - \delta_1), \ldots, \sum_{i \in I_n} f(S_i - \delta_{n-1}), \ldots \]
are independent.

Proof. We will construct the sequence \((\delta_n)_{n \in \mathbb{N}}\) by induction. Define
\[ \delta_1 := \{S_{b+l} - S_b\} - F_\{S_{b+l} - S_b\}(\{S_{b+l} - S_b\}) \]
where, for any random variable \(Y\), \(F_Y\) denotes the distribution function of \(Y\) and \(\{\cdot\}\) denotes fractional part. By the assumptions of our theorem and Theorem 1 of Schatte [18] we have
\[ \sup_t |F_{\{S_{n}\}}(t) - t| \leq Ce^{-\lambda n} \quad n \in \mathbb{N}. \tag{2.4} \]
Since \(S_{b+l} - S_b \overset{d}{=} S_\ell\) for all \(b\) and all \(\ell\), it follows easily that \(|\delta_1| \leq Ce^{-\lambda \ell}\). Furthermore we have
\[ \{S_{p_1} - \delta_1\} = \{S_{p_1} - \{S_{b+l} - S_b\} + F_\{S_{b+l} - S_b\}(\{S_{b+l} - S_b\})\} \]
\[ = \{(X_1 + \cdots + X_b) + (X_{b+\ell+1} + \cdots + X_{p_1}) + F_\{S_{b+l} - S_b\}(\{S_{b+l} - S_b\})\}, \]
since \(\{x\}\) has period 1. Similarly,
\[ \{S_{p_{1+1}} - \delta_1\} = \{X_1 + \cdots + X_b + (X_{b+\ell+1} + \cdots + X_{p_{1+1}}) \]
\[ + F_\{S_{b+l} - S_b\}(\{S_{b+l} - S_b\})\} \]
\[ \vdots \]
\[ \{S_{p_{1+b_1}} - \delta_1\} = \{(X_1 + \cdots + X_b) + (X_{b+\ell+1} + \cdots + X_{p_{1+1}}) \]
\[ + F_\{S_{b+l} - S_b\}(\{S_{b+l} - S_b\})\}. \]
Thus applying Lemma 1 of [17] with
\[ X = (X_1, X_2, \ldots, X_b) \]
\[ U = F_\{S_{b+l} - S_b\}(\{S_{b+l} - S_b\}) \]
\[ (W_1, \ldots, W_{p_{1+b_1}}) = (X_{b+\ell+1} + \cdots X_{p_1}), \ldots, (X_{b+\ell+1} + \cdots + X_{p_{1+b_1}}) \]
\[ W = X_1 + \cdots + X_b \]
it follows that
\[ \sum_{j \in I_1} f(S_j) \text{ is independent of } \sum_{j \in I_2} f(S_j - \delta_1). \]
Now suppose $\delta_1, \ldots, \delta_{n-1}$ have been constructed and define

$$Y_n = \left\{ (S_{pn-1+b_{n-1}+\ell} - S_{pn-1+b_{n-1}}) \right\}, \quad \delta_n = Y_n - F_{Y_n}(Y_n).$$

As before, it follows easily that $|\delta_n| \leq Ce^{-\lambda\ell}$. We let

$$X = (X_1, \ldots, X_{pn-1+b_{n-1}}, \delta_1, \ldots, \delta_{n-1})$$

$$U = F_{Y_n}(Y_n)$$

$$W = X_1 + \cdots + X_{pn-1+b_{n-1}}$$

$$(W_1, \ldots, W_{pn+b_n}) = (X_{pn-1+b_{n-1}+\ell+1} + \cdots + X_{pn}, X_{pn-1+b_{n-1}+\ell+1} + \cdots + X_{pn+b_n}).$$

Then, again by Lemma 1 of [17] it follows that

$$\sum_{i \in I_{n+1}} f(S_i - \delta_n)$$

is independent of

$$\left( \sum_{i \in I_1} f(S_i), \ldots, \sum_{i \in I_n} f(S_i - \delta_{n-1}) \right),$$

which completes the induction step and the proof of the lemma.

Put $\tilde{m}_k = \sum_{j=1}^{m_k} \lfloor j/2 \rfloor$, $\tilde{m}_k = \sum_{j=1}^{m_k} \lfloor j/4 \rfloor$ and let $m_k = \tilde{m}_k + \hat{m}_k$. Using Lemma 1 we can construct sequences $(\Delta_k)_{k \in \mathbb{N}}, (\Pi_k)_{k \in \mathbb{N}}$ of random variables such that $\Delta_0 = 0$, $\Pi_0 = 0$,

$$|\Delta_k| \leq Ce^{-\lambda\sqrt{k}}, \quad |\Pi_k| \leq Ce^{-\lambda\sqrt{k}} \quad (2.5)$$

and

$$T_k^{(f)} := \sum_{j=m_k-1+1}^{m_k-1+\lfloor \sqrt{k} \rfloor} (f(S_j - \Delta_{k-1}) - Ef(S_j - \Delta_{k-1}))$$

$$T_k^{*-f)} := \sum_{j=m_k-1+1+\lfloor \sqrt{k} \rfloor}^{m_k} (f(S_j - \Pi_{k-1}) - Ef(S_j - \Pi_{k-1}))$$

are sequences of independent random variables.

**Lemma 2.** Under the conditions of Lemma 1 we have

$$\sum_{k=1}^{n} \text{Var}(T_k^{(f)}) \sim A^{(f)}\tilde{m}_n$$

$$\sum_{k=1}^{n} \text{Var}(T_k^{*-f)}) \sim A^{(f)}\tilde{m}_n,$$

where $A^{(f)}$ is defined by (2.3).
Proof. Since \( f \) does not change in the proof, we will drop the upper index \( f \) from \( T_k, T_k^* \) and \( A^f \). Clearly
\[
\begin{align*}
\text{Var}(T_k) &= \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} \mathbb{E}f^2(S_j - \Delta_{k-1}) \\
&+ 2 \sum_{g=1}^{\lfloor \sqrt{k} \rfloor} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor \sqrt{k} \rfloor-g} \mathbb{E}f(S_\ell - \Delta_{k-1})f(S_\ell+g - \Delta_{k-1}) - L^{(k)}
\end{align*}
\]
where
\[
L^{(k)} := \left( \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} \mathbb{E}f(S_j - \Delta_{k-1}) \right)^2.
\]
By (2.1), (2.2), (2.4) and (2.5) we have
\[
\|f(S_j - \Delta_{k-1}) - f(S_j)\| \leq C e^{-\lambda \sqrt{k-1}}
\]
and
\[
|\mathbb{E}f(S_j)| = |\mathbb{E}f(S_j) - \mathbb{E}f(F\{S_j\}(\{S_j\}))| \leq C e^{-\lambda j}.
\]
Thus
\[
L^{(k)} \leq C k e^{-\lambda \sqrt{k-1}}.
\]
Let now
\[
\Lambda^{(k)} := \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} \gamma_j, \quad O^{(k)} := \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} \varepsilon_j,
\]
where
\[
\gamma_j = \mathbb{E}f^2(S_j - \Delta_{k-1}) - \mathbb{E}f^2(S_j)
\]
\[
\varepsilon_j = \mathbb{E}f^2(S_j) - \mathbb{E}f^2(F\{S_j\}(\{S_j\})).
\]
Repeating the argument above for the function \( f^2 - \|f\|^2 \), we get
\[
\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} \mathbb{E}f^2(S_j - \Delta_{k-1}) = \|f\|^2 \lfloor \sqrt{k} \rfloor + \Lambda^{(k)} + O^{(k)}
\]
and
\[
|\Lambda^{(k)}| \leq C \sqrt{k} e^{-\lambda(k-1)^{1/4}}, \quad |O^{(k)}| \leq C e^{-\lambda(m_{k-1}+1)}.
\]
We now turn to the cross terms. Define $T^\ell_\theta = X_{\ell+1} + \cdots + X_{\ell+\theta}$ and split the product expectation $\mathbb{E}f(S_{\ell} - \Delta_{k-1})f(S_{\ell+\theta} - \Delta_{k-1})$ into the sum of terms

\[
e^\ell := \mathbb{E}f(S_{\ell} - \Delta_{k-1})f(S_{\ell+\theta} - \Delta_{k-1}) - \mathbb{E}f(S_{\ell})f(S_{\ell+\theta} - \Delta_{k-1})
\]

\[
g^\ell := \mathbb{E}f(S_{\ell})f(S_{\ell+\theta} - \Delta_{k-1}) - \mathbb{E}f(S_{\ell})f(S_{\ell+\theta})
\]

\[
h^\ell := \mathbb{E}f(S_{\ell})f(S_{\ell+\theta} - \Delta_{k-1}) - \mathbb{E}f\left(F_{\{S_{\ell}\}}(\{S_{\ell}\})\right)f(S_{\ell+\theta})
\]

\[
i^\ell := \mathbb{E}f\left(F_{\{S_{\ell}\}}(\{S_{\ell}\})\right)f(S_{\ell} + T^\ell_\theta) - \mathbb{E}f\left(F_{\{S_{\ell}\}}(\{S_{\ell}\})\right)f\left(F_{\{S_{\ell}\}}(\{S_{\ell}\}) + T^\ell_\theta\right)
\]

\[
C^\ell_\theta := \mathbb{E}f\left(F_{\{S_{\ell}\}}(\{S_{\ell}\})\right)f\left(F_{\{S_{\ell}\}}(\{S_{\ell}\}) + T^\ell_\theta\right).
\]

Here $F_{\{S_{\ell}\}}(\{S_{\ell}\})$ is a uniformly distributed variable independent of $T^\ell_\theta$ and thus letting $U$ denote a uniform random variable independent of $(X_\ell)_{\ell \geq 1}$,

\[
C^\ell_\theta = C_\theta = \mathbb{E}f(U)f(U + S_\theta)
\]

does not depend on $\ell$. Exactly as before,

\[
|e^\ell| \leq Ce^{-\lambda(k-1)^{1/4}} |g^\ell| \leq Ce^{-\lambda(k-1)^{1/4}} |h^\ell| \leq Ce^{-\lambda} |i^\ell| \leq Ce^{-\lambda}.
\]

Thus letting

\[
E^{(k)} = 2 \sum_{\ell=1}^{\lfloor \sqrt{k} \rfloor - 1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+|\sqrt{k}|-\theta} e^\ell, \quad G^{(k)} = 2 \sum_{\ell=m_{k-1}+1}^{\lfloor \sqrt{k} \rfloor - 1} \sum_{\ell=1}^{m_{k-1}+|\sqrt{k}|-\theta} g^\ell
\]

\[
H^{(k)} = 2 \sum_{\ell=1}^{\lfloor \sqrt{k} \rfloor - 1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+|\sqrt{k}|-\theta} h^\ell, \quad I^{(k)} = 2 \sum_{\ell=m_{k-1}+1}^{\lfloor \sqrt{k} \rfloor - 1} \sum_{\ell=1}^{m_{k-1}+|\sqrt{k}|-\theta} i^\ell
\]

we have

\[
|E^{(k)}| \leq Ck e^{-\lambda(k-1)^{1/4}} , \quad |G^{(k)}| \leq Ck e^{-\lambda(k-1)^{1/4}}
\]

\[
|H^{(k)}| \leq C\sqrt{k} e^{-\lambda(m_{k-1}+1)} , \quad |I^{(k)}| \leq C\sqrt{k} e^{-\lambda(m_{k-1}+1)}.
\]

Furthermore,

\[
\sum_{\ell=1}^{\lfloor \sqrt{k} \rfloor - 1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+|\sqrt{k}|-\theta} C^\ell_\theta = \sum_{\ell=m_{k-1}+1}^{\lfloor \sqrt{k} \rfloor - 1} \sum_{\ell=1}^{m_{k-1}+|\sqrt{k}|-\theta} C_\theta =
\]

\[
= [\sqrt{k}] \sum_{\ell=1}^{\lfloor \sqrt{k} \rfloor} C_\theta - [\sqrt{k}] \sum_{\ell=\lfloor \sqrt{k} \rfloor}^{\infty} C_\theta - \sum_{\ell=1}^{\lfloor \sqrt{k} \rfloor - 1} C_\theta,
\]
Thus using the independence of the $T_k$ we get
\[
\text{Var} \left( \sum_{k=1}^{n} T_k \right) = \sum_{k=1}^{n} \text{Var}(T_k) = O(1) + \sum_{k=4}^{n} \text{Var}(T_k)
\]
\[
= O(1) + \sum_{k=4}^{n} \left( \sqrt{k} + \Lambda(k) + O(k) + 2E^{(k)} + 2G^{(k)} + 2H^{(k)} + 2I^{(k)} \right)
\]
\[
+ \sqrt{k} \cdot 2 \sum_{e=1}^{\infty} C_e - 2 \sqrt{k} \sum_{e=\lceil \sqrt{k} \rceil}^{\infty} C_e - 2 \sum_{e=1}^{\lfloor \sqrt{k} \rfloor - 1} \Theta C_e - L^{(k)}.
\]

Using the same techniques as before, we get $|C_e| \leq C e^{-\lambda \rho}$. Hence the previously established inequalities yield
\[
\text{Var} \left( \sum_{k=1}^{n} T_k \right) \sim A \tilde{m}_n \sim A m_n.
\]
Similarly,
\[
\text{Var} \left( \sum_{k=1}^{n} T^*_k \right) \sim A \tilde{m}_n,
\]
completing the proof of Lemma 2.

Since
\[
\text{Cov}(T_k^{(f)}, T_k^{(g)}) = \frac{1}{4} \left( \text{Var}(T_k^{(f+g)}) - \text{Var}(T_k^{(f-g)}) \right),
\]
Lemma 2 implies

**Corollary.** We have
\[
\sum_{k=1}^{n} \text{Cov}(T_k^{(f)}, T_k^{(g)}) \sim \frac{1}{4} \left( A^{(f+g)} - A^{(f-g)} \right) \tilde{m}_n
\]
and
\[
\sum_{k=1}^{n} \text{Cov}(T^*_k^{(f)}, T^*_k^{(g)}) \sim \frac{1}{4} \left( A^{(f+g)} - A^{(f-g)} \right) \tilde{m}_n.
\]

From (2.3) it follows that
\[
A^{(f+g)} - A^{(f-g)} = 4 \langle f, g \rangle + 4 \sum_{k=1}^{\infty} \mathbb{E} f(U) g(U + S_k) + 4 \sum_{k=1}^{\infty} \mathbb{E} g(U) f(U + S_k).
\]
Let $0 < t_1 < \ldots < t_r \leq 1$ and put
\[
Y_k = (f_{(0,t_1)}(S_k), f_{(0,t_2)}(S_k), \ldots, f_{(0,t_r)}(S_k))
\]
where $f_{(a,b)} = I_{(a,b)} - (b - a)$. 10
Lemma 3. With \( \mathbb{P} \)-probability 1, the class of limit points of the sequence
\[
\left\{ (2N \log \log N)^{-1/2} \sum_{k=1}^{N} Y_k, \quad N = 1, 2, \ldots \right\}
\]
in \( \mathbb{R}^r \) is the ellipsoid
\[
\left\{ (x_1, \ldots, x_r) : \sum_{i,j=1}^{r} \Gamma(t_i, t_j)x_i x_j \leq 1 \right\}.
\]

Proof. Let
\[
T_k = \left( T_k^{(f(0,t_1))}, \ldots, T_k^{(f(0,t_r))} \right), \quad T_k^{*} = \left( T_k^{*(f(0,t_1))}, \ldots, T_k^{*(f(0,t_r))} \right).
\]
and let \( \Sigma_k \) denote the covariance matrix of the vector \( T_k \). From the Corollary and (2.6) it follows that
\[
m_n^{-1} (\Sigma_1 + \ldots + \Sigma_n) \rightarrow \Sigma
\]
where
\[
\Sigma = (\Gamma(t_i, t_j))_{1 \leq i, j \leq r}.
\]
Clearly
\[
|T_k| \leq C_r k^{1/2} = o(m_k \log \log m_k)^{1/2}
\]
where \( C_r \) is a constant depending on \( r \), showing that the sequences \( (T_k)_{k \geq 1} \) of independent random variables satisfies Kolmogorov’s condition of the LIL. Thus Theorem 1 in Berning [5] implies that the set of limit points of
\[
\left\{ (2m_n \log \log m_n)^{-1/2} \sum_{k=1}^{n} T_k \right\}
\]
is the ellipsoid (2.8). A similar statement holds for the sequence \( (T_k^{*})_{k \geq 1} \), implying that
\[
\left| \sum_{k=1}^{n} T_k^{*} \right| = O(\hat{m}_n \log \log \hat{m}_n)^{1/2} = o(m_n \log \log m_n)^{1/2} \quad \text{a.s.}
\]
Also, for any \( f = f(0,t_i), \ 1 \leq i \leq r \) we have
\[
\max_{m_k+1 \leq \ell \leq m_k+1} \left| \sum_{j=m_k+1}^{\ell} (f(S_j) - \mathbb{E}f(S_j)) \right| = O(m_{k+1} - m_k) = O(k^{1/2}) = o(m_k \log \log m_k)^{1/2}.
\]
From these relations Lemma 3 follows immediately.
Lemma 4. Let $f$ satisfy (2.1), (2.2). Then we have
\[ \mathbb{E} \left( \sum_{k=M+1}^{M+N} f(S_k) \right)^2 \leq C \| f \| N. \]  

(2.10)

Proof. We first show
\[ |\mathbb{E} f(S_k)f(S_\ell)| \leq C e^{-\lambda(\ell-k)} \| f \| \quad (k < \ell). \]  

(2.11)

Indeed, as in the proof of Lemma 1, there exists a r.v. $\Delta$ with $|\Delta| \leq C e^{-\lambda(\ell-k)}$ such that $S_\ell - \Delta$ is a uniform r.v. independent of $S_k$. Hence
\[ \mathbb{E} f(S_\ell - \Delta) = \int_0^1 f(t)dt = 0 \]
and thus
\[ \mathbb{E} f(S_k)f(S_\ell - \Delta) = \mathbb{E} f(S_k)\mathbb{E} f(S_\ell - \Delta) = 0. \]  

(2.12)

On the other hand,
\begin{align*}
|\mathbb{E} f(S_k)f(S_\ell) - \mathbb{E} f(S_k)f(S_\ell - \Delta)| &\leq \mathbb{E} (|f(S_k)||f(S_\ell) - f(S_\ell - \Delta)|) \\
\leq (\mathbb{E} f^2(S_k))^{1/2} (\mathbb{E} |f(S_\ell) - f(S_\ell - \Delta)|^2)^{1/2}.
\end{align*}

(2.13)

Since $X_1$ has a bounded density, by Theorem 1 of Schatte [18] the density $\varphi_n$ of $S_n$ exists for all $n \geq 1$ and satisfies $\varphi_n \to 1$ uniformly on $[0,1]$. Thus
\[ P\{S_n \in I\} \leq C |I| \quad (n \geq 1) \]  

(2.14)

for some constant $C > 0$, whence we get
\[ \mathbb{E} f^2(S_k) \leq C \int_0^1 f^2(t)dt = C \| f \|^2. \]  

(2.15)

On the other hand,
\[ \mathbb{E} |f(S_\ell) - f(S_\ell - \Delta)|^2 \leq C e^{-\lambda(\ell-k)} \]  

(2.16)

which, together with (2.13)–(2.16), gives
\[ |\mathbb{E} f(S_k)f(S_\ell) - \mathbb{E} f(S_k)f(S_\ell - \Delta)| \leq C e^{-\lambda(\ell-k)}. \]

Thus using (2.12) we get (2.11). Now by (2.11)
\[ \left| \sum_{M+1 \leq k < \ell \leq M+N} \mathbb{E} f(S_k)f(S_\ell) \right| \leq CN \| f \| \sum_{\ell \geq 1} \ell^{-2} \leq CN \| f \| \]
which, together with (2.15), completes the proof of Lemma 4.
Lemma 5. Let $f$ satisfy (2.1), (2.2). Then for any $M \geq 0$, $N \geq 1$, real $t \geq 1$ and $\|f\| \geq N^{-1/4}$ we have

$$P\left\{ \left| \sum_{k=M+1}^{M+N} f(S_k) \right| \geq t\|f\|^{1/4}(N \log \log N)^{1/2} \right\} \leq \exp\left(-Ct\|f\|^{-1/2} \log \log N\right) + t^{-2}N^{-1}. \tag{2.17}$$

Proof. We divide the interval $[M + 1, M + N]$ into subintervals $I_1, \ldots, I_L$, with $L \sim N^{19/20}$, where each interval $I_\nu$ contains $\sim N^{1/20}$ terms. We set

$$\sum_{k=M+1}^{M+N} f(S_k) = \eta_1 + \cdots + \eta_L$$

where

$$\eta_\nu = \sum_{k \in I_\nu} f(S_k).$$

We deal with the sums $\sum_2 \eta_{2j}$ and $\sum_2 \eta_{2j+1}$ separately. Since there is a separation $\sim N^{1/20}$ between the even block sums $\eta_{2j}$, we can apply Lemma 4.3 of [4] to get

$$\eta_{2j} = \eta_{2j}^* + \eta_{2j}^{**}$$

where

$$\eta_{2j}^* = \sum_{k \in I_{2j}} f(S_k - \Delta_j)$$

$$\eta_{2j}^{**} = \sum_{k \in I_{2j}} (f(S_k) - f(S_k - \Delta_j)) \tag{2.18}$$

where the $\Delta_j$ are r.v.’s with $|\Delta_j| \leq \psi(N^{1/20}) \leq N^{-10}$ and the r.v.’s $\eta_{2j}^{**}$, $j = 1, 2, \ldots$ are independent. Relation (2.16) in the proof of Lemma 4 shows that the $L_2$ norm of each summand in $\eta_{2j}^{**}$ is $\leq C\psi(N^{1/20}) \leq CN^{-10}$ and thus for $\|f\| \geq N^{-1/4}$ we have

$$\|\eta_{2j}^{**}\| \leq CN^{-9} \leq C\|f\|N^{-8}. \tag{2.19}$$

Thus

$$\left\| \sum_{2j} \eta_{2j}^{**} \right\| \leq C\|f\|N^{-7}$$

and therefore by the Markov inequality

$$P\left( \left| \sum_{2j} \eta_{2j}^{**} \right| \geq t\|f\|^{1/4}(N \log \log N)^{1/2} \right) \leq Ct^{-2}\|f\|^{-1/2}(N \log \log N)^{-1}\|f\|^2N^{-14} \leq t^{-2}N^{-1}. \tag{2.20}$$
Let now $|\lambda| = \mathcal{O}(N^{-1/16})$, then $|\lambda \eta^*_j| \leq C|\lambda|N^{1/20} \leq 1/2$ for $N \geq N_0$ and thus using $e^x \leq 1 + x + x^2$ for $|x| \leq 1/2$ we get, using $E\eta^*_2 = 0$,

$$E\left(\exp\lambda\left(\sum_j \eta^*_j\right)\right) = \prod_j E(e^{\lambda \eta^*_j}) \leq \prod_j E(1 + \lambda \eta^*_j + \lambda^2 \eta^*_j^2)$$

$$= \prod_j (1 + \lambda^2 E\eta^*_j^2) \leq \exp\left(\lambda^2 \sum_j E\eta^*_j^2\right). \tag{2.21}$$

By Lemma 4

$$\|\eta^*_j\| \leq C\|f\|^{1/2}N^{1/40},$$

which, together with (2.19) and the Minkowski inequality, implies

$$\|\eta^*_j\| \leq C\|f\|^{1/2}N^{1/40}.$$ 

Thus the last expression in (2.21) cannot exceed

$$\exp\left(\lambda^2 C\|f\| \sum_j N^{1/20}\right) \leq \exp(\lambda^2 C\|f\|N).$$

Hence choosing

$$\lambda = (\log \log N/N)^{1/2}f^{-3/4}$$

(note that by $\|f\| \geq N^{-1/4}$ we have $|\lambda| = \mathcal{O}(N^{-1/6})$) and using the Markov inequality, we get

$$P\left\{\left|\sum_j \eta^*_j\right| \geq t\|f\|^{1/4}(N \log \log N)^{1/2}\right\}$$

$$\leq \exp\left\{-\lambda t\|f\|^{1/4}(N \log \log N)^{1/2} + \lambda^2 C\|f\|N\right\} \tag{2.22}$$

$$= \exp(-\|f\|^{-1/2} t \log \log N + C\|f\|^{-1/2} \log \log N)$$

$$\leq \exp(-C'\|f\|^{-1/2} \log \log N)$$

completing the proof of Lemma 5.

With Lemmas 1-5 at hand, the proof of the Theorem can be completed easily. Given any $0 < t_1 < \ldots < t_r = 1$, let $B_{\Gamma}(t_1, \ldots, t_r)$ denote the set of points of the form $(g(t_1), \ldots, g(t_r))$, where $g(\cdot) \in B_{\Gamma}$. By the standard method of proving functional laws of the iterated logarithm developed in Strassen [21], Finkelstein [7], it suffices to prove that with probability 1 the sequence $\alpha_N(\cdot)$ is relatively compact in the $D[0,1]$ topology with and for any $0 < t_1 < \ldots < t_r = 1$, $r = 1, 2, \ldots$ the set of limit points of the vector $(\alpha_N(t_1), \ldots, \alpha_N(t_r))$ is identical with the set $B_{\Gamma}(t_1, \ldots, t_r)$. Since $B_{\Gamma}(t_1, \ldots, t_r)$ coincides with the ellipsoid (2.8), the second statement follows from Lemma 3. On the other hand, the equicontinuity statement can be proved by a dyadic chaining argument, similar to the proof of Proposition 3.3.2 in Philipp [16]. Since the necessary modifications are routine, we omit the details.
References


