On trigonometric sums with random frequencies

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Abstract

There is a wide and nearly complete theory of trigonometric series with random coefficients; on the other hand, much less is known on trigonometric series with random frequencies. In this paper we study the asymptotic behavior of $S_N = \sum_{k=1}^N \sin n_k x$ for random sequences $(n_k)_{k\ge 1}$, independent and identically distributed over disjoint intervals $I_k \subset (0, \infty)$ of the same length. As it turns out, the behavior of S_N depends on the size of the gaps Δ_k between the intervals I_k: for small gaps the limit distribution of $S_N/\sqrt{N}$ is mixed Gaussian, for large gaps it is pure Gaussian and for intermediate gaps it is the convolution of a mixed Gaussian distribution and the contribution of an associated nonrandom trigonometric sum which can be nongaussian.

1 Introduction

In their pioneering work "Some random series of functions I-III" , Paley and Zygmund studied trigonometric series

$$\sum_{k=1}^{\infty} X_k \cos(kx + \Phi_k)$$

where (X_k, Φ_k) are independent random vectors. Written equivalently, these are series of the form

$$\sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

where the coefficients a_k, b_k are random. The investigations of Paley and Zygmund were extended in various directions and today we have an extensive and nearly complete theory of such series; for an exposition, see the monograph of Kahane [16]. No

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similar complete theory exists for trigonometric series with random frequencies, i.e. series of the form

$$\sum_{k=1}^{\infty}(a_k \cos n_k x + b_k \sin n_k x) \tag{1.1}$$

where the n_k are random, even though constructions using such series play an important role in harmonic analysis. By a classical result of Salem and Zygmund [20], if $(n_k)_{k\ge 1}$ is a sequence of positive integers satisfying the Hadamard gap condition

$$n_{k+1}/n_k \ge q > 1 \qquad (k = 1, 2, \dots) \tag{1.2}$$

then $(\sin 2\pi n_k x)_{k\ge 1}$ obeys the central limit theorem, i.e.

$$N^{-1/2} \sum_{k=1}^{N} \sin 2\pi n_k x \xrightarrow{d} N(0, 1/2) \tag{1.3}$$

with respect the the probability space $(0, 1)$ equipped with Borel sets and the Lebesgue measure. Erdős [8] showed that (1.3) remains valid if we weaken the Hadamard gap condition (1.2) to

$$n_{k+1}/n_k \ge 1 + c_k k^{-1/2}, \qquad c_k \to \infty \tag{1.4}$$

and this result is sharp, i.e. for any $c > 0$ there exists a sequence (n_k) satisfying

$$n_{k+1}/n_k \ge 1 + c k^{-1/2}, \qquad k = 1, 2, \dots$$

such that the CLT (1.3) is false. For sequences $(n_k)_{k\ge 1}$ growing slower than the speed defined by (1.4), the asymptotic behavior of the partial sums of $\sin 2\pi n_k x$ depends sensitively on the number theoretic properties of (n_k) (see e.g. Erdős [8], Gaposhkin [13]), and deciding the validity of the CLT leads to difficult open problems in additive number theory. (For a discussion see Halberstam and Roth [14], Chapter 3.) Still, the CLT holds for many slowly increasing sequences $(n_k)_{k\ge 1}$. Salem and Zygmund [21] showed that if $X_1, X_2, \dots$ are independent random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ taking the values 0 and 1 with probability $1/2 - 1/2$ and $(n_k)_{k\ge 1}$ denotes the sequence of indices j with $X_j = 1$, then $\mathbb{P}$-almost surely the random subsequence $(\sin 2\pi n_k x)_{k\ge 1}$ satisfies the CLT (1.3). By the theorem on "pure heads", $\limsup_{k\to\infty}(n_{k+1} - n_k)/\log k = 1$, i.e. the gaps in this example are at most logarithmic. Using a different random construction, Berkes [2] showed that here $n_{k+1} - n_k = O(\log k)$ can be replaced by $n_{k+1} - n_k = O(\omega(k))$ for any $\omega(k) \to \infty$ and Bobkov and Götze [5] showed that (1.3) cannot hold for any sequence $(n_k)_{k\ge 1}$ of integers with $n_{k+1} - n_k = O(1)$. On the other hand, Fukuyama [9] showed, using a sophisticated random construction, that for any $0 < \sigma^2 < 1/2$ there exists a sequence $(n_k)_{k\ge 1}$ of integers with bounded gaps such that (1.3) holds with a limiting normal distribution with variance σ^2. For related bounded gap constructions for the LIL and discrepancy LIL we refer to Fukuyama [10], [11], [12] and Aistleitner and Fukuyama

[1]. In a completely different direction, Erdős [7] gave a random construction solving a long standing open problem of Sidon for lacunary trigonometric series; for a detailed discussion we refer again to Halberstam and Roth [14], Chapter 3. Wiener approximation of random trigonometric sums was given in Kaufman [17]. Schatte [22], Weber [23] and Berkes and Weber [4] investigated the case when $(n_k)_{k\geq 1}$ is an increasing random walk, i.e. $n_{k+1}-n_k$ are i.i.d. positive random variables. For ergodic type results we refer to Durand and Schneider [6] and the references therein.

The purpose of the present paper is to investigate the asymptotic behavior of $\sum_{k=1}^N \sin n_k x$ for the simplest random model where n_k are independent random variables having the same distribution over disjoint intervals $I_1, I_2, \ldots$ on $(0,\infty)$ with the same length. For the simplicity of the calculations we will assume that the n_k have a bounded density, even though at the cost of minor complications, the discrete case can be settled in a similar way. To formulate our results, define the probability measure μ on the Borel sets of $\mathbb{R}$ by

$$\mu(A) = \frac{1}{\pi}\int_A \left(\frac{\sin x}{x}\right)^2 dx, \qquad A \subset \mathbb{R}.$$

Throughout this paper, $\xrightarrow{d}$ will denote convergence in distribution over the real line $\mathbb{R}$, equipped with Borel sets and the probability measure μ. Our main result is

Theorem 1. *Let $n_1, n_2, \ldots$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that n_k is uniformly distributed on the interval $[A_k, A_k+B]$, where $A_{k+1}-A_k > B+2$, $k=1,2,\ldots$. Let $\lambda_k(x) = \mathbb{E}(\sin n_k x)$. Then $\mathbb{P}$-almost surely*

$$\frac{1}{\sqrt{N}}\sum_{k=1}^N (\sin n_k x - \lambda_k(x)) \xrightarrow{d} F \tag{1.5}$$

where F is the distribution with characteristic function

$$\phi(\lambda) = \int_{-\infty}^{+\infty} \exp\left(-\frac{\lambda^2}{4}\left(1-\frac{4\sin^2(Bx/2)}{B^2x^2}\right)\right) d\mu(x). \tag{1.6}$$

If in addition we have

$$\frac{1}{\sqrt{N}}\sum_{k=1}^N \lambda_k(x) \xrightarrow{d} G \tag{1.7}$$

with respect to to any interval $E \subset \mathbb{R}$ with positive μ-measure, then $\mathbb{P}$-almost surely

$$\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N (\sin n_k x - \lambda_k(x)), \frac{1}{\sqrt{N}}\sum_{k=1}^N \lambda_k(x)\right) \xrightarrow{d} (F,G) \tag{1.8}$$

where the components of the limit vector are independent.

Remarks.

1. There is a number of variants of Theorem 1 that can be proved by the same argument. If the disjoint intervals $I_k = [A_k, A_k + B]$ in Theorem 1 have length ≥ 1 and n_k is uniformly distributed over the integers of I_k, then Theorem 1 remains valid with μ replaced by normalized Lebesgue measure on $(0, 2\pi)$ and $g(x)$ replaced by an explicitly computable trigonometric function. The case $B = 1$ and $A_k = 2k$ was settled earlier by Bobkov and Götze [5]. Similarly, Theorem 1 remains valid if $n_1, n_2, \dots$ have a common bounded density f on the interval $[A_k, A_k + B]$. In this case $g(x)$ becomes

$$g(x) = \frac{1}{2}\left(1 - \int_0^B \int_0^B \cos(v - \xi) f(v) f(\xi) dv d\xi\right).$$

2. With $\mathbb{P}$-probability 1, i.e. for almost all sequences (n_k) generated by the random algorithm in Theorem 1, (1.5) yields a central limit theorem over $(\mathbb{R}, \mathcal{B}, \mu)$ with a random centering factor and a mixed Gaussian limit distribution, i.e. a normal limit law with random variance. Such random factors are typical in the theory of lacunary series, see e.g. Gaposhkin [13]. In the random frequency case, however, the centering factor $\lambda_k(x)$ in the CLT (1.5) plays a different and substantial role. As we will see, if the gaps $\Delta_k = A_{k+1} - A_k - B$ between the intervals remain constant or if the A_k are integers and $\Delta_k \uparrow \infty$, $\Delta_k = O(k^\gamma)$ with $\gamma < 1/4$ (small gaps), then (1.5) holds with $\lambda_k = 0$, i.e. without a centering factor. At the other end of the spectrum, i.e. for rapidly increasing A_k, the centering factors themselves contribute to the limit distribution, i.e.

$$\frac{1}{\sqrt{N}} \sum_{k=1}^N \lambda_k(x) \tag{1.9}$$

has a nondegenerate limit distribution. More precisely, if A_k satisfies the Erdős gap condition

$$A_{k+1}/A_k \geq 1 + c_k/\sqrt{k}, \qquad c_k \to \infty \tag{1.10}$$

then (1.9) has the limit distribution with characteristic function

$$\phi(\lambda) = \int_{-\infty}^{+\infty} \exp\left(-\frac{\lambda^2}{4} \cdot \frac{4\sin^2(Bx/2)}{B^2 x^2}\right) d\mu(x) \tag{1.11}$$

and thus by the asymptotic independence of the components of (1.8) it follows that

$$N^{-1/2} \sum_{k=1}^N \sin n_k x \tag{1.12}$$

has a pure Gaussian limit distribution $N(0, 1/2)$, i.e. (1.3) holds. Since in this case (n_k) satisfies (1.4), the asymptotic normality of (1.12) follows from Erdős' central limit theorem [8] even for nonrandom (n_k), i.e. in this case Theorem 1 reduces to a result in classical lacunarity theory. In the intermediate case between slowly and rapidly increasing (n_k), the centering factors $\lambda_k(x)$ in (1.5) may or may not contribute to the limit distribution F and F may be nongaussian. Observing that

$$\lambda_k(x) = \frac{2\sin(Bx/2)}{Bx} \sin\left(A_k + B/2\right) x, \tag{1.13}$$

from the results of Berkes [3] it follows that there exist sequences (A_k) satisfying (1.10) with $c_k \to \infty$ replaced by $c_k = c > 0$ such that (1.9) has a nongaussian limit distribution and for any positive sequence $c_k \to 0$ there exist sequences (A_k) satisfying (1.10) such that (1.9) tends to 0 in probability. This shows that nongaussian limits of (1.9) can occur arbitrary close to the gap condition (1.10), i.e. (1.10) is critical in the theory. Theorem 1 and relation (1.13) also show that the limit distribution of (1.12), if exists, is the convolution of a mixed normal distribution and the limit distribution of a normed trigonometric sum with nonrandom frequencies $A_k + B/2$. As pointed out before, the asymptotic behavior of such nonrandom sums is more an arithmetic than a probabilistic problem and thus, apart from a few remarks, in this paper we do not deal with them.

2 Proof of Theorem 1.

Let

$$\varphi_k(x) = \sin n_k x - \mathbb{E}(\sin n_k x)$$

and

$$T_N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \varphi_k(x).$$

Note that

$$\int_{-\infty}^{+\infty} \cos \alpha x \left(\frac{\sin x}{x}\right)^2 dx = 0 \quad \text{for } |\alpha| > 2 \tag{2.14}$$

(see e.g. Hartman [15]) and thus our assumption $A_{k+1} - A_k > B + 2$ $(k = 1, 2, \ldots)$ implies that for any $u_1 \in [A_k, A_k + B]$, $u_2 \in [A_\ell, A_\ell + B]$, $k \neq \ell$, $\sin u_1 x$ and $\sin u_2 x$ are orthogonal in $L^2_\mu(\mathbb{R})$, which implies that $\sin u_1 x$ and $\lambda_\ell(x) = \mathbb{E}(\sin n_\ell x) = B^{-1} \int_{A_\ell}^{A_\ell + B} \sin txdt$ and also φ_k and φ_ℓ are orthogonal in $L^2_\mu(\mathbb{R})$. Thus elementary algebra shows that the $L^2_\mu(\mathbb{R})$ norm of $|T_M - T_{N^3}|$ is at most $C/\sqrt{N}$ for $N^3 \leq M \leq (N+1)^3$ with an absolute constant C. Hence to prove (1.5) it suffices to show that $T_{N^3} \xrightarrow{d} F$ $\mathbb{P}$-a.s.

A simple calculation shows that

$$\begin{aligned} \lambda_k(x) = \mathbb{E}(\sin n_k x) &= \frac{1}{B} \int_{A_k}^{A_k + B} \sin txdt = \frac{1}{Bx}(\cos A_k x - \cos(A_k + B)x) \\ &= \frac{2\sin(Bx/2)}{Bx} \sin\left(A_k + B/2\right) x \end{aligned}$$

and

$$\mathbb{E}(\cos 2n_k x) = \frac{1}{B}\int_{A_k}^{A_k+B} \cos 2tx dt = \frac{\sin Bx}{Bx}\cos(2A_k + B)x.$$

Thus

$$\begin{aligned}\mathbb{E}\varphi_k^2(x) &= \mathbb{E}(\sin^2 n_k x) - \lambda_k^2(x) = \frac{1}{2}(1 - \mathbb{E}(\cos 2n_k x)) - \lambda_k^2(x)\\ &= \frac{1}{2} - \frac{\sin Bx}{2Bx}\cos(2A_k + B)x - \frac{4\sin^2(Bx/2)}{B^2x^2}\sin^2(A_k + B/2)x\\ &= \left(\frac{1}{2} - \frac{2\sin^2(Bx/2)}{B^2x^2}\right) + \left(\frac{2\sin^2(Bx/2)}{B^2x^2} - \frac{\sin Bx}{2Bx}\right)\cos(2A_k + B)x.\end{aligned}$$

From $A_{k+1} - A_k > B + 2$, (2.14) and elementary trigonometric identities it follows that the functions $\cos(2A_k + B)x$ are orthogonal in $L^2_\mu(\mathbb{R})$ and thus the Rademacher-Menshov convergence theorem implies that $\sum_{k=1}^\infty k^{-1}\cos(2A_k + B)x$ converges μ-almost everywhere. Consequently, the Kronecker lemma implies

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N \cos(2A_k + B)x = 0 \qquad \mu - \text{a.e.}$$

and thus

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N \mathbb{E}\varphi_k^2(x) = \frac{1}{2}\left(1 - \frac{4\sin^2(Bx/2)}{B^2x^2}\right) \qquad \mu - \text{a.e.}$$

Since $\varphi_k^2(x) - \mathbb{E}\varphi_k^2(x)$, $k = 1, 2, \dots$ are independent, uniformly bounded, zero mean random variables for fixed x, the strong law of large numbers yields

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N(\varphi_k^2(x) - \mathbb{E}\varphi_k^2(x)) = 0 \qquad \mathbb{P} - \text{a.s.}$$

and thus we conclude that for μ-a.e. x we have $\mathbb{P}$-almost surely

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N \varphi_k^2(x) = \frac{1}{2}\left(1 - \frac{4\sin^2(Bx/2)}{B^2x^2}\right). \tag{2.15}$$

By Fubini's theorem, $\mathbb{P}$-almost surely the last relation holds for μ-almost all $x \in \mathbb{R}$. Fix $\lambda \in \mathbb{R}$. Using $|\varphi_k(x)| \le 2$ and

$$\exp(z) = (1 + z)\exp\left(\frac{z^2}{2} + o(z^2)\right) \qquad z \to 0$$

we get

$$\exp\left(\frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right) = \left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right)\exp\left(-\frac{\lambda^2\varphi_k^2(x)}{2N} + o\left(\frac{\lambda^2\varphi_k^2(x)}{N}\right)\right)$$

as $N \to \infty$, uniformly in x and the implicit variable $\omega \in \Omega$. Thus the characteristic function of T_N with respect to the probability space $(\mathbb{R}, \mathcal{B}, \mu)$ is

$$\phi_{T_N}(\lambda) = \int_{-\infty}^{+\infty} \exp(i\lambda T_N(x))d\mu(x)$$

$$= \int_{-\infty}^{+\infty} \prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right) \exp\left(-(1+o(1))\frac{\lambda^2}{2N}\sum_{k=1}^{N}{\varphi_k}^2(x)\right) d\mu(x).$$

For simplicity let

$$g(x) = \frac{1}{2}\left(1 - \frac{4\sin^2(Bx/2)}{B^2x^2}\right).$$

Using $1 + x \leq e^x$ and $|\varphi_k(x)| \leq 2$ we get

$$\begin{aligned}\left|\prod_{k=1}^{N}\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right)\right| &= \prod_{k=1}^{N}\left(1 + \frac{\lambda^2}{N}{\varphi_k}^2(x)\right)^{1/2} \\ &\leq \exp\left(\frac{\lambda^2}{2N}\sum_{k=1}^{N}{\varphi_k}^2(x)\right) \leq e^{2\lambda^2} \end{aligned} \tag{2.16}$$

and thus the dominated convergence theorem and (2.15) imply $\mathbb{P}$-almost surely

$$\phi_{T_N}(\lambda) = \int_{-\infty}^{+\infty} \prod_{k=1}^{N}\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right) \exp\left(-\lambda^2 g(x)/2\right) d\mu(x) + o(1).$$

Since the characteristic function of F in (1.5) is given by (1.6), it remains to show that letting

$$\Gamma_N = \int_{-\infty}^{+\infty}\left[\prod_{k=1}^{N}\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right) - 1\right] \exp\left(-\lambda^2 g(x)/2\right) d\mu(x)$$

we have

$$\Gamma_{N^3} \xrightarrow{\mathbb{P}-\text{a.s.}} 0.$$

Clearly

$$\begin{aligned}\mathbb{E}|\Gamma_N|^2 = \mathbb{E}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} &\left[\prod_{k=1}^{N}\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right) - 1\right]\left[\prod_{k=1}^{N}\left(1 - \frac{i\lambda}{\sqrt{N}}\varphi_k(y)\right) - 1\right] \\ &\times \exp\left(-\lambda^2 g(x)/2\right)\exp\left(-\lambda^2 g(y)/2\right) d\mu(x)d\mu(y).\end{aligned} \tag{2.17}$$

Now using the independence of the φ_k and $\mathbb{E}\varphi_k(x) = \mathbb{E}\varphi_k(y) = 0$ we get

$$\begin{aligned}&\mathbb{E}\left[\prod_{k=1}^{N}\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right) - 1\right]\left[\prod_{k=1}^{N}\left(1 - \frac{i\lambda}{\sqrt{N}}\varphi_k(y)\right) - 1\right] \\ &= \mathbb{E}\left[\prod_{k=1}^{N}\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x)\right)\left(1 - \frac{i\lambda}{\sqrt{N}}\varphi_k(y)\right)\right] - 1 \\ &= \mathbb{E}\left[\prod_{k=1}^{N}\left(1 + \frac{i\lambda}{\sqrt{N}}\varphi_k(x) - \frac{i\lambda}{\sqrt{N}}\varphi_k(y) + \frac{\lambda^2}{N}\varphi_k(x)\varphi_k(y)\right)\right] - 1\end{aligned}$$

$$= \prod_{k=1}^{N} \left(1 + \frac{\lambda^2}{N}\Psi_k(x,y)\right) - 1,$$

where $\Psi_k(x,y) = \mathbb{E}\varphi_k(x)\varphi_k(y)$. Thus interchanging the expectation with the double integral in (2.17) we get

$$\mathbb{E}|\Gamma_N|^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\prod_{k=1}^{N}\left(1+\frac{\lambda^2}{N}\Psi_k(x,y)\right)-1\right]\times$$
$$\times \exp\left(-\lambda^2 g(x)/2 - \lambda^2 g(y)/2\right) d\mu(x)d\mu(y)$$
$$\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\prod_{k=1}^{N}\left(1+\frac{\lambda^2}{N}\Psi_k(x,y)\right)-1\right| d\mu(x)d\mu(y).$$

Using $|\Psi_k(x,y)| \leq 4$ and $|\log(1+x)-x| \leq Cx^2$ for all $|x| \leq 1$ and some constant $C > 0$, one deduces for all N large enough

$$\left|\log\prod_{k=1}^{N}\left(1+\frac{\lambda^2}{N}\Psi_k(x,y)\right) - \sum_{k=1}^{N}\frac{\lambda^2}{N}\Psi_k(x,y)\right| \leq \frac{16C\lambda^4}{N}.$$

Thus letting

$$G_N(x,y) := \sum_{k=1}^{N}\frac{\lambda^2}{N}\Psi_k(x,y)$$

we get, using $G_N(x,y) \leq 4\lambda^2$, that

$$\prod_{k=1}^{N}\left(1+\frac{\lambda^2}{N}\Psi_k(x,y)\right) = \exp\left\{G_N(x,y) + O(\lambda^4/N)\right\} = 1 + O(|G_N(x,y)|) + O(1/N).$$

Hence

$$\mathbb{E}|\Gamma_N|^2 \leq C_1\left(\frac{1}{N} + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|G_N(x,y)|\, d\mu(x)d\mu(y)\right) \tag{2.18}$$

for some constant C_1. As noted before, φ_k and φ_ℓ are orthogonal in $L^2_\mu(\mathbb{R})$ and since $\Psi_k(x,y)\Psi_\ell(x,y) = \mathbb{E}\varphi_k(x)\varphi_\ell(x)\varphi_k(y)\varphi_\ell(y)$, it follows that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Psi_k(x,y)\Psi_\ell(x,y)d\mu(x)d\mu(y) = 0 \qquad \text{for } k \neq \ell$$

and thus by the Cauchy-Schwarz inequality the last integral in (2.18) is $O(N^{-1/2})$. Hence $\mathbb{E}|\Gamma_N|^2 = O(N^{-1/2})$ and thus $\sum_{N\in\mathbb{N}}\mathbb{E}|\Gamma_{N^3}|^2 < \infty$, implying $\sum_{N\in\mathbb{N}}|\Gamma_{N^3}|^2 < \infty$ and $\Gamma_{N^3} \to 0$ $\mathbb{P}$-a.s., completing the proof of (1.5). To prove the second statement of Theorem 1, consider the integral

$$I_N = I_N(\lambda,\lambda^*) = \int_{-\infty}^{\infty}\exp\left(i\lambda T_N(x) + i\lambda^* V_N(x)\right) d\mu(x)$$

where

$$V_N(x) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \lambda_k(x).$$

Similarly as before, we have $I_N = J_N + o(1)$ $\mathbb{P}$-a.s. as $N \to \infty$, where

$$J_N = \int_{-\infty}^{\infty} e^{-\lambda^2 g(x)/2} \prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x)\right) \exp\left(i\lambda^* V_N(x)\right) d\mu(x).$$

Clearly,

$$\begin{aligned} &J_N \overline{J}_N = \\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\lambda^2 g(x)/2 - \lambda^2 g(y)/2} \prod_{k=1}^{N} \left(1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x)\right) \prod_{k=1}^{N} \left(1 - \frac{i\lambda}{\sqrt{N}} \varphi_k(y)\right) \\ &\qquad \times \exp\left(i\lambda^* V_N(x) - i\lambda^* V_N(y)\right) d\mu(x) d\mu(y) \end{aligned}$$

and thus

$$\begin{aligned} &\mathbb{E} J_N \overline{J}_N \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\lambda^2 g(x)/2 - \lambda^2 g(y)/2} \prod_{k=1}^{N} \left(1 + \frac{\lambda^2}{N} \Psi_k(x,y)\right) \\ &\qquad \times \exp\left(i\lambda^* V_N(x) - i\lambda^* V_N(y)\right) d\mu(x) d\mu(y). \end{aligned}$$

Also,

$$\pi_N := \mathbb{E} J_N = \int_{-\infty}^{\infty} e^{-\lambda^2 g(x)/2} \exp\left(i\lambda^* V_N(x)\right) d\mu(x)$$

and consequently

$$\begin{aligned} &\mathbb{E}(J_N - \pi_N)(\overline{J}_N - \overline{\pi}_N) = \mathbb{E} J_N \overline{J}_N - \pi_N \overline{\pi}_N \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\lambda^2 g(x)/2 - \lambda^2 g(y)/2} \left[\prod_{k=1}^{N} \left(1 + \frac{\lambda^2}{N} \Psi_k(x,y)\right) - 1\right] \\ &\qquad \times \exp\left(i\lambda^* V_N(x) - i\lambda^* V_N(y)\right) d\mu(x) d\mu(y), \end{aligned}$$

which implies

$$\mathbb{E}|J_N - \pi_N|^2 \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\prod_{k=1}^{N} \left(1 + \frac{\lambda^2}{N} \Psi_k(x,y)\right) - 1\right| d\mu(x) d\mu(y) = O(N^{-1/2}),$$

where the last relation was already proved in the estimation of $\mathbb{E}|\Gamma_N|^2$ above. As before, this implies that $J_{N^3} - \pi_{N^3} \to 0$ $\mathbb{P}$-a.s. and consequently, $I_{N^3} - \pi_{N^3} \to 0$ $\mathbb{P}$-a.s. On the other hand, if (1.7) holds with respect to to any interval $E \subset \mathbb{R}$ with positive μ-measure, then

$$\lim_{N\to\infty} \mu(E)^{-1} \int_E \exp\left(itV_N(x)\right) d\mu(x) = \gamma(t), \qquad t \in \mathbb{R},$$

where γ is the characteristic function of G. The last relation and a simple approximation argument yield for any $t, u \in \mathbb{R}$

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \exp\left(itV_N(x)\right) e^{-u^2 g(x)/2} d\mu(x) = \gamma(t) \int_{-\infty}^{\infty} e^{-u^2 g(x)/2} d\mu(x).$$

Thus π_{N^3} and consequently I_{N^3} converge $\mathbb{P}$-a.s. to

$$\gamma(\lambda^*) \int_{-\infty}^{\infty} e^{-\lambda^2 g(x)/2} d\mu(x),$$

which is the characteristic function of the vector with independent marginal distributions F and G appearing in (1.5), (1.7). Thus along the subsequence N^3, the vector (T_N, V_N) converges in distribution to the same vector. As in the first part of the theorem, this convergence holds along the whole sequence of integers, completing the proof of Theorem 1.

In conclusion we prove a claim made after Theorem 1, namely that if the size of the gap Δ_k between the intervals $[A_k, A_k + B]$ and $[A_{k+1}, A_{k+1} + B]$ remains constant or if the A_k are integers and

$$\Delta_k \uparrow \infty, \qquad \Delta_k = O(k^\gamma), \quad \gamma < 1/4 \tag{2.19}$$

then

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{iA_k x} \longrightarrow 0 \qquad \mu - \text{a.s.} \tag{2.20}$$

and thus in view of (1.13), relation (1.7) holds with G concentrated in 0, i.e. (1.5) holds with $\lambda_k(x) = 0$. In the case of constant Δ_k we have $A_k = Dk + D^*$ for some constants $D > 0$ and D^* and (2.20) is obvious by an explicit computation of the sum. In the second case it suffices to verify (2.20) with x replaced by $2\pi x$. Let us break the sum $\sum_{k=1}^N e^{2\pi i A_k x}$ into subsums

$$Z_{N,r} = \sum_{k\le N,\, A_{k+1}-A_k = r} e^{2\pi i A_k x}, \qquad r = 1, 2, \dots . \tag{2.21}$$

Since $A_{k+1} - A_k$ is nondecreasing and integer valued, $Z_{N,r}$ consists of M_r consecutive terms of $\sum_{k=1}^N e^{2\pi i A_k x}$ for some $M_r \ge 0$ and thus in the case $M_r \ge 1$ we have for some integer $P_r \ge 0$,

$$|Z_{N,r}| = \left| \sum_{j=0}^{M_r - 1} e^{2\pi i (P_r + jr)x} \right| = \left| \sum_{j=0}^{M_r - 1} e^{2\pi i j r x} \right| \le \frac{1}{|e^{2\pi i r x} - 1|} \le \frac{C}{\langle rx \rangle},$$

where C is an absolute constant and $\langle t \rangle$ denotes the distance of t from the nearest integer. From a well known result in Diophantine approximation theory (see e.g. Kuipers and Niederreiter [18], Definition 3.3. on p. 121 and Exercise 3.5 on page 130), for every $\varepsilon > 0$ and almost all x in the sense of Lebesgue measure we have $\langle nx \rangle \ge cn^{-(1+\varepsilon)}$ for some constant $c = c(x) > 0$ and all $n \ge 1$. This shows that

$Z_{N,r} = O(r^{1+\varepsilon})$ a.e. and since by (2.19) the largest r actually occurring in breaking $\sum_{k=1}^N e^{2\pi i A_k x}$ into a sum of $Z_{N,r}$'s is at most $C_1 N^\gamma$, we have

$$\left| \sum_{k=1}^N e^{2\pi i A_k x} \right| \le C_2 \sum_{r \le C_1 N^\gamma} r^{1+\varepsilon} = o(\sqrt{N}) \qquad \text{a.e.}$$

by $\gamma < 1/4$, upon choosing ε small enough.

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