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Finite and infinite vertex-transitive cubic graphs and their distinguishing cost and density*

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Dedicated to Marston Conder on the occasion of his 65th birthday.

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Abstract

A set S of vertices in a graph G with nontrivial automorphism group is *distinguishing* if the identity mapping is the only automorphism that preserves S as a set. If such sets exist, then their minimum cardinality is the *distinguishing cost* $\rho(G)$ of G. A closely related concept is the *distinguishing density* $\delta(G)$. For finite G it is the quotient of $\rho(G)$ by the order of G.

We consider connected, vertex-transitive, cubic graphs G and show that either $\rho(G) \leq 5$ or $\rho(G) = \infty$ and $\delta(G) = 0$ if G has one or three arc-orbits, or two arc-orbits and vertex-stabilisers of order at most 2.

For the case of two arc-orbits and vertex stabilizers of order > 2 we show the existence of finite graphs with $\rho(G) > 5$ and infinite graphs with $\delta(G) > 0$.

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We also prove that two well known results about finite, vertex-transitive, cubic graphs hold without the finiteness condition and construct infinitely many cubic GRRs.

Keywords: Distinguishing number, distinguishing cost, vertex-transitive cubic graphs, automorphisms, infinite graphs, asymmetrizing set.

Math. Subj. Class.: 05C15, 05C10, 05C25, 05C63, 03E10.

1 Introduction

This paper is concerned with automorphism breaking of graphs by vertex colorings. Following Albertson and Collins [1], we call a vertex coloring of a graph G *distinguishing* if the identity is the only automorphism of G that preserves it. The smallest number of colors needed is the *distinguishing number* $\mathrm{D}(G)$ of G. One says such a coloring *breaks* the automorphisms of G. When $\mathrm{D}(G) = 2$ each of the two colors induces a set of vertices which is preserved only by the identity automorphism. We call such sets *distinguishing*, but also use the term *asymmetrizing*, which was introduced in 1977 by Babai [2], when referring to results predating Albertson and Collins' paper [1].

The cardinality of a smallest distinguishing set of a graph G is the 2-*distinguishing cost*. It was introduced by Boutin [4] in 2008 and denoted $\rho(G)$. Clearly $0 < \rho(G) \leq |V(G)|/2$. Although we cannot talk of the cost unless we already know that $\mathrm{D}(G) = 2$, when it is clear from the context, we refer to $\rho(G)$ as the *distinguishing cost*, or simply as the *cost*, without adding that $\mathrm{D}(G) = 2$.

For 2-distinguishable graphs G the *distinguishing density* $\delta(G)$, or simply the *density*, is defined in Section 3. For finite G it is the quotient $\rho(G)/|V(G)|$.

Note that $\mathrm{D}(G)$ is 1 for asymmetric graphs and 2 for almost all other finite graphs, because almost all finite graphs that are not asymmetric have just one automorphism, a transposition of two vertices. Such graphs can be distinguished by coloring one of the two vertices black, and all other vertices white. This means that $\rho(G) = 1$ for almost all finite graphs that are not asymmetric.

Here we investigate the distinguishing cost and density of graphs of maximum valence 3, which we call *subcubic*. This is a rich class of graphs that is fully classified with respect to the distinguishing number, see [16], but little is known about its cost and density.

In Section 2 we construct finite, connected, subcubic graphs, whose costs take on numerous values between 1 and $|V(G)|/2$, and in Section 3 infinite, connected graphs of maximum valence 3 with various densities between 0 and $1/2$. With the exception of the obvious bounds $1 \leq \rho \leq |V(G)|/2$ for the cost, and $0 \leq \delta \leq 1/2$ for the density, there seem to be few restrictions on the values.

Beginning with Section 4 we focus on connected, vertex-transitive, cubic graphs. For them the situation is completely different and the main topic of interest. Hence, after Section 3 all graphs will be connected, vertex-transitive and cubic, unless it is otherwise stated or clear from the context.

For their investigation we classify them by the number of arc-orbits and treat the classes separately. *Arcs* are ordered pairs (u, v) of adjacent vertices u, v in G, and the *orbit of an*

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arc (u,v) under the action of $\mathrm{Aut}(G)$ is the set

$$O((u,v)) = \{(\alpha(u),\alpha(v)) \,|\, \alpha \in \mathrm{Aut}(G)\}.$$

Recall that to any two vertices u, w of a vertex-transitive graphs there is an $\alpha \in \mathrm{Aut}(G)$ such that $w = \alpha(u)$. Therefore any arc of the form (w, z) is in the orbit of $(u, \alpha^{-1}(z))$. If G is cubic, then there are only three possibilities for $\alpha^{-1}(z)$, and thus the number of arc-orbits in a vertex-transitive, cubic graph is $1, 2$ or 3.

A closely related, but weaker concept, is that of *edge-orbits*. One says two edges uv and xy are in the same edge-orbit if there exists an automorphism α that maps the unordered pair $\{u, v\}$ into the unordered pair $\{x, y\}$. The relationship between arc- and edge-orbits is described in Section 4.

Graphs with only one arc-orbit are called *arc-transitive*. Four of them are not 2-distinguishable. All other arc-transitive graphs G are 2-distinguishable with cost ≤ 5 unless G is the infinite 3-valent tree, which has infinite cost and zero density; see Theorem 6.1. If G has three arc-orbits, then each automorphism of G that fixes a vertex is the identity, so $\rho(G) = 1$.

For graphs with two arc-orbits we will show in Corollary 4.2 that one of the orbits consists of pairs (u, v) and (v, u), where the edges uv are independent and meet each vertex of G. Because *complete matchings* of a graph G are defined as sets of independent edges that meet each vertex of G, we call this orbit the *matching orbit* and its edges *matching edges*.

If one removes the matching edges, but not their endpoints from G, then the vertices in the remaining graph have valence 2. This implies that its connected components are cycles or 2-sided infinite paths. Because G is vertex-transitive, all components must be either 2-sided infinite paths or cycles of the same lengths. We call this orbit the *cycle-orbit*.

Let uv be a matching edge in a graph with two arc-orbits. Then u, v have no common neighbors unless G is the K_4, which is not 2-distinguishable. If x, y are the neighbors of u different from v, and w, z the neighbors of v different from u, then Figure 1 depicts the subgraph of G consisting of the edges ux, yx,uv,vw and vz. It need not be induced unless G has neither 3- nor 4-cycles. If there are 3-cycles, then they must be in the cycle-orbit and then both uxy and vwz are 3-cycles. If there are 4-cycles, then there may be a pair of independent edges between the sets $\{x, y\}$ and $\{w, z\}$. It is also possible that u and v are in the same cycle of the cycle-orbit.

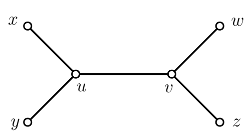

Figure 1: Edges incident with a matching edge uv in a graph with two arc-orbits.

The pairs of arcs $(u, x), (u, y)$ and $(v, w), (v, z)$ are in the cycle-orbit. Hence there exists an $\alpha \in \mathrm{Aut}(G)$ that fixes u and interchanges x, y, and a $\beta \in \mathrm{Aut}(G)$ that fixes v and interchanges w, z. If each automorphism that fixes u and interchanges x, y also interchanges w, z, then we call G *rigid*, and otherwise *flexible*. Note that G is rigid exactly when the order of the group induced by $\mathrm{Aut}(G)$ on the subgraph formed by a matching edge and its four incident edges is 2.

Other important concepts are the *girth* and the *motion* of a graph G. The girth $g(G)$ is the size of a smallest cycle of G, and the motion $\mathrm{m}(G)$ is the smallest number of vertices moved by any non-trivial automorphism of G.

Our results pertaining to cost and density and some of the ensuing questions can be summarized as follows:

Let G be a connected, vertex-transitive, cubic graph different from K_4, $K_{3,3}$, the cube and the Petersen graph.

1. *If G has three arc-orbits, then G has trivial vertex stabilizers and $\rho(G) = 1$.*

2. *If G is arc-transitive, then $\rho(G) \leq 5$, except for the infinite cubic tree, which has infinite distinguishing cost and density* 0. *See* Theorem 6.1.

   For the existence of certain infinite, arc-transitive cubic graphs of girth 6 see Question 6.9.

3. *If G is rigid with two arc-orbits, then $\rho(G) \leq 3$, see* Theorem 7.1.

4. *If G is finite, flexible with two arc-orbits and of girth* 3*, then $\rho(G) \leq 3$. See* Theorem 8.1.

   For the existence of such graphs compare Proposition 8.2.

5. *For two arc-orbit, flexible graphs G of girth* 4 *we construct families of connected finite graphs with cost* > 5 *and infinite graphs of density $\delta(G) = 1/\mathrm{m}(G) > 0$.*

   For both families we have $\delta(G) \leq 1/4$. See Theorems 9.10 *and* 9.11.

   We do not know whether other connected, two arc-orbit, flexible graphs G of girth ≥ 4 with positive density or finite cost > 5 exist. See Question 9.12 and Lemma 9.3.

   If they do not exist, then all finite or infinite, vertex-transitive, cubic graphs on at least 20 vertices have density $\leq 1/4$. See Questions 2.1 and 3.5.

6. *We also construct infinite, two arc-orbit, flexible, cubic graphs G of all girths with $\rho(G) = \infty$ and $\delta(G) = 0$. See* Theorem 7.6 *and* Lemma 9.3.

   We do not know whether two arc-orbit, flexible, cubic graphs G, whose cycle-orbits consist of 2-sided infinite paths, exist. See Question 7.5.

Furthermore, we prove a number of other results. In Section 2 we provide examples of infinite graphs with finite and infinite distinguishing cost. In Section 3 we define and give examples of the distinguishing density of infinite graphs. In these sections the graphs are not necessarily cubic or vertex-transitive. In Section 4 we show that edge-transitivity implies arc-transitivity not only for finite, but also for infinite, vertex-transitive, cubic graphs. Section 5 treats vertex-transitive, cubic graphs with three arc-orbits, that is, vertex-transitive graphs with trivial vertex stabilizers. Such graphs are called Graphical Regular Representations, GRRs for short. In Section 6 Theorem 6.2 extends a result about the existence of connected, vertex-transitive, cubic graphs of girth at most 5 to infinite graphs. Section 9 is concerned with flexible cubic graphs of girth 4. As a byproduct we construct a class of infinitely many cubic GRRs with two edge-orbits. The paper ends with a new characterization of Split Praeger–Xu graphs.

We conclude the introduction with the remark that Babai's paper [2] was not the only one on automorphism breaking that predates Albertson and Collins [1]. After Babai, it was notably Polat and Sabidussi [24, 25] and Polat [22, 23], who published deep and interesting results on the asymmetrization of graphs of any cardinality.

Moreover, the concept of asymmetrization, or distinguishing, extends to groups of permutations on a set. As most results on asymmetrizing sets of groups are also relevant for graphs, let us mention that Gluck [13] showed in 1983 that permutation groups of odd order can be asymmetrized by two colors, and that Cameron, Neumann, and Saxl [6] proved in 1984 that all but finitely many primitive permutation groups other than A_n, S_n can be asymmetrized by two colors. In 1997 Seress [28] classified the remaining ones, taking recourse to the classification of finite simple groups.

2 Cost

We continue with examples on bounds on the distinguishing cost for finite graphs and show that infinite graphs can have finite or infinite cost.

It is important to keep in mind that the distinguishing cost is only defined for graphs that are 2-distinguishable. In other words, the results on cost and density only hold for such graphs. For cubic graphs this is not a strong restriction, because the only vertex-transitive cubic graphs that are not 2-distinguishable are the K_4, the $K_{3,3}$, the cube and the Petersen graph, where the K_4 has distinguishing number 4, and the others 3.

If the distinguishing number of a graph G is 2, then $V(G)$ can be partitioned into two sets V_1, V_2, where the stabilizer of either one is the trivial subgroup. This means, if $\alpha \in \mathrm{Aut}(G)$ and $\alpha(V_i) = V_i$ for $i = 1$ or $i = 2$, then $\alpha = \mathrm{id}$. Recall from the introduction that either of the sets V_1 or V_2 is called distinguishing, and that the smallest possible size of such a set is the distinguishing cost $\rho(G)$ of G.

To fix ideas we shall always choose a minimum distinguishing set of G, color its vertices black and all others white. Thus $\rho(G)$ is the minimal number of black vertices needed to break all automorphisms.

Clearly $0 < \rho(G) \le |V(G)|/2$. The upper bound can easily be attained. The simplest example is a single edge. To construct larger graphs with $\rho(G) = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ we consider binary trees:

Let B_k be the binary tree of height k. B_k is defined as a rooted tree, whose root has valence 2, whose vertices of distance k from the root are leaves, and where all other vertices have valence 3. B_k has $2^{k+1} - 1$ vertices, $\rho(B_k) = 2^k - 1$ and $\delta(B_k) = 1/2 - 1/(2^{k+1} - 1) = \lfloor |V(G)|/2 \rfloor$. By attaching a path to the root one obtains a graph with smaller density. It can be made arbitrarily small.

For cost $|V(G)|/2$ we consider the graph G_k consisting of two binary trees B_k, whose roots are connected by an edge. Clearly $\rho(G_k) = 2^{k+1} - 1 = |V(G_k)|/2$.

If the graphs are vertex-transitive, then we have stricter bounds. In Theorem 6.1 we show that $\rho(G) \le 5$ for finite, connected, arc-transitive, cubic graphs that are different from K_4, $K_{3,3}$, the cube and the Petersen graph. Hence, for finite arc-transitive graphs G on at least 20 vertices $\delta(G) \le 1/4$.

There are 24 connected, vertex-transitive graphs of order < 20, see [27]. As a brief application of results of the paper we show that exactly five of these 24 graphs have density $> 1/4$: two are Möbius ladders, two circular ladders, and the fifth is the Heawood graph. A k-*Möbius ladder* consists of a cycle of length $2k$, together with k diagonals that connect opposite vertices of the cycle. The k diagonals are the rungs of the ladder.

A *circular ladder* or k*-ladder* is a prism over a k-cycle, that is, the Cartesian product of C_k by K_2. For $k \leq 9$ they have < 20 vertices. If they are 2-distinguishable, their cost is 3 by Theorem 7.1, and thus the density is $3/(2k)$. Since we are interested in density $> 1/4$ we can restrict attention to $k \leq 5$.

The 2-Möbius ladder is the K_4, the 3-Möbius ladder the $K_{3,3}$, the 4-Möbius ladder has density $3/8$ and the 5-Möbius ladder density $3/10$. The 2-ladder does not exist, the 3-ladder is the prism over a triangle and has density $1/2$, the 4-ladder is the cube and the 5-ladder has density $3/10$.

For the Heawood graph, which has girth 6 and 14 vertices we show in Section 6 that its cost is at most five. However, it is relatively easy to see that its cost is at least 4, and hence its density $> 1/4$.

The Petersen graph, the Heawood graph, the eight Möbius ladders, and the seven prisms of order < 20 account for 17 vertex-transitive cubic graphs of order < 20.

Two of the remaining seven graphs have girth 3. If one contracts their triangles to single vertices one obtains the K_4 and the $K_{3,3}$. By Theorem 8.1 the costs of the uncontracted graphs are 3 and the densities $1/4$ and $1/6$. Two other graphs are the crossed 3- and 4-ladders of densities $1/4$, see Theorem 9.11. The remaining three graphs have girth 6. One is the Pappus graph. It has 18 vertices, and its cost is 3 by the remark after Lemma 6.8. Another graph on 18 vertices has cost 1. It is the smallest GRR, i.e. graph with three arc-orbits, see Section 5. The last graph has 16 vertices, and thus cannot have three arc-orbits. By Corollary 4.2 it also cannot have two arc-orbits. Therefore it is arc-transitive. By Lemma 6.8 and the following remarks it has cost ≤ 3, because it is not the Heawood graph.

Question 2.1. Are there finite, connected, vertex-transitive, cubic graphs on at least 20 vertices with distinguishing densities $> 1/4$?

If such graphs exist, then they must be flexible by Lemma 5.1, Theorem 6.1 and Theorem 7.1.

The cost can be finite for infinite graphs. This was studied in [3], where it was shown that the cost $\rho(G)$ of connected, locally finite, infinite graphs G is finite if and only if $\mathrm{Aut}(G)$ is countable. (In this statement it is not needed to explicitly assume 2-distinguishability, because all locally finite graphs with countable automorphism group are 2-distinguishable, see [18].)

An example for an infinite graph with countable automorphism group and finite cost is the infinite ladder of Figure 2. It has distinguishing cost 3, as is easily verified.

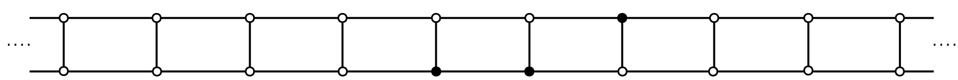

Figure 2: The infinite ladder with a distinguishing coloring.

Contrariwise, the closely related *chain of quadrangles* Q of Figure 3, also called *infinite crossed ladder*, is an example for an infinite, connected graph with uncountable automorphism group, and hence infinite distinguishing cost. Q can be formally defined as a graph consisting of the 2-sided infinite paths $\ldots u_{-2}u_{-1}u_0u_1u_2\ldots$ and $\ldots v_{-2}v_{-1}v_0v_1v_2\ldots$, with the edges $u_{2n}v_{2n+1}, v_{2n}u_{2n+1}$, where $n \in \mathbb{Z}$.

To see that $\mathrm{Aut}(Q)$ is uncountable, observe that we obtain an automorphism of Q for any integer n by simultaneously interchanging u_{2n-1} with v_{2n-1}, and u_{2n-1} with u_{2n}, while fixing all other vertices. This automorphism interchanges the pair of matching edges $u_{2n-1}u_{2n}$ and $v_{2n-1}v_{2n}$. Because Q has infinitely many such pairs of matching edges, and because the edges in each pair can be interchanged independently of the other pairs, $\mathrm{Aut}(Q)$ has at least $2^{\aleph_0}$ elements, and is thus uncountable.

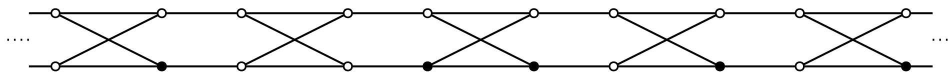

Figure 3: The chain of quadrangles Q with a distinguishing coloring.

A distinguishing 2-coloring is displayed in the figure. The black vertices are u_0 and u_{2n+1}, where n is an integer different from -1. Because u_0, u_1 are the only adjacent black vertices each automorphism must map the set $\{u_0, u_1\}$ into itself. As there is a black vertex of distance 2 from u_1, namely u_3, but no black vertex of distance 2 from u_0, the vertices u_0, u_1, u_3 are fixed individually. It is easy to see that this implies that all four-cycles are fixed setwise, and thus also all pairs of matching edges $u_{2n-1}u_{2n}$ and $v_{2n-1}v_{2n}$. Each such pair has exactly one black vertex, hence all of its vertices must be fixed. Because $V(Q)$ is partitioned by the sets of endpoints of these pairs of matching edges the coloring is distinguishing.

3 Density

To define the *distinguishing density* we follow [17] and begin with the density of sets of vertices. Let S be a set of vertices of a graph G, $v \in G$, and $B_G(v,n) = \{w \in G : d(v,w) \leq n\}$ be the ball of radius n with center v. If no ambiguity arises, we also write $B(v,n)$ for $B_G(v,n)$. Then

$$\delta_v(S) = \limsup_{n\to\infty} \frac{\mid B(v,n) \cap S \mid}{\mid B(v,n) \mid}$$

is the *density* of S at v. If $\delta_v(S)$ exists for all vertices, which is the case for locally finite graphs, then

$$\delta(S) = \sup\{\delta_v(S) : v \in V(G)\}$$

is the density of S in G. Finally,

$$\delta(G) = \inf\{\delta(S) \mid S \text{ is a distinguishing set of } G\}$$

is the *distinguishing density* of G.

Note that $\delta(G) = \rho(G)/|V(G)|$ for finite graphs and zero for infinite graphs with finite distinguishing cost.

Under certain conditions all values of $\delta_v(G)$, $v \in V(G)$, are equal. Then $\delta(G)$ is equal to $\delta_v(G)$ for arbitrarily chosen $v \in V(G)$. We have two such conditions, one for density zero, and a stricter one for positive density. First the condition for density zero see [17, Lemma 1].

Lemma 3.1. *Let G be a connected graph and $v, w \in V(G)$. Suppose there is a constant c such that for all $n \in \mathbb{N}$ we have $|B(w, n+1)| < c|B(w, n)|$. If G has distinguishing density zero at v, then G has distinguishing density zero.*

For an example of a 2-distinguishable graph that does not have distinguishing density zero, but nevertheless distinguishing density zero at some vertex v, see [17]. That graph does not satisfy the conditions of the lemma.

The infinite tree T_k, where each vertex has degree k, is an example of a graph with distinguishing density zero. For the proof and for many other examples of graphs with distinguishing density zero we refer to [17].

For positive distinguishing density we have the following lemma.

Lemma 3.2. *Let G be a connected graph, $S \subseteq V(G)$, $v \in V(G)$, and*

$$\limsup_{n\to\infty} \frac{|B(v,n)|}{|B(v,n+1)|} = 1. \tag{3.1}$$

If there is a vertex u in G with $\delta_u(S) = a \geq 0$, then $\delta_w(S) = a$ for all $w \in V(G)$.

Proof. We first observe that (3.1) implies that

$$\limsup_{n\to\infty} \frac{|B(v,n)|}{|B(v,n+d)|} = 1, \tag{3.2}$$

for all natural numbers d.

Let $\delta_v(S) = a$, $w \in G$, and $d = d(v, w)$. Clearly

$$|B(v,n)| \leq |B(w,n+d)| \leq |B(v,n+2d)|,$$

and hence

$$\limsup_{n\to\infty} \frac{|B(v,n)|}{|B(v,n+2d)|} \leq \limsup_{n\to\infty} \frac{|B(w,n+d)|}{|B(v,n+2d)|} \leq \limsup_{n\to\infty} \frac{|B(v,n+2d)|}{|B(v,n+2d)|} = 1.$$

By (3.2)

$$\limsup_{n\to\infty} \frac{|B(w,n+d)|}{|B(v,n+2d)|} = 1. \tag{3.3}$$

Clearly

$$|B(v,n) \cap S| \leq |B(w,d+n) \cap S| \leq |B(v,n+2d) \cap S|,$$

and hence

$$\frac{|B(v,n) \cap S|}{|B(v,n)|} \frac{|B(v,n)|}{|B(v,n+2d)|} \leq \frac{|B(w,n+d) \cap S|}{|B(w,n+d)|} \frac{|B(w,n+d)|}{|B(v,n+2d)|} \leq \frac{|B(v,n+2d) \cap S|}{|B(v,n+2d)|}.$$

By (3.2) the $\limsup$ of the term on the left is equal to the $\limsup$ of the term on the right, and by (3.3)

$$\delta_w(S) = \limsup_{n\to\infty} \frac{|B(w,n+d) \cap S|}{|B(w,n+d)|} = \limsup_{n\to\infty} \frac{|B(v,n) \cap S|}{|B(v,n)|} = \delta_v(S)$$

for all $w \in V(G)$. As we assume the existence of a vertex $u \in V(G)$ with $\delta_u(S) = a$, we infer that $\delta_w(S) = a$ for all $w \in V(G)$, i.e. $\delta(G) = a$. $\square$

Lemma 3.2 will be useful in Subsection 3.1. Interestingly we do not need it for the chain of quadrangles.

Lemma 3.3. *The distinguishing density of the chain of quadrangles Q is $1/4$.*

Proof. In Section 2 we already observed that any distinguishing set S of Q must contain at least one vertex in each pair of matching edges connecting two quadrangles. It is easy to see that this implies that $\delta_v(S) \geq 1/4$ for arbitrary S and v. Hence $\delta(G) \geq 1/4$. If we choose for S the set of black vertices in Figure 3, then $\delta_v(S) = 1/4$ for all v, and therefore the distinguishing density of Q is indeed $1/4$. □

Let us note nonetheless that Q satisfies the conditions of Lemma 3.2, because

$$|B_Q(v, n+1)| = |B_{Q)}(v, n)| + 4$$

for $n \geq 4$.

We wish to add that it is relatively easy to construct graphs with non-zero distinguishing density that are not vertex-transitive, see [17].

3.1 Upper bounds for the density

For infinite graphs the situation is similar. If we take a one-sided infinite path P and connect each of its vertices by an edge to the root of a copy of the binary tree B_k, then we obtain a graph, say H_k, of distinguishing cost $\rho(H_k) = 1/2 - 1/2^{k+1}$. In this way we can reach densities that are arbitrarily close to $1/2$, but not $1/2$.

To reach density $1/2$ we have to be more careful. Let $P_{1/2}$ be the graph obtained from the one-sided infinite path $P = v_1v_1v_2\dots$ as follows: Connect v_1 to the root of B_1 by an edge, then each of v_2, v_3 by an edge to the root of distinct copies of B_2. Continue by connecting v_4, v_5, v_6 to the roots of distinct copies of B_3, and so on.

Lemma 3.4. $\delta(P_{1/2}) = 1/2$.

Proof. Each minimal 2-distinguishing coloring of $P_{1/2}$ leaves P and its neighbors white. The remaining vertices come in pairs $\{u, v\}$ of vertices of equal distance from v_1, where one vertex has to be colored black, and the other white. Let S be the set of black vertices of such a distinguishing coloring. Then $|B(v_1, n) \cap S| = (|B(v_1, n)| - 2n - 1)/2$, and $|B(v, n) \cap S|/|B(v_1, n)| = 1/2 - (2n - 1)/(2 \cdot |B(v, n)|)$. The supremum of the latter expression is $1/2$ if the growth of $B(v_1, n)$ is more than linear. This is achieved by our construction. It is not hard to see that $P_{1/2}$ satisfies the conditions of Lemma 3.2. We conclude that $\rho(P_{1/2}) = 1/2$. □

It seems that one can construct connected, subcubic infinite graphs with arbitrary density between 0 and $1/2$ in similar ways.

But we know of no infinite, connected, vertex-transitive, cubic graph with density larger than $1/4$.

Question 3.5. Are there infinite, connected, vertex-transitive, cubic graphs with distinguishing densities larger than $1/4$?

As we already observed after Question 2.1, any examples of such graphs must be flexible.

4 Arc- versus edge-orbits

This section describes the relationship between arc- and edge-orbits, respectively between arc- and edge-transitivity.

Edge-transitive graphs need not be vertex-transitive, as the $K_{2,3}$ shows. But even a vertex-transitive graph G that is also edge-transitive need not be arc-transitive. The existence of such graphs was shown in 1966 by Tutte [33]. He also proved that any finite graph of this type must be regular of even degree. The first examples were given by Bouwer [5], and the smallest graph of this type is the Doyle graph. It is quartic and has 27 vertices, see [15] and [26] .

By Tutte's result each finite, vertex-transitive, cubic graph that is edge-transitive is arc-transitive. In 1989 Thomassen and Watkins [30] extended Tutte's result to infinite graphs of subexponential growth. They proved that each vertex- and edge-transitive graph of odd valence with subexponential growth is 1-transitive. For infinite cubic graphs we do not need the growth condition:

Theorem 4.1. *Let G be a finite or infinite, connected, vertex-transitive, cubic graph. If it is edge-transitive, then it is also arc-transitive.*

Proof. Let G satisfy the assumptions of the Theorem. If it is not arc-transitive, then it has two or three arc-orbits.

Let u be an arbitrary vertex of G with the incident edges uv, uw, ux. Suppose first that G has three arc-orbits. By edge-transitivity there is an $\alpha \in \mathrm{Aut}(G)$ that maps uv into uw and a $\beta \in \mathrm{Aut}(G)$ that maps uw into ux. Because G has three arc-orbits $\alpha(u) = w$, $\alpha(v) = u$, and $\beta(w) = u$, $\beta(u) = x$. Hence $\beta\alpha$ maps the arc (u, v) into the arc (u, x), a contradiction.

Suppose now that G has two arc-orbits, where (u, w) and (u, x) are in the same arc-orbit. Let O_1 be the arc-orbit of (u, v), O_2 the orbit of (u, w), and D be the digraph on $V(G)$ whose arcs are the arcs of O_2. Because G is edge-transitive at least one of the arcs (u, v) or (v, u) must be in O_2. By assumption $(u, v) \notin O_2$, hence $(v, u) \in O_2$. Therefore, in D, the vertex u has one incoming arc and two outgoing arcs. By vertex-transitivity this holds for all vertices of G. But then G cannot be finite, because the total number of incoming arcs in D has to be the same as the total number of outgoing arcs.

Hence G is infinite. It cannot be a tree, otherwise it would be the infinite cubic tree T_3, which has only one arc-orbit. Therefore G has a cycle, say C. By vertex-transitivity we can assume that u is in C. Because, u has two outgoing arcs in D, one of them, say (u, w), must be in C. Continuing this argument one sees that C is a directed cycle in D. By edge-transitivity each arc is in directed cycle in D.

Let C' be a cycle containing (u, x). Clearly both C and C' contain the arc (v, u). Let P be the longest path in $C \cap C'$ that contains (v, u). One of its endpoints is u, let the other one be z. Clearly z has one outgoing arcs in D, and two incoming arcs, which is not possible. □

Corollary 4.2. *Let G be a finite or infinite, connected, vertex-transitive, cubic graph. If it has two arc-orbits, then one of the arc-orbits consists of pairs (u, v), (v, u), where the edges uv are independent and meet each vertex of G.*

Proof. Let G be a finite or infinite, connected, vertex-transitive, cubic graph with two arc-orbits and u be an arbitrary vertex of G with the incident edges uv, uw, ux. As in the proof of Theorem 4.1 we assume that (u, v) is in O_1 and that $(u, w), (u, x)$ are in O_2. In the proof of the theorem the assumption that (v, u) was in O_2 led to a contradiction to the number of arc-orbits of G. Hence, $(v, u) \in O_1$. By vertex-transitivity each vertex $y \in V(G)$ meets exactly two arcs of the form $(y, z), (z, y) \in O_1$. Clearly the corresponding edges yz are independent and meet each vertex of G. □

Corollary 4.3. *Let G be a finite or infinite, connected, vertex-transitive, cubic graph. If it has two arc-orbits, then it has two edge-orbits, but if it has three arc-orbits, it may have two or three edge-orbits.*

Proof. By definition each edge-orbit consists of the undirected arcs of the union of one or more arc-orbits. We have seen that G has only one arc-orbit if G has only one edge-orbit. Hence, if G has two arc-orbits, they must be in different edge-orbits, and thus G has two edge-orbits.

If G has three arc-orbits, then it has two or three edge-orbits, because the case of one edge-orbit is not possible by Theorem 4.1. □

Recall from the introduction that we called O_1 the matching orbit. The other orbit was called cycle-orbit. By definition O_1 and O_2 are arc-orbits, but by the corollary they correspond to edge-orbits. By abuse of language we do not distinguish between O_1 or O_2 as and arc- or edge-orbits. In this sense O_2 is the disjoint union of cycles of the same length or of 2-sided infinite paths.

5 Three arc-orbits

Let G be a connected, vertex-transitive, cubic graph with three arc-orbits. If we fix a vertex v, then all neighbors of v are also fixed. As G is connected, this implies that all vertices of G are fixed if one is fixed. To break its automorphisms it suffices to color one vertex black and leave all others white. The distinguishing cost is 1.

Lemma 5.1. *The distinguishing cost of connected, vertex-transitive, cubic graphs with three arc-orbits is* 1.

Recall that Graphical Regular Representations, or GRRs, are vertex-transitive graphs with trivial vertex stabilizers. They have been widely investigated and although GRRs are abundant, it is interesting to explicitly describe special classes.

From Corollary 4.3 we know that the case of three arc-obits allows two or three edge-orbits. The smallest example of a GRR has 18 vertices. It has two edge-orbits and girth 6. The smallest example for the case of three edge-orbits is the truncated cuboctahedron. It is the skeleton of an Archimedean solid that was already described by Kepler, where each vertex is in one cube, one hexagon and one octagon. For both graphs we refer to a list of cubic GRRs with at most 120 vertices from 1981 by Coxeter, Frucht and Powers, see [8].

A series of infinitely many such graphs was constructed by Godsil in 1983 [14], the smallest of order $19!/2$. In Section 9 we also construct infinitely many such graphs, see Corollary 9.5. The smallest has 48 vertices and is also listed in [8].

Let us note in passing that the same publication lists two GRRs of girth 5, both of them on 110 vertices, whereas there is only one finite 2-distinguishable arc-transitive cubic graph of girth 5, the dodecahedron, and no infinite one. See Theorem 6.2.

We continue with graphs with one and two arc-orbits. Clearly the cost in these cases is at least 2.

6 One arc-orbit

By definition such graphs are arc-transitive. Tutte [31] also calls them *symmetric*, but the notation is not uniform and *symmetric* is also used for edge-transitive graphs that are not arc-transitive. To avoid confusion we only speak of arc-transitive graphs. But we need a refinement of the concept.

Following Tutte [31] we call a sequence of distinct vertices $v_0, v_1, \dots, v_s \in V(G)$ an *s-arc* if $v_i v_{i-1} \in E(G)$ for $1 \leq i \leq s$, but $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. Then G is *s-arc-transitive* if $\mathrm{Aut}(G)$ is transitive on the set of all s-arcs on G. Moreover, we call G *s-arc-regular* if for any two s-arcs $v_0 v_1 \dots v_s$ and $w_0 w_1 \dots w_s$ there is a unique automorphism φ that maps $v_0 v_1 \dots v_s$ into $w_0 w_1 \dots w_s$ and respects the order of the vertices.

In this section we shall prove the following theorem.

Theorem 6.1. *Let G be an arc-transitive, cubic graph different from K_4, $K_{3,3}$, the cube and the Petersen graph. If it has finite girth, then $\rho(G) \leq 5$, otherwise it is the infinite cubic tree T_3, which has infinite distinguishing cost and distinguishing density* 0.

We will break up the proof of Theorem 6.1 into several parts, but begin with the remark that there is only one connected acyclic cubic graph. It is the infinite cubic tree T_3, of which we already mentioned that it has distinguishing density zero.

It thus remains to prove the theorem for graphs with finite girth. We begin with a structure theorem about graphs of girth at most 5. For girth 3 and 4 the theorem is folklore, for finite graphs of graphs of girth 5 it was shown by Glover and Marušić [12]. Here is a concise proof for all cases.

Theorem 6.2. *The only connected, arc-transitive, cubic graphs of girth at most* 5 *are K_4, $K_{3,3}$ the cube, the Petersen graph and the dodecahedron.*

Proof. Let G be a connected, arc-transitive, cubic graphs of girth at most 5. Because we forbid multiple edges the smallest girth is 3 and the arcs incident to an arbitrary vertex induce a $K_{1,3}$. Let a, b, c be the arcs incident with a vertex v of G. We show first that there is an automorphism α of G that rotates a, b, c. In other words, there is an automorphism α whose cycle representation of its action on $\{a, b, c\}$ is (abc) or (acb). If this were not the case, there would exist automorphisms whose actions on $\{a, b, c\}$ are of the type (ab), (ac), because of arc-transitivity. Then their product $(ab)(ac) = (abc)$ is the desired rotation.

Let x, y, z be the endpoints of a, b, c different from v, and let the notation be chosen such that there exists an automorphism α of G that rotates the arcs a, b, c and their endpoints.

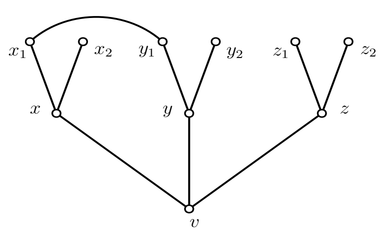

Figure 4: Basic structure for girth 5.

Girth 3. If G has a triangle we can assume without loss of generality that it is vxy. By applying α twice we see that G also contains the edges yz and zx. Hence $G = K_4$.

Girth 5. We first observe that the neighbors of x, y, z that are different from v are distinct. Let them be x_1, x_2, y_1, y_2, z_1, z_2 and vxx_1y_1y a pentagon, see Figure 4.

Rotating clockwise it is seen that there is a pentagon $vy\alpha(x_1)z_1z$, where $\alpha(x_1) \in \{y_1, y_2\}$. Suppose $\alpha(x_1) = y_1$. Notice that we have two pentagons that share two edges. The rotation moves z_1 into a neighbor of x. It must be x_2, because we have no triangles. By further rotations we obtain the 6-cycle $x_1y_1z_1x_2y_2z_2$, see Figure 5. Clearly G is the Petersen graph.

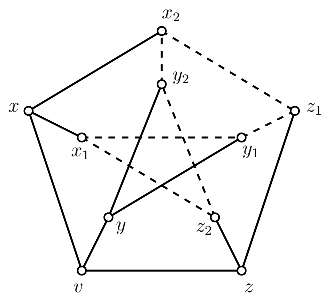

Figure 5: The Petersen graph.

Suppose now that no two pentagons share two edges. Then $\alpha(x_1) = y_2$, which implies $\alpha(x_2) = y_1$. Furthermore, we can choose the notation such that $\alpha(y_1) = z_2$, and $\alpha(y_2) = z_1$. Thus $z_2 = \alpha(y_1) = \alpha^2(x_2)$, and hence $\alpha(z_2) = x_2$ and $\alpha(z_1) = x_1$. By rotation we obtain the pentagons vyy_2z_2z and vzz_1x_2x from vxx_1y_1y. Let H be the union of these pentagons, see Figure 6. Note that v is in three pentagons, hence, by vertex-transitivity, this holds for all vertices. More important, any two incident arcs determine exactly one pentagon.

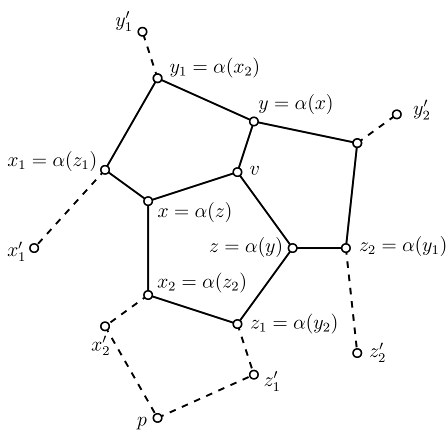

Figure 6: The subgraph H (solid lines).

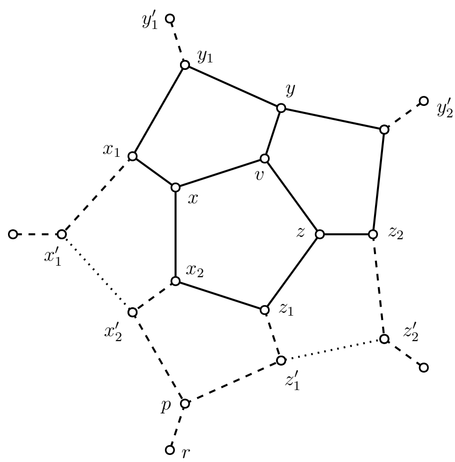

Figure 7: The graph H'.

The vertices of valence 2 in H form the set $S = \{x_1, y_1, y_2, z_2, z_1, x_2\}$. If $u \in S$, let u' be the third neighbor of u. Let $S' = \{x_1', y_1', y_2', z_2', z_1', x_2'\}$. By the girth condition and the condition that two pentagons share only one edge we infer that the sets S' and H are disjoint and that any two vertices of S' distinct.

Now consider x. It is in two pentagons in H, but has to be in a third in G. Neither x_1x_1' nor x_2x_2' can be in the first two pentagons, hence $x_1x_2' \in E(G)$. Similarly we find that $z_1'z_2'$ and $y_1'y_2'$ are in $E(G)$. The new graph, say H', is depicted in Figure 7.

Consider the path $x_2'x_2z_1z_1'$. Its edges must be in a pentagon that is not in H', which is only possible if there exists a vertex, say p, that is adjacent to x_2' and z_1'. Because G has girth 5 the vertex p cannot be a neighbor of x_1', nor of z_2', and because G is cubic it cannot be any vertex of H'.

Now consider the neighbors of x_1', p and z_2' that are not in $V(H') \cup \{p\}$, and denote them by q, r, s. Because of the girth condition and because G is cubic they are pairwise distinct and not in $V(H') \cup \{p\}$.

Finally, because all vertices have to be in three pentagons we find four additional edges that complete the graph to a dodecahedron. □

Corollary 6.3. *The only* 2*-distinguishable, finite or infinite, connected, arc-transitive, cubic graph of girth at most* 5 *is the dodecahedron. Its distinguishing cost is* 3.

Proof. We already know that $K_4, K_{3,3}$, the cube and the Petersen graph are not 2-distinguishable. Hence, by the theorem, the only 2-distinguishable graph of girth at most five is the dodecahedron. It is easily seen that its distinguishing cost is 3. □

For girth > 5 we will invoke a result of Tutte for finite graphs and its extension to infinite graphs by Djokovic and Miller. First the result from [32] for finite graphs.

Theorem 6.4. *Let G be a connected, finite, arc-transitive, cubic graph. Then G is s-arc-regular for some $s \leq 5$.*

And now the extension to infinite graphs from [9].

Theorem 6.5. *Every connected, infinite, arc-transitive, cubic graph is s-arc-regular for some $s \leq 5$ with the exception of the infinite cubic tree.*

We begin with a result for 1-arc-regular graphs.

Lemma 6.6. *Each connected, 1-arc-regular cubic graph has distinguishing cost 2.*

Proof. Let G be a connected, 1-arc-regular cubic graph. Then the action of $\mathrm{Aut}(G)$ on the arcs incident with an arbitrary vertex v cannot have involutions. To see this, let a, b, c be the arcs incident with v, and $\alpha \in \mathrm{Aut}(G)$ induce the involution (cb) on $\{a, b, c\}$. Then $\alpha(a) = a$, which contradicts 1-regularity.

Furthermore, by Theorem 6.2 the girth of G must be at least 5, because the graphs K_4, $K_{3,3}$, the cube, the Petersen graph and the dodecahedron are not 1-arc-regular. We now choose two vertices u, v of distance 2 in G and color them black. Because $g(G) > 4$ there is a unique vertex w that is adjacent to both u and v. Therefore any color preserving automorphism α fixes w. If it interchanges u with v, then it interchanges two arcs incident with w and fixes the third, and thus induces an involution on the arcs incident with w. If α fixes u, then it fixes the arc uw and is the identity by 1-arc-regularity. □

The existence of such graphs is guaranteed by a result of Frucht [11], who provided an example of a 1-arc-regular cubic graph of girth 12 with 432 vertices. It can be embedded in a surface of genus 55 so as to form a map of 108 dodecagons.

For the girth of s-arc-regular cubic graphs we will use the bound

$$2s \leq g(G) + 2 \tag{6.1}$$

from [31].

We first consider girth 7 and then girth 6.

Lemma 6.7. *Let G be a connected, s-arc-regular, cubic graph of girth at least 7. If $s = 1$, then $\rho(G) = 2$, otherwise $\rho(G) \leq 3$, unless $s = 4$ and $g(G) = 7$, then $\rho(G) \leq 4$.*

Proof. Because of arc-transitivity we can invoke Theorems 6.4 and 6.5. They imply that our graphs are s-arc-regular for some $s \leq 5$.

By Lemma 6.6 we can assume that $s > 1$. For $s = 2$ or 3 we choose a path $uxvw$ in G. This is possible because the girth is > 6. We color u, v, w black as visualized in Figure 8. Each color preserving automorphism φ fixes u because it is the only black vertex without black neighbors. As v and w have different distances from u, they are also fixed. Hence φ fixes the s-arcs ux, uxv and $uxvw$, where $s = 2, 3$, respectively. By s-arc-regularity φ is the identity.

Now, let $s = 4$. For girth $g > 7$ we choose a path $uxyvw$ and color u, v and w black as in Figure 8. Then we argue as before to prove that the 4-arc $uxyvw$ is fixed by all color preserving automorphisms. If the girth is 7 this coloring allows that both v and w have distance 3 from u. In this case it suffices to color y black to fix the 4-arc $uxyvw$ by all color preserving automorphisms.

For $s = 5$ we first observe that the girth is at least 8 by Equation (6.1). We choose a path $uxyzvw$ of length 5 and color u, v and w black. If the girth is different from 9 this coloring fixes the 5-arc $uxyzvw$. If the girth is 9, then v, w could be interchanged by color preserving automorphisms. □

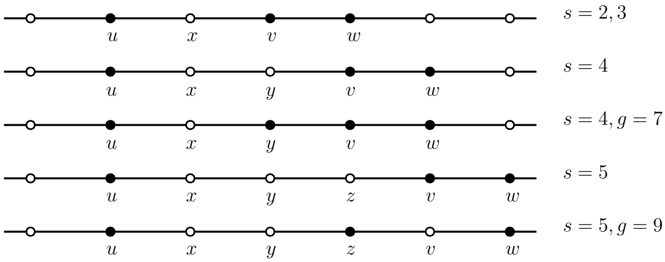

Figure 8: Colorings of s-arcs.

Lemma 6.8. *Let G be a connected, arc-transitive, cubic graph of girth 6. Then G is at most 4-arc-regular. If G is 1-arc-regular, then $\rho(g) = 2$, if G is 2 or 3-arc-regular, then $\rho(G) \leq 3$, otherwise $\rho(G) \leq 5$.*

Proof. That G is at most 4-arc-regular follows from Equation 6.1. When G is 1-arc-regular we invoke Lemma 6.6 and when G is 2- or 3-arc-regular we can use the same coloring as in the proof of Lemma 6.7.

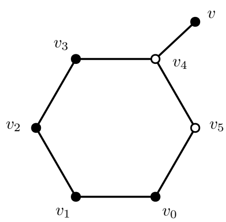

Figure 9: Coloring for $s = 4$ and girth 6.

If G is 4-arc-regular we choose an arbitrary 6-gon $v_0 \cdots v_5$ and color $v_0, \ldots, v_3$ black as well as the neighbor of v_4 that is not in the cycle. Let this neighbor be v, see Figure 9. Clearly v cannot be adjacent to any vertex of the cycle, because G has girth 6. Hence v is fixed by the coloring. Furthermore, v cannot have distance 2 from v_0, otherwise G would have a cycle of length 5. As v has distance 2 from v_3, this implies that the coloring fixes v_3. Invoking the girth condition we see that v_4 is also fixed, because it is on a path of length 2 between the fixed vertices v_3 and v. This fixes the entire 4-arc $v_0v_1v_2v_3v_4$. By 4-arc-regularity this coloring is distinguishing. □

Together with Corollary 6.3, Lemma 6.7 and Theorem 6.5 this also completes the proof of Theorem 6.1.

Let us mention that, except for the interchange of black and white, our coloring for the 4-arc-regular case is the same as that for a graph described in [16, Theorem 7.3].

We also observe that there exists only one finite, connected, 4-arc-regular, cubic graph of girth 6. It is the Heawood graph, see [7, 10], also known as Tutte's 6-cage [31]. It is not know whether there exist infinite, connected, 4-arc-regular, cubic graphs of girth 6.

Similarly, by [7] there are only two finite, connected, 3-arc-regular cubic graphs of girth 6, namely the Pappus graph and the generalized Petersen graph $GP(10, 3)$.

Question 6.9. Are there any infinite, 3-arc- or 4-arc-regular, cubic graphs of girth 6?

7 Two arc-orbits

Now we turn to connected, vertex-transitive, cubic graphs G with two arc-orbits, say O_1 and O_2. We choose the notation such that O_1 is the matching orbit and O_2 the cycle-orbit. By the *elements* of O_1 and O_2 we mean their connected components, and recall that, by Lemma 4.2, the case of two arc-orbits coincides with the case of two edge-orbits. Hence, in this case we need not distinguish between arc- and edge-orbits, but we need the following observation about the action $\mathrm{Aut}(G)$ on the elements of $O1$ and O_2.

The group of automorphisms of G acts on the elements of O_2 by reflections and vertex-transitively, that is, $\mathrm{Aut}(G)$ induces the full group of automorphisms on each element of O_2. The full groups of automorphisms of finite cycles are called *dihedral*, and the full group of 2-sides infinite paths is called *infinite dihedral*. It acts by reflections and translations. On the elements of O_1 the group $\mathrm{Aut}(G)$ acts by reflections.

First a Theorem about connected, vertex-transitive, cubic graphs with two arc-orbits that are rigid. Recall that graphs with two arc-orbits are rigid if the order of the group induced by $\mathrm{Aut}(G)$ on the subgraph formed by a matching edge and its four incident edges is 2.

Theorem 7.1. *The distinguishing cost of connected, vertex-transitive, rigid cubic graphs G with two arc-orbits is* 2*, unless G is an infinite ladder, a k-ladder or a k-Möbius ladder with $k > 3$. Then $\rho(G) = 3$.*

Proof. Let G be a connected, vertex-transitive, rigid cubic graph with matching orbit O_1 and cycle-orbit O_2. Suppose there is a quadrangle $uvwz$, where uv and wz are in O_1. If we color u, v, w black and leave all other vertices white, then w is fixed by all color preserving automorphism α, and hence also u, because uv is a matching edge. Hence α also fixes w, and therefore the element of the cycle-orbit that contains uw, say A.

If an automorphism α of a rigid graph fixes an element A of O_2, be it a cycle or a 2-sided infinite path, then α fixes all neighboring elements of A in O_2, and because G is connected all elements of O_2, and thus the entire graph G.

It is easily seen that the only graphs that satisfy the assumptions of the lemma and that contain such a quadrangle $uvwz$ are an infinite ladder, or a k-ladder, resp. a k-Möbius ladder. As the 3-ladder is the cube and the 3-Möbius ladder the $K_{3,3}$, both of which are not 2-distinguishable, we have to require that $k > 3$.

In all other cases one considers a matching edge uv together with a neighbor w of v, and colors u and w black. As there is no quadrangle $uvwz$ in G, the vertex v is the only common neighbor of u and w, and hence fixed. Now we conclude as before that the coloring is distinguishing. □

We have two corollaries, one about the order of the vertex-stabilizers of rigid and of flexible graphs, and one about planar graphs.

Corollary 7.2. *Let G be a connected, vertex-transitive, cubic graph with two arc-orbits. When G is rigid, then the order of its vertex-stabilizers is* 2*, otherwise, that is when G is flexible, it is at least* 4*.*

Proof. The validity of the assertion for rigid G is clear by the proof of the theorem, and for flexible G it follows directly from the definition. □

Corollary 7.3. *Let G be a 3-connected, planar, vertex-transitive, cubic graph with two arc-orbits. Then G is rigid and $\rho(G) = 2$ unless G is a k-ladder or a k-Möbius ladder with $k > 3$.*

Proof. Let G satisfy the assumptions of the corollary. By Whitney's theorem, which by [29] also holds for infinite graphs, G is uniquely embeddable into the plane in the sense that any automorphism that reverses the order of the edges incident with an arbitrary vertex reverses this order for all vertices. Hence G is rigid, and $\rho(G) = 2$ by Theorem 7.1. □

For flexible cubic graphs we have the following lemma.

Lemma 7.4. *Let G be a connected, vertex-transitive, flexible cubic graph with two arc-orbits, say O_1 and O_2, where O_2 consists of finite cycles. Then:*

(i) *One arc-orbit of G, say O_1, consists of arcs (u,v) and (v,u), where the edges uv form a complete matching.*

(ii) *The number of cycles in O_2 is a least 2.*

(iii) *O_2 consists of finite cycles of equal lengths without diagonals.*

(iv) *The number of edges between pairs of adjacent cycles is either 1 for all pairs, or 2 for all pairs.*

(v) *If the number of edges between adjacent cycles is 2, then the edges between neighboring cycles connect pairs of opposite vertices.*

(vi) *If the length ℓ of the cycles in the cycle-orbit is at least 6, then $g(G) = \ell$.*

Proof. Item (i) is Lemma 4.2. As usual we let O_1 be the matching orbit.

For (ii) we begin with the case when O_2 has only one connected component. Suppose it is the cycle $C_n = u_0u_1 \dots u_{n-1}$ of length n. By (i) each vertex of C_n is incident with exactly one edge of O_1. Let u_0u_j be the edge of O_1 that is incident with u_0, and α be the reflection of O_2 that fixes u_0. Clearly $\alpha(u_j) = u_{n-j}$. Because G is cubic, $u_0u_j = u_0u_{n-j}$, which is only possible if n is even, say $n = 2k$, and $j = k$. By vertex-transitivity the other edges of O_1 are of the form u_iu_{i+k}. The resulting graph is the k-Möbius ladder. As we have already seen, for $k = 3$ it is the $K_{3,3}$, which is not 2-distinguishable, and for $k > 3$ it is rigid. This proves (ii).

So we turn to the case when O_2 has at least two cycles, and all cycles of O_2 have the same length, say ℓ. If a cycle has a diagonal, then this diagonal must be in O_1. By vertex-transitivity this implies all elements of O_1 are diagonals, but then G is disconnected. This proves (iii).

Now, let k be the number of edges between neighboring cycles. Clearly k is a divisor of ℓ. When $k = \ell$, then G is the prism over the k-cycle and rigid.

When $3 \le k < \ell$, consider a cycle $u_0u_1 \dots u_\ell$ in O_2 and a neighboring cycle $u'_0u'_1 \dots u'_\ell$ in O_2, and then the induced subgraph of G that contains the paths from $u_{\ell-\ell/k}$ to u_ℓ and from $u'_{\ell-\ell/k}$ to u'_ℓ. It consists of two cycles of length $4\ell/k + 2$ with the common edge $u_0u'_0$, which is in O_1. When u_0 is fixed and $u_{\ell-\ell/k}$ is interchanged with u_ℓ, then $u'_{\ell-\ell/k}$ is interchanged with u'_ℓ. In this case G is rigid. This means that k is 1 or 2 if G is flexible, which proves (iv).

If $k = 2$, then ℓ is even. Furthermore, the matching edges between two neighboring cycles must connect pairs of opposite edges in the cycles. To see this, let $C = v_0v_1 \dots v_\ell$ be an element of O_2, C' a neighboring cycle, and v_0, v_j be the origins of the matching edges form C to C'. Because the edges $v_0v_{\ell-1}$ and v_0v_1 are in the same arc-orbit, there is an automorphism α that fixes v_0 and interchanges $v_{\ell-1}$ with v_1. Clearly α preserves C' and maps v_j into $v_{\ell-j}$, which is only possible if $v_{\ell-j} = v_j$, because there are only two edges between C and C'. Hence $\ell = 2j$. This proves (v).

It is easy to check (vi). □

Lemma 7.4 excludes graphs whose cycle-orbits O_2 consist of 2-sided infinite paths. If such a graph exists, then it is easily seen that O_2 contains at least two connected components, and that there can be at most one matching edge between any two components, otherwise G would be rigid. Furthermore G cannot be G acyclic, because then it is the infinite cubic tree T_3, which has only one arc-orbit. This leads to the following question:

Question 7.5. Are there any two arc-orbit, flexible, cubic graphs G whose cycle-orbits consist of 2-sided infinite paths?

It is not hard to show that the girth of any such graph, if it exists, must be at least 9.

We conclude this part with a theorem on tree-like graphs. We call an infinite, flexible cubic graph G *tree-like* if its cycle-orbit consist of finite cycles, and if the graph G^* that is obtained from G by contraction of the cycles in the cycle-orbit to single vertices and by replacement of double edges, if they occur, by single edges, is a tree of valence at least 3.

Theorem 7.6. *Let G be a tree-like infinite, flexible cubic graph. Then $\rho(G) = \infty$ and $\delta(G) = 0$.*

Proof. Suppose G satisfies the assumptions of the theorem, and let k be the length of the cycles in the cycle-orbit. Then G^* is an infinite tree T_r of valence $r = k$ or $r = k/2$, where $r \geq 3$. By [17] T_r is 2-distinguishable, has uncountable automorphism group, infinite distinguishing cost and distinguishing density zero. It is easy to see that the coloring c of G that is obtained from a distinguishing coloring c^* of G^* by coloring an arbitrary preimage of each black vertex of G^* black is distinguishing, has infinitely many black vertices, and distinguishing density 0.

To complete the proof we have to show that $\rho(G)$ is infinite. We know that G^* has uncountable group and any automorphism of G^* is induced by one of G. If α and β are two automorphisms of G that induce different automorphisms of G^* then they cannot leave all k-cycles invariant and must be distinct. Hence $\mathrm{Aut}(G)$ is uncountable. □

Tree like graphs are the basis for the construction of connected, vertex-transitive, flexible cubic graphs of girth four with two arc-orbits, infinite cost and zero density. See Lemma 9.3.

8 Flexible graphs of girth 3

Let G be a connected, vertex-transitive, cubic graph with two arc-orbits and girth 3. Then the cycle-orbits of G are triangles. Let G^* be the graph obtained by contracting each such triangle to a single vertex, making the matching edges of G the edges of G^*.

This implies that $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G^*)$ are isomorphic. To see this, let α be an automorphism of a graph H. It induces an incidence preserving permutation on $E(H)$ that maps each edge uv of H into $\alpha(u)\alpha(v)$. We denote it by α_E and the group of incidence preserving permutations of $E(H)$ by $\mathrm{Aut}_E(H)$. If H is a nontrivial graph, then $\mathrm{Aut}(H)$ and $\mathrm{Aut}_E(H)$ are isomorphic if and only if H has at most one isolated vertex and K_2 is not a component. It is not hard to prove directly, but actually it is a theorem about the automorphism group of the line graph of a graph, see e.g. [19, Theorem 1.2].

This means that $\mathrm{Aut}(G^*) \cong \mathrm{Aut}_E(G^*)$. Furthermore, each $\varphi \in \mathrm{Aut}(G)$ induces an incidence preserving permutation φ^*_E of $E(G^*)$. Clearly the mapping $\varphi \mapsto \varphi^*_E$ is bijective and thus $\mathrm{Aut}(G)$ and $\mathrm{Aut}_E(G^*)$ are isomorphic. By the above this implies that $\mathrm{Aut}(G) \cong \mathrm{Aut}(G^*)$. We also observe:

1. G^* is arc-transitive. By Theorems 6.4 and 6.5 this implies that it is s-arc-regular for some $s \leq 5$, unless it is the infinite cubic tree.

2. If $G^* \neq T_3$, then G^* is s-arc-regular for $s > 2$ if and only if G is flexible. In particular, if G^* is 3-connected and planar, then $s = 2$ and G is rigid. Observe that the assertion about s for G^* is a consequence of the unique embeddability of 3-connected planar graphs into the plane. Compare the proof of Corollary 7.3.

3. $\rho(G) \leq \rho(G^*)$, because if coloring the vertices of a set $U \subseteq V(G^*)$ black distinguishes G^*, then coloring one vertex black in each triangle corresponding to a vertex in U distinguishes G. A similar argument was used in the proof of Theorem 7.6.

4. In some cases one can show that $\rho(G) < \rho(G^*)$ by using the extra freedom of three choices for vertices in each triangle. Our approach is as follows. Let c be a coloring of the vertices of G and $\mathrm{Aut}(G)_c$ be the group of the color preserving automorphisms of G. Recall that $\mathrm{Aut}(G^*) \cong \mathrm{Aut}(G)$. So $\mathrm{Aut}(G)_c$ is isomorphic to a subgroup, say $(\mathrm{Aut}(G)_c)^*$, of $\mathrm{Aut}(G^*)$. If G^* is s-regular and if $(\mathrm{Aut}(G)_c)^*$ fixes an s-arc, then $(\mathrm{Aut}(G)_c)^*$ must be the trivial group, and thus also $\mathrm{Aut}(G)_c$.

   Therefore, if c is a 2-coloring with n black vertices, where the other vertices are white, and $n < \rho(G^*)$, then $\rho(G) < \rho(G^*)$ if $(\mathrm{Aut}(G)_c)^*$ is trivial.

Theorem 8.1. *Let G be a finite, vertex-transitive, flexible cubic graph with two arc-orbits and girth* 3*. Then* $\rho(G) = 2$*, unless* $g(G^*) = 4, 6, 8$*. In these cases* G^* *is the* $K_{3,3}$*, the Heawood graph or the Tutte-Coxeter graph and* $\rho(G) = 3$*.*

Proof. Suppose $g(G^*) \leq 5$, so G^* is K_4, the cube, the dodecahedron, the Petersen graph, or $K_{3,3}$. The first three are 3-connected planar and thus G is rigid, contrary to assumption. When G^* is the Petersen graph or $K_{3,3}$, then G is flexible, because both graphs are 3-regular by [7].

By Theorems 6.4 and 6.5 G^* is s-arc-regular with $s \leq 5$. Given a G^* with $s \leq 5$, we choose an s-arc whose arcs $a^*_1, \dots a^*_s$ in G^* correspond to matching arcs $a_1, \dots a_s$ in G. Set $a_i = u_i u'_i$, and color u_1, u_s black. Let u^*_1, u^*_s be the vertices in G^* corresponding to u_1, u_s in G.

If the girth of G^* is at least $2s - 1$ then $u_1 u'_1 u_2 u'_2 \cdots u_s$ is the unique shortest u_1, u_s-path in G. Because $u_1 u'_1$ is a matching edge, but not $u'_{s-1} u_s$ each color preserving automorphism of G must fix u_1 and u_s. But if u_s is fixed, then u'_s is also fixed. Let $\varphi \in \mathrm{Aut}(G)$ fix u_1 and u'_s. Then it fixes $a_1, \dots a_s$ in G and φ^* fixes $a^*_1, \dots a^*_s$ in G^*. Because G^* is s-arc regular φ^* is the identity, and thus also φ. Hence the coloring of G is distinguishing and $\rho(G) = 2$, see Figure 10 for $s = 5$.

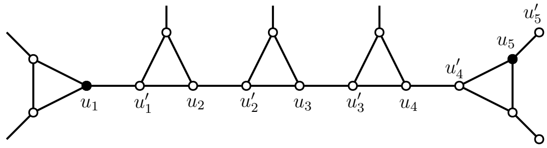

Figure 10: Distinguishing coloring of G for $s = 5$, $g(G) \geq 9$.

Therefore, cost 2 is assured for $s = 3, 4, 5$ when $g(G^*) \geq 5, 7, 9$, respectively. This leaves the cases $g(G^*) = 4, 6, 8$ for $s = 3, 4$ and 5.

If $s = 3$, then $g(G^*) = K_{3,3}$, which is the only remaining graph $g(G^*)$ of girth at most 5. It is Tutte's 4-cage. Furthermore, at the end of Section 6 we noted that the only finite, 4-arc-regular, cubic graph of girth 6 is the Heawood graph, also known as Tutte's 6-cage. Finally, in 1991 M. J. Morton [21] showed that the Tutte-Coxeter graph, also called Tutte's 8-cage, is the only finite graph with $s = 5$ and girth 8.

Using the same notation for all three cages, we color u_1, u_s and u_s' black. As $d_G(u_1, u_s) = 2s - 2$ and $d_G(u_1, u_s') = 2s - 3$, all black vertices have to be fixed by any color preserving $\varphi \in \mathrm{Aut}(G)$. As before we conclude that φ^* fixes $a_1^*, \ldots a_s^*$ in G^*. Hence the coloring of G is distinguishing, see Figure 11 for $s = 4$.

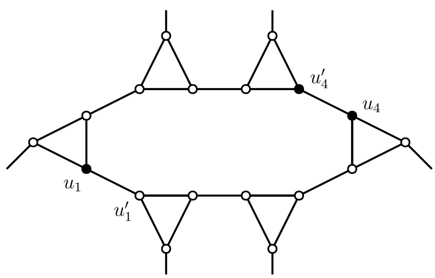

Figure 11: Distinguishing coloring of G for $s = 4$, $g(G) = 6$.

It remains to show that $\rho(G) > 2$ when G^* is a Tutte $4, 6$ or 8-cage, that is when $s = 3, 4$ or 5. We first show that each path of length $s - 1$ is in two cycles of length $2s - 2$. Let C be a cycle of length $2s - 2$ in G^* and $a_1^*, \ldots a_s^*$ be the arcs of an s-arc in C. By arc-transitivity there is $\alpha \in \mathrm{Aut}(G^*)$ that fixes $a_1^*, \ldots a_{s-1}^*$, but not a_s^*. It maps C into another cycle $\alpha(C)$. Clearly the $C \cap \alpha(C) = a_1^*, \ldots a_{s-1}^*$. By arc-transitivity this implies that any path of length $s - 1$ is in two cycles of length $2s - 2$.

To show that $\rho(G) > 2$ it suffices to show that to any vertices u,v in G there is an automorphism of G that interchanges them or a nontrivial automorphism that fixes them. If u,v are in the same triangle, then there clearly is an automorphism that interchanges them. Because the diameter of the cages is $s-1$ we can thus assume that $1 < d_{G^*}(u^*, v^*) \leq s-1$.

Suppose $d_{G^*}(u^*, v^*) = s - 1$ and that u^* and v^* are the endpoints of an $s - 1$-arc $a_1^*, \ldots a_{s-1}^*$. Clearly there are at least three automorphisms of G^* that interchange u^* and v^*: one that reverses the $s - 1$-arc $a_1^*, \ldots a_{s-1}^*$, another one that rotates C, and one that rotates $\alpha(C)$. Therefore, if $d_G(u, v) = 2s - 2$ or $2s$, then an inflection interchanges u with v, and if $d_G(u, v) = 2s - 1$ a rotation interchanges them.

This leaves the case $d = d_{G^*}(u^*, v^*) \leq s-2$. If $d_G(u,v) = 2d-1$ or $2d+1$ then we can interchange u, v by an inflection. Hence we can assume that $d_G(u,v) = 2d$. Then each shortest u, v-path either begins or ends with a matching edge. We choose the notation such that it begins with a matching edge. In G^* there clearly is an s-arc $a_1^*, \dots a_s^*$ where u^* is the origin of a_2^* and v^* the origin of a_{d+2}^*. Note that $d+2 \leq s$. As before we let a_i denote the matching edge in G that corresponds to a_i^* and set $a_i = u_i u_i'$. By our assumptions $u = u_2$ and $v = u_{d+2}$. There is an $\alpha^* \in \mathrm{Aut}(G^*)$ that moves a_1^*, but fixes $a_2^*, \dots, a_s^*$. The automorphism α^* uniquely extends to an $\alpha \in \mathrm{Aut}(G)$ that fixes $u_2, u_2', \dots u_{d+2}, u_{d+2}'$. Figure 12 depicts the case when d is maximal, i.e. when $v = u_{d+2} = u_s$. Observe that α interchanges the neighbors of u_2 that are different from u_2', fixes all other depicted vertices, but the non-matching edges incident with u_s' can be interchanged. □

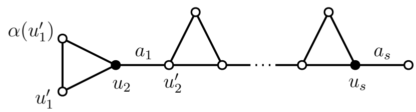

Figure 12: Nontrivial α fixing $u = u_2$ and $v = u_s$.

We conclude the section with a remark about the construction of connected, vertex-transitive, flexible cubic graph with two arc-orbits and girth 3. The process of contracting triangles in G to get G^* can be reversed by *truncation*, namely replacing each vertex by a triangle. In this case, we begin with G^* and construct G by truncating G^*. This way every cubic graph is a G^* for some cubic graph G. For example, in the proof of Theorem 8.1, the graph G with $G^* = K_{3,3}$ is flexible since $K_{3,3}$ is. In general, we have

Proposition 8.2. *Let G^* be an s-arc regular cubic graph, where $s \geq 3$. Then its truncation G is flexible of girth* 3.

Proof. Clearly, G has two arc-orbits, one orbit consisting of arcs in the triangles and a matching orbit corresponding to arcs in G^*. □

9 Flexible graphs of girth 4

In this section we mainly consider connected, flexible, vertex-transitive, cubic graphs G with two arc-orbits and girth 4. We show that this class contains finite graphs of arbitrarily large distinguishing cost and infinite graphs with positive distinguishing densities.

For our constructions we need two operations on G, one we call *folding*, and the other *unfolding*. The first folds the 4-cycles of a graph G into edges of a new graph $F(G)$, and the second unfolds the matching edges of a graph G into 4-cycles of a new graph $U(G)$.

Figure 13 shows how a matching edge uv of the graph on the left is unfolded into a 4-cycle in the graph on the right, respectively it shows how a 4-cycle in the graph on the right is folded into an edge of the graph on the left.

To be more precise, let G be a cubic graph with a complete matching M and C be the subgraph of G with the edge-set $E(G) - M$. The edges not in M form a subgraph C in which each edge has valence 2. Hence, each component of C is a finite cycle or a 2-sided infinite path. When G is vertex-transitive only one of these options is possible, and if there are finite cycles, then all have the same length.

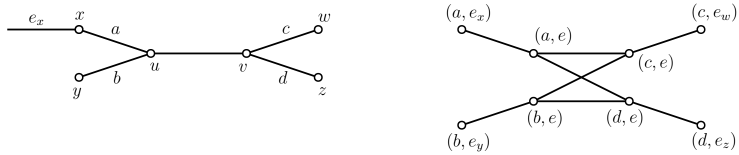

Figure 13: Unfolding and folding.

For a formal description of the new operations we denote incidence of edges a, b by $a|b$, and non-incidence by $a \nmid b$. The *unfolded graph* $U(G)$ is then defined by:

$$V(U(G)) = \{(a, e) : a \in E(C), e \in M, a|e\},$$
$$E(U(G)) = \{(a, e)(c, f) : a = c, a|e, a|f, e \neq f \text{ or } e = f, e|a, e|c, a \neq c, a \nmid c\}.$$

Each vertex u of G is incident with exactly one matching edge, say e, and two edges from C, say a,b. It thus gives rise to two vertices of $U(G)$, we say each vertex is split into two vertices. Furthermore, if $(a, e)(b, f) \in E(U(G))$, then either a connects an endpoint of e with one of f, or e connects an endpoint of a with one of b. Hence, identification of the two vertices in each pair of split vertices is a homomorphism from $U(G)$ onto G.

Clearly unfolding is well defined for all cubic graphs G with a matching M. The folding operation, however, is only defined for cubic graphs G with a set of disjoint of 4-cycles. Given such a graph G, the *folded* graph $F(G)$ is obtained by identification of opposite vertices in the 4-cycles of G, and replacement of the ensuing multiple edges from the 4-cycles by single edges.

Observe that we did not require vertex-transitivity for our operations.

Now we recall that any $\alpha \in \mathrm{Aut}(G)$ induces an incidence preserving permutation on $E(G)$ that maps each edge uv of G into $\alpha(u)\alpha(v)$. We denote it by α_E and, as before, we let $\mathrm{Aut}_E(G)$ be the group of incidence preserving permutations of $E(G)$. We already observed that $\mathrm{Aut}(G) \cong \mathrm{Aut}_E(G)$ if and only if G has at most one isolated vertex and K_2 is not a component.

Lemma 9.1. *Let G be a connected, vertex-transitive, cubic graph with a complete matching M, and let C denote the subgraph of G consisting of the edges not in M. Then*

(i) *$U(G)$ is a connected, cubic graph that consists of disjoint quadrangles, which form a spanning subgraph, and a set of independent edges that form a complete matching. The quadrangles arise from the edges in M, and the matching edges of $U(G)$ are in one-one correspondence with the edges of C.*

(ii) *If all quadrangles of G are components of C, then any two adjacent quadrangles in $U(G)$ are connected by exactly one edge.*

(iii) *There is an isomorphism of $\mathrm{Aut}_E(G)$ into $\mathrm{Aut}(U(G))$. If G has three arc-orbits, then $U(G)$ is not vertex-transitive.*

(iv) *If G is vertex-transitive with two arc-orbits, then $U(G)$ is also vertex-transitive, but may have three arc-orbits, see* Corollary 9.5.

(v) *If G is vertex-transitive and flexible with two arc-orbits, then $U(G)$ is vertex-transitive with two arc-orbits, but may be rigid. See* Corollary 9.8 *and the remark after its proof.*

Proof. Let $e = uv \in M$, and $a, b, c, d \in V(C)$, where a, b are incident with u, and c, d incident with v. Then $(a, e)(c, e)(b, e)(d, e)$ is a quadrangle in $U(G)$. Note that (a, e) is adjacent to (c, e) and (d, e) in $U(G)$, but not to (b, e).

If $a = ux \in M$ and e_x is the matching edge incident with x, then $(a, e)(a, e_x) \in E(U(G))$. Hence the the neighbors of (a, e) are (c, e), (d, e) and (a, e_x). This proves (i).

For (ii) let A, B be two neighboring quadrangles in $U(G)$ that are joined by (at least) two edges. By (i) they are matching edges, say e, f. Let a_e, a_f be the endpoints of e, resp. f, in A and b_e,b_f be the endpoints in B. If a_e and a_f arise from the same vertex in G, then G has a double edge or a triangle, depending on whether b_e and b_f arise from the same vertex in G or not. Hence a_e, a_f, b_e, b_f arise from different vertices in G. It also means that $a_e a_f$ and $b_e b_f$ arise from matching edges, while e, f arise from non-matching edges. But then G has a quadrangle that is not in C, contrary to assumption. This proves (ii).

To prove (iii) we observe that the mapping $(a, e) \mapsto (\alpha_E(a), \alpha_E(e))$ is an automorphism of $U(G)$, say α^*. It is then easy to see that the mapping $\alpha_E \mapsto \alpha^*$ from $\mathrm{Aut}_E(G)$ into $\mathrm{Aut}(U(G))$ is an injective isomorphism. If G has three arc-orbits and if a, b are the non-matching edges that are incident with a vertex v and if e is the matching edge incident with v, then there is automorphism of G that fixes v and interchanges a with b, and thus no automorphism of $U(G)$ that maps (a, e) into (b, e). This proves (iii).

To prove (iv) consider two vertices (a, e), (b, f) of $U(G)$. Let u be the common vertex of a and e, and v the common vertex of b and f. If G is vertex-transitive with two arc-orbits, then there exists an automorphism α that maps u into v, where $\alpha_E(e) = f$ and $\alpha_E(a)$ is incident with v. If $\alpha_E(a) \neq b$, then there is a $\beta \in \mathrm{Aut}(G)$ that fixes v and maps $\alpha_E(a)$ into b. Then $(\beta\alpha)^*(a, e) = (b, f)$, and $U(G)$ is vertex-transitive. This proves (iv).

Finally, consider a vertex (a, e) and its two non-matching neighbors, say (c, e), (d, e). If $e = uv$, then c, d are non-matching edges of G that are incident with v. Because G is flexible with two arc-orbits, there is an automorphism α that fixes a, u, e and v and where α_E interchanges c, d. Clearly α^* fixes (a, e) and interchanges (c, e) with (d, e). □

Lemma 9.2. *Let G be a cubic graph with a complete matching M and a set C of quadrangles, where any two neighboring quadrangles of C are connected by one edge of M. Then there is a natural isomorphism between* $\mathrm{Aut}(G)$ *and* $\mathrm{Aut}(U(G))$.

Proof. Suppose G satisfies the conditions of the lemma. Then all quadrangles of G must be in the set C. If not there must be a quadrangle $uvwx$ that contains a matching edge, say $e = uv$. Vertices u and v are in different quadrangles from C, say A and B. Then ux is in A, vw in B, and wx is a second edge that connects A with B.

By Lemma 9.1 this implies that any two neighboring quadrangles in $U(G)$ are connected by exactly one edge, and thus the partition of $E(U(G))$ into a set of matching edges and a set of quadrangles is unique. Hence the set of pairs of split vertices arising in the construction of $U(G)$ from G is exactly the set of pairs of opposite vertices of the quadrangles in $U(G)$. Thus there is only one way to fold $U(G)$, and $G = F(U(G))$.

Each automorphism α of $U(G)$ preserves the pairs of opposite vertices of the quadrangles in $U(G)$ and thus acts as a permutation α' on $V(G)$. As adjacences are preserved α' is an automorphism. Moreover, $(\phi\psi)' = \phi'\psi'$, and thus $\phi \mapsto \phi'$ is a homomorphism.

If there were two distinct automorphisms ϕ and ψ of $U(G)$ that induce the same action on G, they would have to coincide on these pairs as sets, but in at least one pair $\{u, v\}$ they would have to act differently. Then $\psi^{-1}\phi(u)$ is not the identity on $\{u, v\}$, and thus interchanges u and v. But then the matching edge e_u incident with u has to be interchanged with the matching edge e_v incident with v. As e_u and e_v lead to different quadrangles, not all pairs of opposite vertices in the quadrangles can be preserved. Hence $\phi \mapsto \phi'$ is injective, and hence also the mapping $\phi \mapsto (\phi')_E$

The observation that $\mathrm{Aut}(G) \cong \mathrm{Aut}_E(G)$ and that there is an injective isomorphism of $\mathrm{Aut}_E(G)$ into $\mathrm{Aut}(U(G))$ completes the proof. □

As an application of Lemma 9.1 we have the following lemma.

Lemma 9.3. *There exist infinitely many connected, vertex-transitive, flexible cubic graphs of girth four with cost ≤ 5 or with infinite cost and zero density.*

Proof. Let G be the truncation of an s-transitive cubic graph, where $s \geq 3$. By Lemmas 8.2 and 8.1 it is flexible with cost ≤ 5 or zero density. Then $U(G)$ has girth 4 and it is easy to see that it is flexible with cost ≤ 5 or zero density.

Similarly, if G is one of the tree-like graphs of Theorem 7.6, then $U(G)$ is flexible of girth 4 and its density is zero. □

9.1 The graphs $Q(n)$ and $Q_k(n)$

In Section 2 we defined the chain of quadrangles Q, which we also called the infinite crossed ladder, as a graph with vertex set $V(Q) = \{u_i, v_i : i \in \mathbb{Z}\}$ and edge set

$$E(Q) = \{u_i u_{i+1}, v_i v_{i+1}, u_{2i} v_{2i+1}, v_{2i} u_{2i+1} : i \in \mathbb{Z}\}.$$

If we replace $\mathbb{Z}$ in the definition of Q by $\mathbb{Z}_{2k}$ and take indices modulo $2k$, then we obtain the *crossed* $2k$*-ladder*, which we will denote by Q_k. As the crossed 4-ladder is the cube, which is not 2-distinguishable, we only consider crossed ladders Q_k for $k \geq 3$.

Q and Q_k for $k \geq 3$ consist of a set M of matching edges and a set C of disjoint quadrangles, where adjacent quadrangles are connected by two matching edges. Note that M consists of the edges of G that are not in the quadrangles and that no matching edge is in a 4-cycle of G. Hence, if G is Q or a crossed $2k$-ladder for $k \geq 3$, then $U(G)$ is well-defined if we unfold with respect to M. By Lemma 9.1 $U(G)$ also consists of a set of matching edges and a set of quadrangles. This property is retained by all graphs that are obtained by iterations of the unfolding process if we unfold with respect to the edges that are not in the quadrangles.

We set $Q(1) = Q$, $Q_k(1) = Q_k$, and, for $k \geq 3$, $n > 1$, iteratively define $Q_k(n)$ by $Q_k(n) = U(Q_k(n-1))$, and $Q(n) = U(Q(n-1))$. We always unfold with respect to the edges that are not in the quadrangles. They are uniquely defined, and hence so are also $Q(n)$ and $Q_k(n)$.

$Q(1)$ and $Q_k(1)$, $k \geq 3$, are vertex-transitive and flexible with two arc-orbits. As we unfold with respect to the matching orbit we infer by Lemma 9.1 that $Q(n)$ and $Q_k(n)$, $k \geq 3$, $2 \leq n < k-1$, are connected, vertex-transitive, flexible cubic graphs with two arc-orbits, where the cycle orbit consists of 4-cycles, and where any two adjacent 4-cycles are connected by exactly one edge in the matching orbit.

Furthermore, by Corollary 9.8 we shall see that $Q_k(k-1)$ is rigid with two arc-orbits.

As there is a unique matching orbit in $Q_k(k-1)$ the graph $Q_k(k) = U(Q_k(k-1))$ is vertex-transitive by Lemma 9.1(v). That it has three arc-obits is asserted in Corollary 9.5.

Because we keep unfolding with respect to the edges that are not in quadrangles, the graphs $Q_k(n)$ are still uniquely defined for $n > k$, but not vertex-transitive any more.

In the definition of $F(G)$ we only merged multiple edges that arose from folded 4-cycles, but not multiple edges that arise from matching edges. Such cases may occur, for example when folding the chain of quadrangles Q, which we now call $Q(1)$, see Figure 3. The folding process maps $Q(1)$ into a chain of alternating single and double edges and $Q_k(1)$ into a cycle of alternating single and double edges. We set $Q(0) = F(Q(1))$ and $Q_k(0) = F(Q_k(1))$. For $Q(0)$ see Figure 14, and for a part of $Q(0)$ and how it is unfolded, see Figure 15.

Figure 14: $Q(0)$.

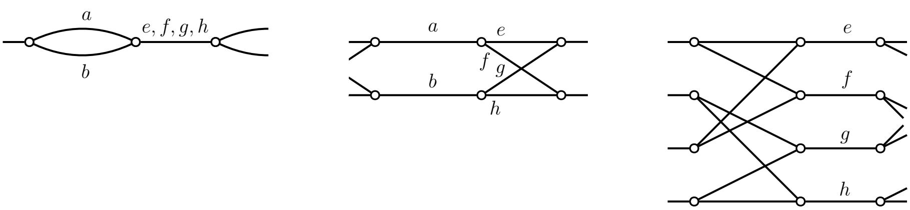

Figure 15: A part G of $Q(0)$ and how it is unfolded.

Clearly all unfolded graphs of $Q(0)$ are isomorphic up to $\mathrm{Aut}_F(Q(0))$, and the graphs $Q_k(0)$ are isomorphic up to $F(Q_k(0))$. Moreover, $\mathrm{Aut}(Q(1)) \cong \mathrm{Aut}_F(Q(0))$, and $\mathrm{Aut}(Q_k(1)) \cong \mathrm{Aut}_F(Q_k(0))$. Recall that $Q_2(1) = U(Q_2(0))$ is the cube.

Theorem 9.4. *The order of* $\mathrm{Aut}(Q_k(n))$*,* $k \geq 3$*,* $1 \leq n \leq k$*, is* $2k \cdot 2^k$*.*

Proof. Clearly the order of $\mathrm{Aut}_E(Q_k(0))$ is $2k \cdot 2^k$. By Lemma 9.1(iii) this is also the order of all groups $\mathrm{Aut}(Q_k(n))$, $k \geq 3$, $1 \leq n \leq k$. □

Corollary 9.5. *For* $k \geq 3$ *the graphs* $Q_k(k)$ *are GRRs.*

Proof. By the lemma $|\mathrm{Aut}(Q_k(k))| = 2k \cdot 2^k$, which equals $|V(Q_k(k))|$. Because $Q_k(k)$ is vertex-transitive this implies that its vertex stabilizers are trivial. Hence $Q_k(k)$ has three arc-orbits and is a GRR. □

Note that the smallest such graph is $Q_3(3)$ and has 48 vertices. The graph $Q_4(4)$ already has 128 vertices.

Finally, observe that the graphs $Q_k(n)$ for $n > k$ are not vertex-transitive by Lemma 9.1(iii).

One calls a partition of the vertex set of a vertex-transitive graph a *system of imprimitivity* if it is preserved by all automorphisms. The partition of $V(G)$ into sets of size one, $\{\{v\} : v \in V(G)\}$ and into just one set of size $|V(G)|$ are called *trivial partitions*. We have seen that the sets of pairs of opposite vertices in the quadrangles of $Q(1)$, resp. $Q_k(1), k \geq 3$, form sets of imprimitivity. Each such pair is the preimage of a single vertex in $Q(0)$, resp. $Q_k(0), k \geq 3$.

Lemma 9.6. *Let $n \in \mathbb{N}$ and G be one of the graphs $Q(n)$ or $Q_k(n)$, $k \geq 3$. Furthermore, let φ denote the mapping from $V(G)$ into $V(F(G))$. Then the preimages of the vertices of $Q(0)$, resp. $Q_k(0)$, $k \geq 3$, with respect to φ^n, are a system of imprimitivity in $Q(n)$, resp. $Q_k(n)$, $k \geq 3$.*

Proof. The assertion of the lemma is true for $n = 1$. Suppose it is true for $n \geq 1$. Let G be $Q(0)$ or $Q_k(0)$, $k \geq 3$, and $\{\varphi^{-n}(v) : v \in V(G)\}$, where $\varphi^{-n}(v)$ is the preimage of v with respect to φ^n. By the induction hypothesis $\{\varphi^{-(n-1)}(v) : v \in V(G)\}$ is a system of imprimitivity in $U^{(n-1)}(G)$. The set $\varphi^{-n}(v)$ arises from $\varphi^{-(n-1)}(v)$ by splitting each of its vertices into a pair of opposite vertices in the quadrangles of $U(U^{(n-1)}(G))$. The observation that $\mathrm{Aut}(U(U^{(n-1)}(G)))$ preserves the set of these pairs concludes the proof. □

Let G be $Q(0)$ or $Q_k(0)$, $3 \leq k$, and $n \geq 1$. Then we call the sets $\varphi^{-n}(v)$, $v \in V(G)$, the *columns* of $U^n(G)$, and denote them by c_v^n. Setting $V(Q(0)) = \mathbb{Z}$ and $V(Q_k(0)) = \mathbb{Z}_{2k}$, $k \geq 3$, the columns of $U^n(G)$ are thus c_i^n, $i \in \mathbb{Z}$ or $\mathbb{Z}_{2k}$.

Lemma 9.7. *Let $n \in \mathbb{N}$ and $c_{-n+2}^n, \dots c_{n+1}^n$ be $2n$ columns in $Q(n)$ or $Q_k(n)$, where $k-1 \geq n$. Then there exists an $\alpha \in \mathrm{Aut}(Q(n))$, respectively $\alpha \in \mathrm{Aut}(Q_k(n))$, that moves all vertices in columns $c_{-n+2}^n, \dots, c_{n+1}^n$ and fixes all other vertices.*

Proof. We proceed by induction with respect to n and note that the assertion of the lemma is true for $n = 1$. Let G be $\mathrm{Aut}(Q(n))$ or $\mathrm{Aut}(Q_k(n))$ and suppose the assertion is true for $n \geq 1$. Then there exists an $\alpha \in \mathrm{Aut}(G)$ that moves all vertices in $c_{-n+2}^n, \dots, c_{n+1}^n$ and fixes all other vertices.

Recall from Lemma 9.1, if G is a connected, vertex-transitive cubic graph with a complete matching M, then $\mathrm{Aut}(G) \cong \mathrm{Aut}_E(G) \cong \mathrm{Aut}(U(G))$. Furthermore, a vertex u of G that is incident with a matching edge e and the non-matching edges f, g gives rise to the vertices (f,e) and (g,e) in $U(G)$. Each $\alpha \in \mathrm{Aut}(G)$ then gives rise to an automorphism α^* of $U(G)$ defined by $\alpha^*(f,e) = (\alpha_E(f), \alpha_E(e))$. This means that all elements in $c_{-n+2}^{n+1}, \dots, c_{n+1}^{n+1}$ are moved by α^*. As the edges between c_{-n+1}^{n+1} and c_{-n+2}^{n+1} are matching edges, as well as the edges between c_{n+1}^{n+1} and c_{n+2}^{n+1}, the vertices in c_{-n+1}^{n+1} and c_{n+2}^{n+1} are also moved.

As α fixes all vertices in c_{-n}^n and c_{n+3}^n, as well as the vertices in the neighboring columns, all edges incident with vertices in c_{-n}^n and c_{n+3}^n are fixed, and thus α^* fixes all vertices in c_{-n}^{n+1} and c_{n+3}^{n+1}.

It is easily seen that all other vertices of $U(G)$ that are not in $c_{-n+1}^{n+1}, \dots, c_{n+2}^{n+1}$ are also fixed. But, we wish to point out that columns c_{-n}^{n+1} and c_{n+3}^{n+1} can be adjacent. In this case $n = k-1$, c_{-n}^{n+1}, c_{n+3}^{n+1} are connected by matching edges and are the only columns whose elements are fixed by α^*. □

Corollary 9.8. *The graphs $Q(n)$ and $Q_k(n)$, $n \in \mathbb{N}$, $k-2 \leq n$, are flexible, but $Q_k(k-1)$ is rigid.*

Proof. Let $\alpha \in \mathrm{Aut}(Q(n))$ or $\mathrm{Aut}(Q_k(n))$ be the automorphism that moves all vertices in the columns $c^n_{-n+2}, \dots, c^n_{n+1}$ and fixes all other vertices. Observe that the edges between c^n_{-n+2} and c^n_{-n+3} are matching edges, and hence also the edges between c^n_{-n} and c^n_{-n+1}, which are fixed. Let v be a vertex in c^n_{-n+1}. It is fixed and has two neighbors, say w,z in c^n_{-n+2} that are both moved. The other neighbor of v, say u is in c^n_{-n}, and uv is a matching edge. Clearly the other two neighbors of u, say x,y are in c^n_{-n-1} and fixed, unless $k-1 = n$, because then $c_{-n-1} = c_{-n+2k} = c_{n+1}$, all of whose elements are moved, which means that $Q_k(k-1)$ is rigid. $\square$

The fact that $Q_k(n)$ is flexible for $1 \leq n \leq k-2$ and rigid for $n = k-1$ is also a consequence of Theorem 9.4 and Corollary 7.2. To see this, observe that $|V(Q(k,n))| = 2k{\cdot}2^n$. The graphs are vertex-transitive and the order of their automorphism groups is $2k{\cdot}2^k$ by Theorem 9.4. Hence the order of their vertex-stabilizers is at least 4 when $n \leq k-2$ and 2 when $n = k-1$. Now an application of Corollary 7.2 completes the argument.

Theorem 9.9. *Let G be $Q(n)$ or $Q_k(n), k-1 \geq n$. Then the motion $\mathrm{m}(G)$ of G is $n \cdot 2^{n+1}$, and to each automorphism α of G that stabilizes the columns of G there is a row of $2n$ columns, all of whose vertices are moved by α.*

Proof. Let G be $Q(n)$ or $Q_k(n), n \leq k-1$. By Lemma 9.7 there exists an automorphism that moves all vertices in the $2n$ columns $c^n_{-n+2}, \dots, c^n_{n+1}$ of G. Because each column contains 2^n vertices, $\mathrm{m}(G) \leq n \cdot 2^{n+1}$.

If an automorphism of G moves a column into another one, then it has to move all columns. This is easily seen, because after n foldings G maps onto $Q(0)$ or $Q_k(0)$, whose automorphisms move all vertices if they move at least one vertex. Hence, if not all columns are stabilized, then at least $2k$ columns are moved, and thus at least $2k{\cdot}2^n > 2n{\cdot}2^n$ vertices.

We can thus restrict attention to automorphisms that stabilize the columns. We shall show the existence of a row of $2n$ columns to each automorphism α of G, where α moves all vertices in the columns, and where the edges between the outermost pairs of columns are matching.

This is true for $n = 1$. We wish to show that it holds for n under the assumption that it holds for $n-1 \geq 1$. Each automorphism of G is of the form α^*, where $\alpha \in F(G)$, where $F(G)$ is $Q(n-1)$ or $Q_k(n-1), n \leq k-1$. Hence there are $2n-2$ columns,all of whose vertices are moved by α. By vertex-transitivity we can assume that the columns are $c^{n-1}_{-n+3}, \dots, c^{n-1}_n$. (This also assures that the edges between the outermost pairs of columns are matching.)

Then α^* moves all vertices in $c^n_{-n+3}, \dots, c^n_n$. As the edges between columns c^n_{-n+2}, c^n_{-n+3}, and c^n_n, c^n_{n+1} are matching, all vertices in the columns $c^n_{-n+2}, \dots, c^n_{n+1}$ are moved. $\square$

Theorem 9.10. *The graphs $Q(n)$, $n \in \mathbb{N}$, are connected, vertex-transitive, flexible cubic graphs with two arc-orbits, girth 4, and motion $\mathrm{m}(Q(n)) = n \cdot 2^{n+1}$. Furthermore, $\rho(Q(n)) = \infty$ and $\delta(Q(n)) = 1/\mathrm{m}(Q(n))$.*

Proof. By Lemma 9.7, Theorem 9.9 and Corollary 9.8 we only have to find a distinguishing coloring with density $1/(n \cdot 2^{n+1})$. We choose the columns c^n_{2nj}, $j \in \mathbb{Z} - \{0\}$ and color one vertex in each column black. Then we choose a vertex in c^n_0, say u. It is incident with a matching edge e, say uv. Instead of coloring u, we color v black. Let c be this coloring. Clearly the projection of c into $Q(0)$ distinguishes $\mathrm{Aut}(Q(0))$, where $\mathrm{Aut}(Q(0))$

is the group of permutations of $V(Q(0))$ that preserve double and single edges. Hence any color preserving automorphism of $Q(n)$ stabilizes the columns, and thus fixes all black vertices. But, because e is a matching edge, it also fixes u, and thus any color preserving automorphism α fixes at least one vertex in any row of $2n$ columns. By Theorem 9.9 α must be the identity. □

Theorem 9.11. *The graphs $Q_k(n)$, $1 \leq n \leq k-2$, are connected, vertex-transitive, flexible cubic graphs with two arc-orbits, girth 4 and motion $\mathrm{m}(Q(n)) = n \cdot 2^{n+1}$. Their distinguishing cost is $\rho(G) = \lceil \frac{k}{n} \rceil$, unless $\frac{k}{n} = 2$; then $\rho(G) = 3$. In all cases $\delta(Q_k(n)) \leq 1/4$.*

Proof. We choose the columns c^n_{2nj}, $1 \leq j \leq \lceil \frac{k}{n} \rceil$, and color one vertex in each column black. For c^n_0 we proceed differently. We choose a $u \in c^n_0$. It is incident with a matching edge $e = uv$, and instead of coloring u, we color v black. Let c be this coloring.

If $k/n \neq 2$ the projection of c into $Q_k(0)$ distinguishes $\mathrm{Aut}(Q_k(0))$, hence any color preserving automorphism of $Q_k(n)$ stabilizes the columns, and thus fixes all black vertices. But, because e is a matching edge, it also fixes u, and thus any color preserving automorphism α fixes at least one vertex in any row of $2n$ columns. By Theorem 9.9 α must be the identity.

If $k/n = 2$ we need three black vertices in $Q_k(0)$ to break $\mathrm{Aut}(Q_k(0))$. It is easily seen, that it suffices to color one vertex in each of the columns c^n_0, c^n_1, and c^n_n black, in order to obtain a distinguishing coloring. Hence $\rho(G) \leq 3$.

But we still have to show that two black vertices do not suffice. Suppose two black vertices suffice, say u,v. When $k/n = 2$ we have $4n$ columns, and because there are automorphisms that move $2n$ contiguous columns and fix all vertices of the other columns, we have to place the black vertices in columns of distance $2n$, say $u \in c^n_1$ and $v \in c^n_{2n+1}$. We do not need consider the case that the black vertices are in c^n_0 and c^n_{2n}, because there is a reflection that interchanges the pair c^n_1, c^n_{2n+1} with c^n_0, c^n_{2n}.

By vertex-transitivity there is an automorphism β with $\beta(u) = v$. Set $w = \beta(v)$. If $u = w$, then our coloring is not distinguishing, hence $w \neq u$. If we can find an automorphism that ψ that fixed v and maps w into u, then $\psi\beta(u) = v$ and $\psi\beta(v) = \psi(w) = u$, and the coloring is not distinguishing.

Let n be fixed and $1 \leq m \leq n$. If $m = 1$, then c^m_1 consists of just two vertices, u and w and one sees directly that one can interchange them while fixing the vertices in c^m_ℓ, for $2m+1 \leq \ell \leq 4n-2m-1$. i.e. also v. We continue by induction with respect to m and assume that for $m-1 \geq 1$ and any two vertices $x, y \in c^{m-1}_1$ there is an automorphism ψ, such that $x = \psi(y)$ and where ψ fixes all vertices in c^m_ℓ, $2m+1 \leq \ell \leq 4n-2m-1$. Given $u, w \in c^m_1$ we consider their images under the folding homomorphism φ. Let $x = \varphi(u)$ and $y = \varphi(w)$. There is an automorphism ψ of $Q(m-1, k)$ that maps y into x and fixes c^m_ℓ, $2m-1 \leq \ell \leq 4n-2m+1$ pointwise. Then ψ^* moves the preimage of y into the preimage of x and fixes all vertices in the columns c^m_ℓ for $2m+1 \leq \ell \leq 4n-2m-1$. If ψ^* does not move w into u, then it moves it into a vertex u', where u, u' are opposite vertices in a quadrangle. But then there is an automorphism that moves u' into u and fixes all vertices in c^m_ℓ for $2m+1 \leq \ell \leq 4n-2m-1$. □

Question 9.12. We wonder whether there exist any infinite, connected, vertex-transitive cubic graphs with positive density other than $Q(n)$, where $1 \leq n$, or finite connected, vertex-transitive cubic graphs with cost > 5 other than $Q_k(n)$, where $1 \leq n \leq k-2$ and $\lceil \frac{k}{n} \rceil > 5$. If they exist, they must be flexible with two arc-orbits.

9.1.1 Split Praeger–Xu graphs

The graphs $Q_k(n)$, $1 \leq n \leq k-1$, $k \geq 3$, are also know as Split Praeger–Xu graphs $\mathrm{SPX}(2,k,n)$, see [20]. Their vertex sets are $\mathbb{Z}_2^n \times \mathbb{Z}_k \times \{+,-\}$ and the edge-sets consists of the pairs

$$(i_0,\ldots,i_{n-1},x,+)(i_1,\ldots,i_n,x+1,-) \text{ and } (i_0,\ldots,i_{n-1},x,+)(i_0,\ldots,i_{n-1},x,-)$$

for $i_j \in \mathbb{Z}_2, x \in \mathbb{Z}_k$.

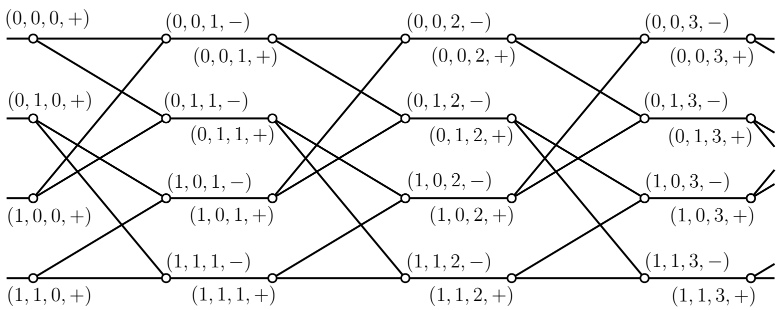

Figure 16: Part of $\mathrm{SPX}(2,k,2)$ for large k.

For $\mathrm{SPX}(2,k,2)$, where k is large, compare Figure 16. We leave it to the reader to verify that $Q_k(n) = \mathrm{SPX}(2,k,n)$. Note that the graphs $\mathrm{SPX}(2,k,n)$, $k \geq 3$, are flexible for $1 \leq n \leq k-2$, rigid for $n = k-1$ by Corollary 9.8, and not defined as SPX-graphs for $n = k$.

By our preceding results on $Q_k(n)$ the following theorem characterizes the Split Praeger–Xu graphs.

Theorem 9.13. *Let* $1 \leq n \leq k-1$ *and* $k \geq 3$. *Then the Split Praeger–Xu graphs* $\mathrm{SPX}(2,k,n)$ *are exactly those cubic graphs G that have a spanning subgraph consisting of disjoint* 4*-cycles and can be folded onto* $Q_k(0)$ *by n foldings.*

We could have based our presentation of $Q_k(n)$ on that of [20], but preferred the more graph theoretic approach. It allowed us to directly treat the graphs $Q(n)$, which are infinite, and to illustrate the role of motion. Besides, it also led to a new series of GRRs.

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