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Margaret Archibald, Vasco Brattka, and Clemens Heuberger

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Margaret Archibald

Laboratory of Foundational Aspects of Computer Science

Department of Mathematics & Applied Mathematics

University of Cape Town, Rondebosch 7701, South Africa

Margaret.Archibald@uct.ac.za

Vasco Brattka*

Laboratory of Foundational Aspects of Computer Science

Department of Mathematics & Applied Mathematics

University of Cape Town, Rondebosch 7701, South Africa

Vasco.Brattka@uct.ac.za

Clemens Heuberger†

Institut für Optimierung und Diskrete Mathematik

Technische Universität Graz, 8010 Graz, Austria

Clemens.Heuberger@tugraz.at

Abstract. The ordinary notion of algorithmic randomness of reals can be characterised as Martin-Löf randomness with respect to the Lebesgue measure or as Kolmogorov randomness with respect to the binary representation. In this paper we study the question how the notion of algorithmic randomness induced by the signed-digit representation of the real numbers is related to the ordinary notion of algorithmic randomness. We first consider the image measure on real numbers induced by the signed-digit representation. We call this measure the signed-digit measure and using the Fourier transform of this measure and the Riemann-Lebesgue Lemma we prove that this measure is not absolutely continuous with respect to the Lebesgue measure. We also show that the signed-digit measure can be obtained as a weakly convergent convolution of discrete measures and therefore, by a classical Theorem of Jessen and Wintner the Lebesgue measure is not absolutely continuous with

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respect to the signed-digit measure. Finally, we provide an invariance theorem which shows that if a computable map preserves Martin-Löf randomness, then its induced image measure has to be absolutely continuous with respect to the target space measure. This theorem can be considered as a loose analog for randomness of the Banach-Mazur theorem for computability. Using this Invariance Theorem we conclude that the notion of randomness induced by the signed-digit representation is incomparable with the ordinary notion of randomness.

Keywords: Algorithmic randomness, computable analysis, signed-digit representation, Stern-Brocot tree, Farey fractions, Stern's diatomic sequence.

1. Introduction

The *signed-digit representation* is a representation of reals in base 2 with coefficients $-1, 0, 1$. This representation, which is obviously redundant due to the extra digit -1, has found a tremendous number of applications and has at least been studied since Cauchy. It plays, for instance, a crucial role in computer arithmetic [1, 23], where it is used for fast parallel algorithms, in coding theory [27] and in cryptography [22], where also the fact is exploited that the representation is balanced, symmetric and redundant. In computable analysis it turned out to be useful as it is a topologically well-behaved representation that is suitable for the definition of computational complexity on the reals [28]. In this paper we want to study the question whether the signed-digit representation is also well-behaved from the measure theoretic point of view and thus suitable for the study of algorithmic randomness.

It is convenient to consider the signed-digit representation just as a representation of the unit interval $I := [-1, 1]$. In the following we will work with the alphabet $\Sigma := \{-1, 0, 1\}$ and by Σ^ω we denote the set of sequences over Σ. We endow Σ^ω with the usual Cantor topology (the product topology of the discrete topology on Σ) and with the uniform Borel measure γ that is induced by $\gamma(w\Sigma^\omega) = |\Sigma|^{-|w|}$ for all open balls $w\Sigma^\omega$ with $w \in \Sigma^*$.

Definition 1.1. (Signed-digit representation)
The *signed-digit representation* is the map $\rho : \Sigma^\omega \to I$, defined by

$$\rho(p) := \sum_{j=0}^{\infty} p(j) 2^{-j-1}$$

for all $p \in \Sigma^\omega$. By μ we denote the image measure induced by the uniform measure γ on Σ^ω, i.e. $\mu(A) := \gamma(\rho^{-1}(A))$ for all measurable $A \subseteq I$.

By a *measurable* set $A \subseteq I$ we always mean a Borel measurable set here (i.e. a set in the σ–algebra on I generated by the open intervals). In the following we will call μ the *signed-digit measure* on I. The map ρ has a number of important properties. First of all, it is continuous and computable (with respect to the standard topologies and computability notions on Σ^ω and I, respectively), it is proper (i.e. preimages of compact sets are compact) and it is a so-called *admissible* representation (which means that it is maximal among all continuous representations of I with respect to continuous reducibility, see [28], essentially this means topological well-behavedness).

The crucial point for this paper is the question how the class of random reals induced by the signed-digit representation is related to the class of ordinary random reals. It is known that the ordinary random reals can either be characterised as Martin-Löf random reals with respect to the Lebesgue measure λ [21] or as reals that have a random name with respect to the binary representation in the sense of Kolmogorov randomness [16, 17]. Indeed, the classes of randomness reals induced by different base expansion all coincide [7, 14] (and see [25] for further properties of Kolmogorov randomness and base expansions). For background information on algorithmic randomness, see [6, 20].

In the following we will denote by I_λ the set of ordinary random reals (in I). The main question of this paper is how the classes of ordinary random reals I_λ and Martin-Löf random reals I_μ with respect to the signed-digit measure μ are related. The main result is that the classes are mutually incomparable. For completeness, we also study the class I_ρ of reals that have a random name with respect to the signed-digit representation and we show that this class is also incomparable with I_λ. We leave the question open whether I_ρ and I_μ are identical, but we show that $I_\rho \subseteq I_\mu$ holds.

In the following section 2 we prove that the signed-digit measure μ is not absolutely continuous with respect to the Lebesgue measure λ and in the following section 3 we conclude with the help of the Theorem of Jessen-Wintner that also the inverse relation does not hold true. These facts could also be derived from results in [9, 10], but in order to present the required techniques in a fully self-contained way, we prefer to include the complete proofs here. In section 4 we prove an Invariance Theorem that states that under certain mild conditions the inclusion of two Martin-Löf classes of random reals implies absolute continuity of the corresponding measures. Somewhat more general, the theorem states that under some conditions any computable map that preserves randomness has the property that its image measure is absolutely continuous with respect to the target space measure. This statement about randomness has some loose similarity with the classical Banach-Mazur Theorem [2] that states that functions, which preserve computable sequences are necessarily continuous on computable inputs (this is an early version of the Ceitin Theorem, see the discussions in [12, 13]). In both cases the preservation of a local property, computable sequences on the one hand and random points on the other hand, guarantees a global property of the function, continuity on the one hand, absolute continuity of related measures on the other hand. The Invariance Theorem together with the results on absolute continuity of measures imply that I_μ and I_λ are incomparable with respect to inclusion. Finally, in section 5 we prove that also the classes I_ρ and I_λ are incomparable and we provide some further interesting properties of random reals with respect to the signed-digit measure. In particular, we prove that I_μ satisfies some natural condition that any set of random reals should satisfy. For instance, I_μ only contains transcendental numbers.

In order to understand the measure theoretic properties of the signed-digit measure μ it turns out to be useful to analyse the semantic name space of the signed-digit representation that is illustrated in Figure 1. Any path in the tree corresponds to a name $p \in \Sigma^\omega$ of a real number $x = \rho(p)$ and nodes from which one can reach exactly the same real numbers are identified. The weight assigned to a node indicates the number of paths leading to that node. The whole tree can be considered as a ternary version of Pascal's triangle and it is also known as *Stern-Brocot tree* or *Farey tree* in the literature [18] (in the Farey tree the nodes typically carry fractions that are closely related to the weights in our nodes). In row n of the tree the weights $f_n(x)$ of nodes that correspond to a number x can be calculated by

$$f_n(x) := |V_n(x)| \text{ with } V_n(x) := \left\{ v \in \Sigma^n : x = \sum_{j=0}^{n-1} v(j) 2^{-j-1} \right\}$$

for all $x \in \mathbb{R}$. We will use the abbreviation

$$w_{n,a} := f_n(a2^{-n})$$

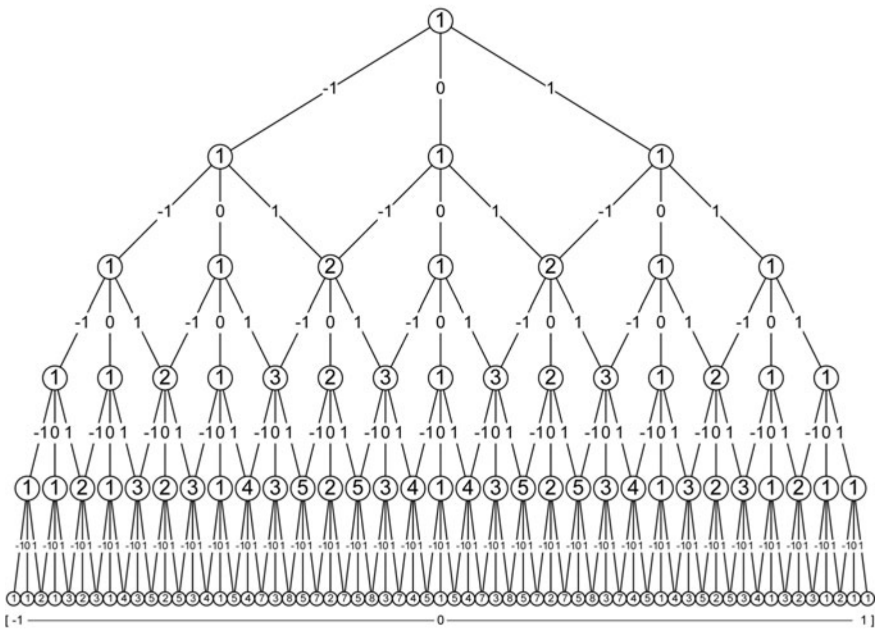

Figure 1. The Stern-Brocot tree

for all $n \in \mathbb{N} := \{0, 1, 2, ...\}$ and integers $a \in \mathbb{Z}$. For values, other than a in

$$A_n := \{-(2^n - 1), ..., 2^n - 1\}$$

one obtains $w_{n,a} = 0$. Intuitively, the values $w_{n,a}$ correspond to the weights in the tree in Figure 1 in the n–th row (where the counting starts with 0) and the a–th position (where the middle position has number 0 and starting from there the positions on the left are numbered with negative integers and the positions on the right with positive integers). The fact that the Stern-Brocot tree is a ternary version of Pascal's triangle is reflected in the following lemma, which describes the inductive rule according to which the weights can be calculated (each weight as the sum of the weights of its predecessors). It is an immediate consequence that the maximal value on layer n is the *Fibonacci number* F_n. We recall that $F_0 := 0$, $F_1 := 1$ and $F_{n+2} := F_n + F_{n+1}$ for all $n \in \mathbb{N}$.

Lemma 1.1. For any $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ we obtain $w_{n,a} = 0$ if $a \notin A_n$ and

(1) $w_{n+1,a} = w_{n,a/2}$, if a is even,

(2) $w_{n+1,a} = w_{n,(a-1)/2} + w_{n,(a+1)/2} = w_{n+1,a-1} + w_{n+1,a+1}$, if a is odd,

(3) $F_{n+1} = \max\{w_{n,a} : a \in \mathbb{Z}\}$.

Here the given rule for the odd weights is correct at the boundary (i.e. for minimal and maximal $a \in A_n$) as well, as all weights outside the tree are 0. Many other properties of the weights are known and have already been proved by Stern and others [26, 19]. For instance, Stern proved that the maximal multiplicity with which a number k will appear in odd positions of some row of the tree is $\varphi(k)$, where φ denotes Euler's totient function (and this multiplicity is reached exactly from row $n = k$ onwards). The weights in the nodes of the Stern-Brocot tree are also know as *Stern's diatomic sequence* (see also A002487 in the On-Line Encyclopedia of Integer Sequences) and the values in nodes in odd positions correspond to the values of denominators of Farey tree fractions (A007306 in the On-Line Encyclopedia of Integer Sequences):

$$1, 2, 3, 3, 4, 5, 5, 4, 5, 7, 8, 7, 7, 8, 7, 5, 6, 9, 11, 10, 11, 13, 12, 9, 9, 12, 13, 11, 10, 11, 9,$$
$$6, 7, 11, 14, 13, 15, 18, 17, 13, 14, 19, 21, 18, 17, 19, 16, 11, 11, 16, 19, 17, 18, 21, 19, ...$$

In some sense our results can also be interpreted as a combinatorial property of this sequence and we discuss this aspect in an epilogue of the paper.

2. Continuity of the Signed-Digit Measure

The purpose of this section is to prove that the signed-digit measure μ is not absolutely continuous with respect to the Lebesgue measure λ. Using the above observations on the Stern-Brocot tree we formulate some straightforward properties of the signed-digit measure μ.

Lemma 2.1. Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ and consider $x_{n,a} := (2a+1)2^{-(n+1)}$ and $I_{n,a} := [a2^{-n}, (a+1)2^{-n}]$. Then

$$\mu(I_{n,a}) = \frac{1}{2 \cdot 3^n} \cdot f_{n+1}(x_{n,a}).$$

Proof:
Let $v \in V_{n+1}(x_{n,a})$. Then v is a path leading to a node in the Stern-Brocot tree in an odd position of row $n+1$ and it is easy to see that $\rho(v\Sigma^\omega) = I_{n,a}$. By Lemma 1.1(2) the weight of any node in an odd position in the Stern-Brocot tree is the sum of its neighbours, i.e.

$$f_{n+1}(x_{n,a}) = w_{n+1,2a+1} = w_{n+1,2a} + w_{n+1,2a+2}.$$

In order to calculate the measure of the set $\rho(v\Sigma^\omega)$ we also have to take into account all paths in $V_{n+1}(x_{n,a})$, not only v, leading to value $x_{n,a}$, but also all paths coming through the neighbour nodes. We obtain

$$\begin{aligned}
\mu(I_{n,a}) &= \gamma(\rho^{-1}\rho(v\Sigma^\omega)) \\
&= \sum_{i=1}^{\infty} \frac{w_{n+1,2a}}{3^{n+1+i}} + \frac{w_{n+1,2a+1}}{3^{n+1}} + \sum_{i=1}^{\infty} \frac{w_{n+1,2a+2}}{3^{n+1+i}} \\
&= \sum_{i=0}^{\infty} \frac{w_{n+1,2a+1}}{3^{n+1+i}} \\
&= \frac{1}{2 \cdot 3^n} \cdot f_{n+1}(x_{n,a}).
\end{aligned}$$

□

We recall that a measure μ_1 is called *absolutely continuous* with respect to another measure μ_2, in symbols $\mu_1 \ll \mu_2$, if $\mu_2(A) = 0$ implies $\mu_1(A) = 0$ for all measurable sets A. It is known that for a finite measure μ_1 this is equivalent to the condition that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all measurable A

$$\mu_2(A) < \delta \Longrightarrow \mu_1(A) < \varepsilon$$

holds true (see Theorem 2.9.6 in [3]). All measures we are interested in here are finite. If the second measure μ_2 is finite (or σ–finite), then by the Radon-Nikodym Theorem (see Theorem 2.9.8 in [3]) $\mu_1 \ll \mu_2$ holds if and only if μ_1 has a density relative to μ_2, i.e. if and only if there exists a non-negative measurable function $f : I \to \mathbb{R}$ such that $\mu_1 = f\mu_2$, which means $\mu_1(A) = \int_A f\, d\mu_2$ for all measurable A.

In a first step we want to approximate the image measure μ of the signed-digit representation ρ by simpler measures, using similar techniques as in [11]. Therefore, we use the *unit mass* δ_a at a, which is the measure defined by

$$\delta_a(A) := \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases}$$

for all measurable $A \subseteq I$. Now we can assign a measure μ_n to any row n of the Stern-Brocot tree by

$$\mu_n := \frac{1}{2 \cdot 3^n} \sum_{a \in \mathbb{Z}} f_{n+1}(x_{n,a}) \delta_{x_{n,a}}$$

(and this actually defines a measure, see Example 1.3.3 in [3]; the sum is in fact finite, as $f_{n+1}(x_{n,a}) = 0$ for $a \notin \{-2^n, ..., 2^n - 1\}$). We now show that this sequence of measures weakly converges to μ. We recall that a sequence $(\mu_n)_{n\in\mathbb{N}}$ of finite measures on I is called *weakly convergent* to some measure μ on I, if

$$\lim_{n\to\infty} \int g\, d\mu_n = \int g\, d\mu$$

for all continuous $g : I \to \mathbb{R}$.

Lemma 2.2. The sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures converges weakly to μ.

Proof:
Let $g : I \to \mathbb{R}$ be some continuous function. We consider the intervals $I_{n,a} := [a2^{-n}, (a+1)2^{-n}]$ for $a = -2^n, ..., 2^n - 1$ and $n \geq 0$ and their characteristic functions $\chi_{I_{n,a}} : I \to \mathbb{R}$. We obtain with Lemma 2.1

$$\mu(I_{n,a}) = \frac{1}{2 \cdot 3^n} \cdot f_{n+1}(x_{n,a}).$$

Now the sequence $(g_n)_{n\in\mathbb{N}}$ of step functions $g_n : I \to \mathbb{R}$

$$g_n := \sum_{a=-2^n}^{2^n-1} g(x_{n,a}) \chi_{I_{n,a}}$$

converges uniformly to g. This implies

$$\begin{aligned}\int g\, d\mu &= \lim_{n\to\infty}\int g_n\, d\mu\\ &= \lim_{n\to\infty}\sum_{a=-2^n}^{2^n-1} g(x_{n,a})\mu(I_{n,a})\\ &= \lim_{n\to\infty}\frac{1}{2\cdot 3^n}\sum_{a=-2^n}^{2^n-1} g(x_{n,a})f_{n+1}(x_{n,a})\\ &= \lim_{n\to\infty}\int g\, d\mu_n,\end{aligned}$$

which was to be proved. □

In the next step we want to determine the *characteristic function* of the measure μ, that is the Fourier transform

$$\widehat{\mu}:\mathbb{R}\to\mathbb{C}, t\mapsto\int e^{itx}d\mu(x).$$

Lemma 2.3. We obtain for all $t\in\mathbb{R}$

$$\widehat{\mu}(t)=\prod_{j=1}^{\infty}\left(1-\frac{4}{3}\sin^2(t2^{-(j+1)})\right).$$

Proof:
We first determine the Fourier transforms of the measures μ_n. As the paths $v\in\Sigma^{n+1}$ leading to $x_{n,a}=(2a+1)2^{-(n+1)}$ are necessarily such that $v(n)\neq 0$, i.e. $v\in\Sigma^n\{1,-1\}$, we obtain

$$\begin{aligned}\widehat{\mu}_n(t) &= \int e^{itx}d\mu_n(x)\\ &= \frac{1}{2\cdot 3^n}\sum_{a=-(2^n-1)}^{2^n-1}\exp(itx_{n,a})f_{n+1}(x_{n,a})\\ &= \frac{1}{2\cdot 3^n}\sum_{v\in\Sigma^n\{1,-1\}}\exp\left(it\sum_{j=1}^{n+1}v(j-1)2^{-j}\right)\\ &= \frac{\exp(-it2^{-(n+1)})+\exp(it2^{-(n+1)})}{2}\cdot\prod_{j=1}^{n}\frac{\exp(-it2^{-j})+1+\exp(it2^{-j})}{3}\\ &= \cos(t2^{-(n+1)})\cdot\prod_{j=1}^{n}\left(\frac{1+2\cos(t2^{-j})}{3}\right)\\ &= \cos(t2^{-(n+1)})\cdot\prod_{j=1}^{n}\left(1-\frac{4}{3}\sin^2(t2^{-(j+1)})\right),\end{aligned}$$

where we use $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $1 = \cos^2(x) + \sin^2(x)$ for the last equation. As $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ by Lemma 2.2 it follows by the Continuity Theorem for Fourier Transforms (see Theorem 8.2.7 in [3]) that $(\widehat{\mu}_n)_{n\in\mathbb{N}}$ converges uniformly to $\widehat{\mu}$ on every compact subset of $\mathbb{R}$ and we obtain

$$\widehat{\mu}(t) = \prod_{j=1}^{\infty} \left(1 - \frac{4}{3}\sin^2(t2^{-(j+1)})\right)$$

as claimed. □

Now we are prepared to prove the main result of this section.

Theorem 2.1. The measure μ is not absolutely continuous with respect to the Lebesgue measure λ on I.

Proof:
Let us assume that μ is absolutely continuous with respect to the Lebesgue measure λ on I. Then by the Radon-Nikodym Theorem μ has a density relatively to λ, i.e. there exists a measurable function $f : I \to \mathbb{R}$ such that

$$\mu(A) = \int_A f \, d\lambda.$$

Thus we obtain (by Theorem 2.9.3 in [3])

$$\widehat{\mu}(t) = \int e^{itx} d\mu(x) = \int e^{itx} f(x)\, d\lambda(x) = \widehat{f}(t),$$

i.e. the Fourier transform of μ is just the Fourier transform of its density with respect to the Lebesgue measure. By the Theorem of Riemann-Lebesgue (see Theorem 8.2.2 in [3]) we obtain that $\lim_{t\to\infty}\widehat{\mu}(t) = \lim_{t\to\infty}\widehat{f}(t) = 0$. However, according to Lemma 2.3 we have

$$\widehat{\mu}(t) = \prod_{j=1}^{\infty} \left(1 - \frac{4}{3}\sin^2(t2^{-(j+1)})\right)$$

and hence $\widehat{\mu}(2^{k+1}\pi) = \widehat{\mu}(2\pi)$ since the first k factors of the product with $t = 2^{k+1}\pi$ are equal to 1 and the remaining ones are equal to those of the product for $t = 2\pi$. That is $\lim_{t\to\infty}\widehat{\mu}(t) = \widehat{\mu}(2\pi)$. As $u_j := \frac{4}{3}\sin^2(\pi 2^{-j}) < 1$ for $j \geq 2$, it follows by Theorem 15.5 in [24] that $\prod_{j=2}^{\infty}(1 - u_j) > 0$ if and only if $\sum_{j=2}^{\infty} u_j < \infty$. But $\sin(x) \leq x$ for $0 \leq x \leq 1$ implies

$$\sum_{j=2}^{\infty} u_j = \sum_{j=2}^{\infty} \frac{4}{3}\sin^2(\pi 2^{-j}) \leq \frac{4}{3}\pi^2 \sum_{j=2}^{\infty} 2^{-2j} \leq \frac{\pi^2}{3}$$

and, in particular, the series $\sum_{j=2}^{\infty} u_j$ converges and we can conclude that the product $\prod_{j=2}^{\infty}(1 - u_j)$ is positive. Since $(1 - u_1) = -\frac{1}{3}$ it follows that $\widehat{\mu}(2\pi)$ is a strictly negative number. Contradiction! Thus, μ is not absolutely continuous with respect to λ. □

The same technique has been used by other authors to obtain similar results. For instance, we could also derive Theorem 2.1 as a consequence of Proposition 5.3 of [9] (for $q = 3$ and $d = 2$).

3. Singularity of the Signed-Digit Measure

In this section we prove that the signed-digit measure μ is singular with respect to the Lebesgue measure λ. We recall that two measures μ_1 and μ_2 on the same σ–algebra are called *singular* with respect to each other, in symbols $\mu_1 \perp \mu_2$, if there exists a measurable set A such that $\mu_1(A) = 0$ and $\mu_2(A^c) = 0$ (see, for instance, [8]). If a measure is singular with respect to the Lebesgue measure λ, then it is simply called *singular*. We recall that a probability measure ν on a set X is called *purely discontinuous*, if its *point spectrum*, i.e. the set P of points $x \in X$ with positive measure $\nu(x) > 0$ has itself full measure, i.e. $\nu(P) = 1$.

By a classical Theorem of Jessen and Wintner (see Theorem 35 in [15]) a measure ν that can be obtained as a weakly convergent infinite convolution of purely discontinuous measures is either purely discontinuous, singular or absolutely continuous with respect to the Lebesgue measure. We prove that the signed-digit measure μ can be obtained as weak limit of a converging infinite convolution of discrete measures. We recall that the *convolution* $\mu_1 * \mu_2$ of two finite Borel measures is just defined to be the image measure of the product measure $\mu_1 \otimes \mu_2$ under the addition map (see Definition 3.4.1 in [3]). The convolution operation is known to be associative, commutative and it distributes with sums: $\nu * (\mu_1 + \mu_2) = \nu * \mu_1 + \nu * \mu_2$. Moreover, $\mu_1 * (\alpha\mu_2) = (\alpha\mu_1) * \mu_2 = \alpha(\mu_1 * \mu_2)$ for positive real numbers α and $\delta_a * \delta_b = \delta_{a+b}$ for the unit masses (see Section 3.4 in [3]). We directly obtain the following result.

Proposition 3.1. The signed-digit measure μ is the weak limit of the infinite convolution product of the discrete measures

$$\nu_n := \frac{1}{3}(\delta_{-2^{-n}} + \delta_0 + \delta_{2^{-n}})$$

for $n \geq 1$.

Proof:
One easily proves by induction that

$$\mu'_n := \nu_1 * \nu_2 * ... * \nu_n = \frac{1}{3^n} \sum_{a \in \mathbb{Z}} w_{n,a} \delta_{a2^{-n}}.$$

For $n = 1$ this is obviously correct, and in the induction step we obtain with the help of Lemma 1.1

$$\begin{aligned}
\nu_{n+1} * \mu'_n &= \frac{1}{3}(\delta_{-2^{-n-1}} + \delta_0 + \delta_{2^{-n-1}}) * \frac{1}{3^n} \sum_{a \in \mathbb{Z}} w_{n,a} \delta_{a2^{-n}} \\
&= \frac{1}{3^{n+1}} \left(\sum_{a \in \mathbb{Z}} w_{n,a} (\delta_{(2a-1)2^{-n-1}} + \delta_{2a2^{-n-1}} + \delta_{(2a+1)2^{-n-1}}) \right) \\
&= \frac{1}{3^{n+1}} \left(\sum_{a \in 2\mathbb{Z}+1} (w_{n,\frac{a-1}{2}} + w_{n,\frac{a+1}{2}}) \delta_{a2^{-n-1}} + \sum_{a \in 2\mathbb{Z}} w_{n,\frac{a}{2}} \delta_{a2^{-n-1}} \right) \\
&= \frac{1}{3^{n+1}} \sum_{a \in \mathbb{Z}} w_{n+1,a} \delta_{a2^{-n-1}} \\
&= \mu'_{n+1}.
\end{aligned}$$

Almost exactly as in Lemma 2.3 one checks

$$\widehat{\mu'_n}(t) = \prod_{j=1}^{n} \left(1 - \frac{4}{3}\sin^2(t2^{-(j+1)})\right).$$

That is, the sequences of Fourier transforms $(\widehat{\mu_n})_{n\in\mathbb{N}}$ and $(\widehat{\mu'_n})_{n\in\mathbb{N}}$ have the same limit $\widehat{\mu}$ and by uniqueness of the Fourier transforms (see Theorem 8.2.4 in [3]) it follows that $(\mu'_n)_{n\in\mathbb{N}}$ weakly converges to μ. $\square$

In the next step we prove that μ is not purely discontinuous.

Lemma 3.1. For any $x \in I$ we obtain $\mu(\{x\}) = 0$ and hence the signed-digit measure μ is, in particular, not purely discontinuous.

Proof:
Any point $x \in I$ is included in intervals of the form $I_{n,a}$ for all $n \in \mathbb{N}$ and suitable $a \in \mathbb{Z}$. By Lemma 1.1 and Lemma 2.1 we obtain

$$\mu(\{x\}) \leq \mu(I_{n,a}) \leq \frac{1}{2 \cdot 3^n} F_{n+2} \leq \left(\frac{2}{3}\right)^n$$

for all those n and a, where the last inequality follows as $F_{n+1} \leq 2^n$ holds for the Fibonacci numbers F_n. As the inequality above holds for all n, it follows that $\mu(\{x\}) = 0$. $\square$

It is clear that the measures ν_n in Proposition 3.1 are purely discontinuous. As the Theorem of Jessen and Wintner states that any measure that is the weak limit of an infinite convolution of purely discontinuous measures and that is neither purely discontinuous, nor absolutely continuous with respect to Lebesgue measure, has to be singular, we directly obtain the following corollary of Proposition 3.1, Lemma 3.1 and Theorem 2.1.

Corollary 3.1. The signed-digit measure μ is singular and hence the Lebesgue measure λ restricted to I is not absolutely continuous with respect to μ.

A measure related to μ has also been studied as so-called $(2, 3)$–Bernoulli convolution and a result corresponding to our corollary has also been observed in [10].

4. Randomness Preservation

In this section we will discuss randomness with respect to different measures and we prove a theorem that formulates a necessary condition in terms of measures that any randomness preserving computable map has to satisfy. Throughout the remainder of the paper X and Y are both supposed to denote the Euclidean interval I or the Cantor space Σ^ω. All results formulated for X and Y simultaneously hold for both cases. We start with recalling the Martin-Löf style definition of randomness with respect to measures. We will only consider Borel measures, i.e. measures defined on the Borel σ–algebra of X, which is the σ–algebra generated by the open subsets.

Definition 4.1. (Randomness)
Let ν be a Borel measure on X. A point $x \in X$ is called *non-random* (with respect to ν) if it admits a Martin-Löf test, i.e. if there exists a computable sequence $(U_n)_{n\in\mathbb{N}}$ of c.e. open sets $U_n \subseteq X$ such that $x \in \bigcap_{n=0}^{\infty} U_n$ and $\nu(U_n) < 2^{-n}$ for all $n \in \mathbb{N}$. A point $x \in X$ is called *random* (with respect to ν) if it is not non-random. We denote the set of random points with respect to ν by X_ν.

Computability of open subsets is understood as computability in the sense of computable analysis (see [28]). Roughly speaking, an open subset $U \subseteq X$ is called *computably enumerable open* (c.e. open for short) if one can computably enumerate a sequence of basic open sets B_j whose union is U. In case of $X = \Sigma^\omega$, the basic open sets are the balls $B_j = v\Sigma^\omega$ with $v \in \Sigma^*$ and in case of $X = I$ the basic open balls are the rational intervals $B_j = (r,s) \cap I$ with $r, s \in \mathbb{Q}$. There exists a representation $\vartheta :\subseteq \Sigma^\omega \to \mathcal{O}(X)$ of the set $\mathcal{O}(X)$ of open subsets of X which has the property that the operations union and intersection are computable (see [28, 5], the symbol "$\subseteq$" indicates that the map is partial). A name $p \in \Sigma^\omega$ of an open set U is just an enumeration of numbers j such that the union of the corresponding B_j is U. A *computable sequence of c.e. open sets* is just a sequence that is computable with respect to ϑ. From now on we consider $(B_j)_{j\in\mathbb{N}}$ as some fixed standard enumeration of the basic open sets of X. The sequences that are random in Σ^ω with respect to the uniform measure γ are also just called random sequences (and they are also random in the sense of Kolmogorov, i.e. in terms of Kolmogorov complexity). The real numbers in I that are random with respect to the Lebesgue measure λ are also just called the random reals. We say that a measure ν is *upper semi-computable*, if the set

$$\left\{(n,k) \in \mathbb{N}^2 : \nu\left(\bigcup_{i\in D_n} B_i\right) < 2^{-k}\right\}$$

is computably enumerable. Analogously, we say that ν is *lower semi-computable*, if the set with "$>$" instead of "$<$" is computably enumerable (see [14]). Here $D_{\sum_{k\in A} 2^k} = A$ for all finite $A \subseteq \mathbb{N}$. We prove a lemma that shows that the image measure of a lower semi-computable measure under a computable function is lower semi-computable again.

Lemma 4.1. Let μ_X be a lower semi-computable measure on X and let the function $T : X \to Y$ be computable. Then the image measure $T(\mu_X)$ is a lower semi-computable measure on Y.

Proof:
For all open sets $U \subseteq Y$ the preimage $T^{-1}(U)$ is open as any computable T is continuous. More than this, for computable T and given U (with respect to ϑ) we can even effectively determine a sequence $(B_{i_n})_{n\in\mathbb{N}}$ such that $T^{-1}(U) = \bigcup_{n\in\mathbb{N}} B_{i_n}$ (see for instance Theorem 6.2.4.1 in [28]). We obtain for all $m \in \mathbb{N}$

$$\begin{aligned} & T(\mu_X)(U) = \mu_X(T^{-1}(U)) > 2^{-m} \\ \iff \quad & (\exists i_1, ..., i_n)\ \mu_X(B_{i_1} \cup ... \cup B_{i_n}) > 2^{-m} \end{aligned}$$

as any measure μ_X is continuous from below (see Theorem 1.3.2 in [3]). As μ_X is lower semi-computable, the relation $\mu_X(B_{i_1} \cup ... \cup B_{i_n}) > 2^{-m}$ is c.e. in $i_1, ..., i_n$. Altogether, this shows that the image measure $T(\mu_X)$ is lower semi-computable. $\square$

Now we are prepared to prove the main result of this section, which shows that if a computable function preserves randomness on measure spaces, then its image measure has to be absolutely continuous with respect to the target measure.

Theorem 4.1. (Invariance Theorem)
Let μ_X and μ_Y be finite Borel measures on X and Y, respectively, that are lower and upper semi-computable respectively, and let the function $T : X \to Y$ be computable Then

$$T(X_{\mu_X}) \subseteq Y_{\mu_Y} \Longrightarrow T(\mu_X) \ll \mu_Y.$$

Proof:
We consider the contraposition. If $T(\mu_X) \not\ll \mu_Y$, then there exists an $\varepsilon > 0$ such that

$$(\forall \delta > 0)(\exists \text{ measurable } A \subseteq Y)\ \mu_Y(A) < \delta \text{ and } T(\mu_X)(A) > \varepsilon.$$

Without loss of generality, we can assume that ε has the form $\varepsilon = 2^{-m}$. As Y is a Polish space, every finite Borel measure μ_Y on Y is outer regular (by Theorem 7.3.3 in [3]). That is

$$\mu_Y(A) = \inf\{\mu_Y(U) : A \subseteq U, U \in \mathcal{O}(Y)\}.$$

Thus, for any A as above there exists some open subset U with $A \subseteq U$ and $\mu_Y(U) < \delta$. By monotonicity, we obtain $T(\mu_X)(U) > \varepsilon$. That is, we can replace the condition above by

$$(\forall k)(\exists U \in \mathcal{O}(Y))\ \mu_Y(U) < 2^{-k} \text{ and } T(\mu_X)(U) > \varepsilon.$$

Any open set $U \in \mathcal{O}(Y)$ is a countable union $U = \bigcup_{j=0}^{\infty} B_{i_j}$ of basic open sets B_{i_j} and if $T(\mu_X)(U) > \varepsilon$, then there exists an $n \in \mathbb{N}$ such that $T(\mu_X)(\bigcup_{j=0}^{n} B_{i_j}) > \varepsilon$ since any measure $T(\mu_X)$ is continuous from below (see Theorem 1.3.2 in [3]). Due to monotonicity of μ_Y it follows that $\mu_Y(U) < 2^{-k}$ implies $\mu_Y(\bigcup_{j=0}^{n} B_{i_j}) < 2^{-k}$ for any $n \in \mathbb{N}$. Thus, the conditions above are equivalent to the condition

$$(\forall k)(\exists i_{k,0}, ..., i_{k,n_k})\ \mu_Y\left(\bigcup_{j=0}^{n_k} B_{i_{k,j}}\right) < 2^{-k} \text{ and } T(\mu_X)\left(\bigcup_{j=0}^{n_k} B_{i_{k,j}}\right) > \varepsilon.$$

Since μ_X is lower semi-computable and T is computable, it follows by Lemma 4.1 that $T(\mu_X)$ is lower semi-computable as well. This and the fact that μ_Y is upper semi-computable implies that we can effectively find corresponding $B_{i_{k,0}}, ..., B_{i_{k,n_k}}$ as above for any k and we can also compute the union

$$U_m = \bigcup_{k=m+1}^{\infty} \bigcup_{j=0}^{n_k} B_{i_{k,j}}.$$

We claim that $(U_m)_{m\in\mathbb{N}}$ is a Martin-Löf test with respect to μ_Y. Due to the algorithm described here, it is clear that the sequence is a computable sequence of c.e. open sets. Moreover, we obtain by subadditivity of μ_Y

$$\mu_Y(U_m) \leq \sum_{k=m+1}^{\infty} \mu_Y\left(\bigcup_{j=0}^{n_k} B_{i_{k,j}}\right) < \sum_{k=m+1}^{\infty} 2^{-k} = 2^{-m}.$$

Let $A = \bigcap_{m=1}^{\infty} U_m$. As μ_X is finite, it follows that $T(\mu_X)$ is finite and we obtain (by Corollary 2.7.2 in [3])

$$\mu_X(T^{-1}(A)) = T(\mu_X)(A) \geq \limsup_{m\to\infty} T(\mu_X)(U_m) \geq \varepsilon > 0.$$

In particular, $T^{-1}(A)$ is non-empty and $T^{-1}(A) \cap X_{\mu_X} \neq \emptyset$ (as the set of non-random elements with respect to μ_X has measure 0 by Proposition 3.11.1 in [14]). This implies $A \cap T(X_{\mu_X}) \neq \emptyset$. On the other hand, the Martin-Löf test $(U_m)_{m\in\mathbb{N}}$ is a Martin-Löf test for all points in A with respect to μ_Y. That is $A \cap Y_{\mu_Y} = \emptyset$. This implies that $T(X_{\mu_X}) \not\subseteq Y_{\mu_Y}$. □

This theorem and the previous lemma could easily be extended to the case that T is a partial computable map such that $\mu_X(X \setminus \mathrm{dom}(T)) = 0$. Applying thy Invariance Theorem 4.1 to the identity, yields the following corollary.

Corollary 4.1. If μ_1 and μ_2 are finite Borel measures on X that are lower and upper semi-computable, respectively, then

$$X_{\mu_1} \subseteq X_{\mu_2} \Longrightarrow \mu_1 \ll \mu_2.$$

Now we show that all measures considered here are lower and upper semi-computable.

Lemma 4.2. The measures λ, μ and γ are lower and upper semi-computable.

Proof:
Both properties are easy to prove for λ and γ. Let us consider the case of λ. Given a finite set of rational intervals $B_{i_n} = (a_n, b_n)$ with $n = 0, ..., k$ one can assume without loss of generality that these intervals are disjoint, as one can easily replace two overlapping intervals by a single one and then one can compute

$$\lambda(B_{i_0} \cup ... \cup B_{i_k}) = (b_0 - a_0) + ... + (b_n - a_n).$$

The proof for γ is similar but with balls $w\Sigma^\omega$ instead of intervals. As the signed-digit representation $\rho : \Sigma^\omega \to I$ is a computable map, it follows with Lemma 4.1 that $\mu = \rho(\gamma)$ is lower semi-computable as well. It remains to be proved that μ is upper semi-computable. As for Lebesgue measure λ above it is sufficient to consider one rational interval $J = (a, b)$ and to approximate $\mu(J)$ from above. In fact, as μ is continuous from above (see Theorem 1.3.2 in [3]) we obtain for any $m \in \mathbb{N}$

$$\mu(J) < 2^{-m} \iff (\exists n)(\exists A \subseteq A_{n+1}) \left(J \subseteq \bigcup_{a\in A} I_{n,a} \text{ and } \mu\left(\bigcup_{a\in A} I_{n,a}\right) < 2^{-m} \right)$$

and by Lemma 2.1

$$\mu\left(\bigcup_{a\in A} I_{n,a}\right) = \sum_{a\in A} \mu(I_{n,a}) = \frac{1}{2\cdot 3^n} \sum_{a\in A} f_{n+1}(x_{n,a}),$$

which can easily be computed. Thus, μ is also upper semi-computable. □

From this lemma together with Corollary 4.1, Theorem 2.1 and Corollary 3.1 we can directly conclude that the the classes I_μ and I_λ of random points with respect to the signed-digit measure and with respect to the Lebesgue measure (i.e. the ordinary random reals) are mutually incomparable.

Corollary 4.2. There exist random reals with respect to the signed-digit measure that are not random in the ordinary sense and vice versa. That is, we obtain $I_\mu \not\subseteq I_\lambda$ and $I_\lambda \not\subseteq I_\mu$.

5. Randomness with respect to the Signed-Digit Representation

In this section we will discuss another way how one can define randomness via the signed-digit representation and we will show that the resulting set of random reals is also incomparable with the ordinary random reals. Usually, a real number $x \in I$ is said to be *computable* with respect to some representation $\delta :\subseteq \Sigma^\omega \to I$, if it admits a computable name $p \in \Sigma^\omega$ with $\delta(p) = x$. Similarly, one could define a number x to be non-random with respect to δ, if it admits a non-random name. However, this approach does not lead to anything useful with respect to the signed-digit representation, as any number admits a non-random name. We even prove a computable version of this observation that is interesting by itself and which shows that given an arbitrary signed-digit name of a real number $x \in I$, we can effectively compute a non-random signed-digit name of the same real number.

Proposition 5.1. There exists a computable function $L : \Sigma^\omega \to \Sigma^\omega$ such that $\mathrm{range}(L)$ contains only non-random sequences and $\rho(p) = \rho L(p)$ for all $p \in \Sigma^\omega$.

Proof:
We describe a Turing machine M that computes some function L as follows. Upon input $p \in \Sigma^\omega$ the machine inspects the input until it finds a 1 followed by a finite or infinite sequence s of symbols -1. The machine replaces these symbols by symbols 0 followed by a 1 in case s is finite. All other input symbols are just copied to the output. More precisely, M replaces finite and infinite sequences

$$1(-1)^n a \text{ and } 1(-1)^\omega$$

with $a \neq -1$ and $n \geq 1$ by finite and infinite sequences

$$0^n 1a \text{ and } 0^\omega, \text{ respectively.}$$

If $a = 1$ and the next input symbol is a -1 again, then a is not written on the output, but the replacement above is applied repeatedly. Obviously, this function L is computable and as

$$2^{-k} - \sum_{i=1}^{n} 2^{-k-i} = 2^{-k-n} \text{ and } 2^{-k} - \sum_{i=1}^{\infty} 2^{-k-i} = 0,$$

respectively, it is clear that $\rho(p) = \rho L(p)$ for all $p \in \Sigma^\omega$. It is clear that by construction all sequences in the range of L do not contain the consecutive symbols $1, -1$. As any non-empty string $v \in \Sigma^*$ necessarily has to occur infinitely often in any random sequence by Theorem 6.49 in [6], it follows that the range of L only contains non-random sequences. $\square$

Now we directly obtain the following corollary.

Corollary 5.1. For any $x \in I$ there exists a non-random $p \in \Sigma^\omega$ such that $\rho(p) = x$.

The next option is to define a real number $x \in I$ to be *random with respect to a representation* δ if x admits a random name $p \in \Sigma^\omega$ with $\delta(p) = x$. It is known that the set of random numbers with respect to the ordinary base b representations coincide for all integer $b \geq 2$ and, in fact, they are all equal to the standard set of random numbers I_λ (see Theorem 6.111 in [6], or the original paper [7] and for a shorter proof [14]).

Let us denote the set of random numbers with respect to the signed-digit representation by I_ρ. We will show that I_ρ is also incomparable with I_λ. From Corollary 4.2 we know that $I_\mu \not\subseteq I_\lambda$ holds. However, by applying the Invariance Theorem 4.1 and a related positive result from [4] directly to ρ we even get a stronger result.

Theorem 5.1. We obtain $I_\rho \subseteq I_\mu$ and $I_\rho \not\subseteq I_\lambda$.

Proof:
In [4] it is proved that any point with a random name $p \in \Sigma^\omega_\gamma$ with respect to some total representation ρ is also random with respect to the randomness space induced by the representation, i.e. with respect to the induced image measure $\rho(\gamma) = \mu$. This implies $I_\rho \subseteq I_\mu$.

By applying the Invariance Theorem 4.1 to $\rho : \Sigma^\omega \to I$ we can conclude that $I_\rho \not\subseteq I_\lambda$ as $\rho(\gamma) = \mu \not\ll \lambda$. Here we exploit the fact that ρ is computable in the standard sense (i.e. we use the identity as representation of Σ^ω and ρ as representation for I). $\square$

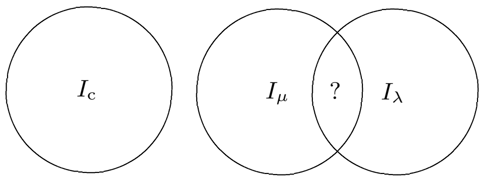

Figure 2. Classes of reals

We leave the question open whether I_ρ is equal to I_μ. We close this section by showing that the class of random reals I_μ (and therefore also I_ρ) shares some properties with I_λ. In particular, we prove that computable reals cannot be in I_μ. Let us denote by I_c the computable numbers in the interval I. Using the same idea as in Lemma 3.1 we can prove the following result that shows that computable numbers are not random with respect to both measure that we have considered in this paper.

Proposition 5.2. We obtain $I_c \subseteq (I \setminus I_\mu) \cup (I \setminus I_\lambda)$.

Proof:
Given a computable point $x \in I$, there is a computable sequence $(a_n)_{n\in\mathbb{N}}$ of integers such that $x \in (I_{n,a_n} \cup I_{n,a_n+1})^\circ$. By Lemma 1.1 and Lemma 2.1 we obtain

$$\mu(I_{n,a_n} \cup I_{n,a_n+1}) \leq \frac{1}{3^n} F_{n+2} \leq 2\left(\frac{2}{3}\right)^n$$

for all those n. We choose some computable function $r : \mathbb{N} \to \mathbb{N}$ such that $(2/3)^{r(n)} < 2^{-n-1}$ for all $n \in \mathbb{N}$. Then $(U_n)_{n\in\mathbb{N}}$ with

$$U_n := (I_{r(n),a_{r(n)}} \cup I_{r(n),a_{r(n)}+1})^\circ$$

is a computable Martin-Löf test for x and hence $x \in I \setminus I_\mu$. It is known that all computable reals are not random with respect to the Lebesgue measure (see for instance [14]). $\square$

As all algebraic numbers are computable (see for instance Corollary 6.3.10 in [28]), the previous result, in particular, implies that all types of random numbers considered in this paper are necessarily transcendental.

Corollary 5.2. The set $I_\mu \cup I_\lambda$ contains only transcendental numbers.

Figure 2 summarises our current knowledge on the classes I_c, I_μ and I_λ and their mutual relation. In fact, we have not studied the question whether I_μ and I_λ have non-empty intersection at all. The class I_μ could replace the class I_ρ in the figure and we do not know whether the two classes are identical.

6. Epilogue: Stern's Diatomic Sequence

We close this paper by showing that our main result can also be interpreted as a combinatorial property of Stern's diatomic sequence. Our result says that there is some $k \in \mathbb{N}$ with the following property: if we calculate the sum of the smallest 2^{n-k} weights in odd positions of row $n+1$ of the Stern-Brocot tree and we divide this sum by 3^n, then the resulting fractions will converge to 0 for $n \to \infty$. This is made precise in the following result.

Proposition 6.1. There exists a $k \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{2\cdot 3^n} \min\left\{\sum_{a\in A} f_{n+1}(x_{n,a}) : A \subseteq A_{n+1}, |A| = 2^{n-k}\right\} = 0.$$

Proof:
For any $n, k \in \mathbb{N}$ and any set $A \subseteq A_{n+1}$ we obtain by Lemma 2.1

$$\frac{1}{2\cdot 3^n}\sum_{a\in A} f_{n+1}(x_{n,a}) = \sum_{a\in A}\mu(I_{n,a}) = \mu\left(\bigcup_{a\in A} I_{n,a}\right).$$

Moreover, as $\lambda(I_{n,a}) = 2^{-n}$, it follows that

$$\lambda\left(\bigcup_{a\in A} I_{n,a}\right) = 2^{-k} \text{ for } |A| = 2^{n-k}.$$

By Corollary 3.1 λ is not absolutely continuous with respect to μ. Thus, there is some $k \in \mathbb{N}$ such that for any $\delta > 0$ there exists some measurable set $U \subseteq I$ with $\mu(U) < \delta$ and $\lambda(U) > 2^{-k}$. As I is a complete separable metric space and hence a Radon space, the finite Borel measures μ is automatically outer regular (see Theorem 7.3.3. in [3]). In particular, it suffices to consider open sets $U \subseteq I$ above. But if $U = \bigcup_{i=0}^{\infty} U_i$ is an arbitrary open set $U \subseteq I$ with intervals U_i, then continuity of λ from below (see Theorem 1.3.2(b) in [3]) implies that whenever $\lambda(U) > 2^{-k}$, then there exists an $m \in \mathbb{N}$ such that $\lambda\left(\bigcup_{i=0}^{m} U_i\right) > 2^{-k}$. But then we can replace any U_i by a potentially slightly smaller union of intervals $I_{n,a}$ with sufficiently large n, i.e. we can select an $n \in \mathbb{N}$ and $A \subseteq A_{n+1}$ such that $V := \bigcup_{a\in A} I_{n,a} \subseteq$

$\bigcup_{i=0}^{m} U_i$ and hence $\mu(V) \leq \mu(U) < \delta$, whereas $\lambda(V) > 2^{-k}$. In other words, we have proved that there is a $k \in \mathbb{N}$ such that for any $\delta > 0$ there exists some $n \in \mathbb{N}$ and $A \subseteq A_{n+1}$ such that

$$\mu\left(\bigcup_{a\in A} I_{n,a}\right) < \delta \text{ and } \lambda\left(\bigcup_{a\in A} I_{n,a}\right) > 2^{-k}.$$

But this obviously implies the claim as we can just choose a subset of A of size 2^{n-k}. $\square$

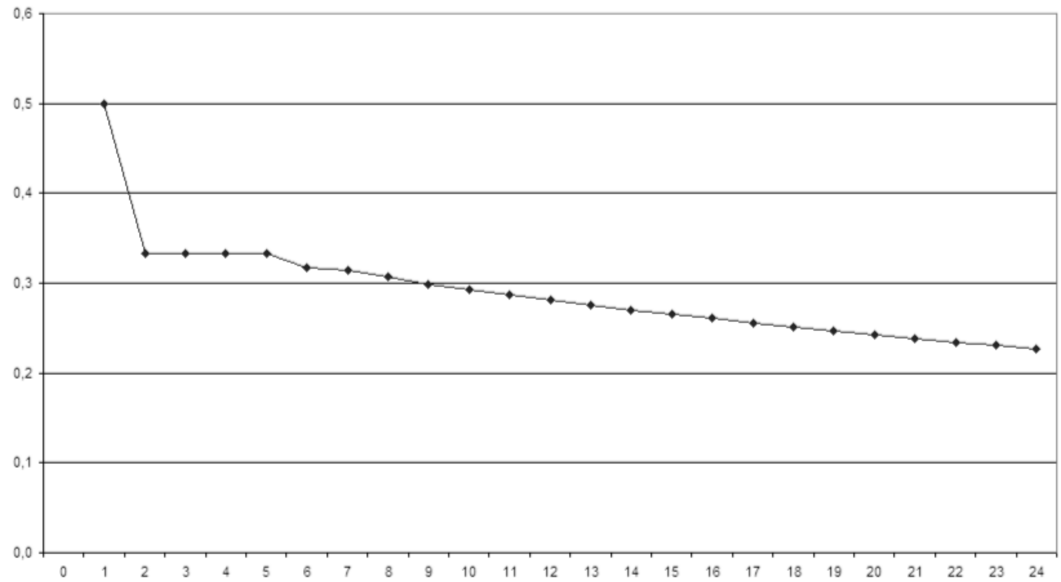

Figure 3. The values $\mu(I_n)$ in dependency of n

We did not provide any specific such $k \in \mathbb{N}$ with this property. The following table shows a calculation of some values for $k = 1$. We assume that $A \subseteq A_{n+1}$ is a set with $|A| = 2^{n-k}$ such that the sum $s_n := \sum_{a\in A} f_{n+1}(x_{n,a})$ is minimal and that $I_n := \bigcup_{a\in A} I_{n,a}$ is the corresponding set. Then we obtain the following values for s_n and $\mu(I_n)$ on the rows $n = 1, ..., 24$ of the Stern-Brocot tree.

n	s_n	$\mu(I_n)$
1	1	0.50000
2	2	0.33333
3	6	0.33333
4	18	0.33333
5	54	0.33333
6	154	0.31687
7	458	0.31412
8	1342	0.30681
9	3910	0.29797
10	11522	0.29268
11	33846	0.28659
12	99490	0.28081
13	292854	0.27552
14	862462	0.27047
15	1269145	0.26534
16	7477054	0.26054
17	22031670	0.25590
18	64907818	0.25130
19	191332926	0.24693
20	564119174	0.24268
21	1663395270	0.23852
22	4906147998	0.23451
23	14473205398	0.23060
24	42702260354	0.22679

Obviously, the values are only slowly converging and the calculation allows no clear conjecture whether the values converge to 0 or not, see Figure 3. However, Proposition 6.1 guarantees that for sufficiently large k the values will converge to 0.

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