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Smoothness analysis of subdivision schemes on regular grids by proximity

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SMOOTHNESS ANALYSIS OF SUBDIVISION SCHEMES ON REGULAR GRIDS BY PROXIMITY

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Abstract. Subdivision is a powerful way of approximating a continuous object $f(x,y)$ by a sequence $((S^l p_{i,j})_{i,j\in\mathbb{Z}})_{l\in\mathbb{N}}$ of discrete data on finer and finer grids. The rule S, that maps an approximation on a coarse grid, $S^l p$, to the approximation on the next finer grid, $S^{l+1}p$, is called subdivision scheme. If for a given scheme S every continuous object $f(x,y)$ constructed by S is of C^k smoothness, then S is said to have smoothness order k.

Subdivision schemes are well understood if they are linear. However, for various applications the data have values in a manifold which is not a vector space (for example when our data are positions of a moving rigid body). Under these circumstances, subdivsion schemes become nonlinear and much harder to analyze. One way of analyzing such schemes is to relate them to a given linear scheme and establishing a so-called proximity condition between the two schemes, which helps in proving that the two schemes share the same smoothness.

The present paper uses this method to show the C^1-smoothness of a wide class of nonlinear multivariate schemes.

Key words. linear subdivision schemes, nonlinear subdivision schemes, smoothness, multivariate subdivision, wavelets

AMS subject classifications. 41A05, 41A25, 26B05, 22E05, 53B20, 68U05

1. Introduction.

Subdivision is a powerful way of approximating a continuous object $f(x,y)$ by a sequence $((S^l p_{i,j})_{i,j\in\mathbb{Z}})_{l\in\mathbb{N}}$ of discrete data on finer and finer grids. The rule S, that maps an approximation on a coarse grid, $S^l p$ to the approximation on the next finer grid $S^{l+1}p$ is called subdivision scheme. Generally this rule depends on l but we will assume that it is always the same rule. The subdivision scheme is then called stationary; this is the class of schemes we are interested in. If for a given scheme S every continuous object $f(x,y)$ constructed by S is of C^k-smoothness, we say that the smoothness order of S is k. The fact that in this introduction we use a bivariate function $f(x,y)$ is immaterial, this paper is concerned with the general multivariate case.

To store a continuous object which is constructed via subdivision, one only needs to store a coarse approximation and the subdivision scheme constructing the object. This fact makes subdivision a valuable tool in computer aided geometric design. Another application of subdivision schemes is data compression and noise removal via wavelet-type transforms associated with the subdivision scheme [4, 2]. All these concepts can be applied to manifold valued data using nonlinear subdivision schemes if the underlying nonlinear scheme is of sufficient smoothness [10, 3].

Subdivision schemes are especially well understood if they are linear. However, for various applications the data have values in a manifold which is not a vector space (for example when our data are positions of a moving rigid body). Under these circumstances, subdivsion schemes become nonlinear and much harder to analyze. One way of analyzing such schemes is to relate them to a given linear scheme and establishing a so-called proximity condition between the two schemes, which helps in proving that the two schemes share the same smoothness. In [13, 12] this method has been used to prove smoothness properties for certain nonlinear curve schemes. The problem in proving higher order smoothness is to verify sufficient proximity conditions. In [15] it

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has been shown that on the sphere and related manifolds any nonlinear scheme that is constructed from an interpolatory linear scheme via closest point projection shares the smoothness of the linear scheme.

All these results only apply to univariate schemes. The present paper treats the general multivariate case and shows the C^1-smoothness of a wide class of nonlinear multivariate schemes. We follow the lines of [13] and also remove the necessity of a condition on the decay rates of the linear scheme needed in this work.

1.1. An example: The Kobbelt scheme and its log-exp analogue. First we describe, as an example, the construction of a nonlinear subdivision scheme for matrix groups which is analogous to a well known interpolatory linear scheme (we construct the log-exp version of Kobbelt's scheme [8]).

The Kobbelt scheme is the bivariate tensor product rule of the interpolatory four point scheme [6] with itself. It is defined as

$$\begin{aligned} Sp_{2i,2j} &= p_{i,j}, \\ Sp_{2i+1,2j} &= p_{i,j} - \frac{1}{16}(v_{-1,0} + 9v_{1,0} - v_{2,0}), \\ Sp_{2i,2j+1} &= p_{i,j} - \frac{1}{16}(v_{0,-1} + 9v_{0,1} - v_{0,2}), \text{ and} \\ Sp_{2i+1,2j+1} &= p_{i,j} + \frac{1}{256}(v_{2,2} - 9v_{2,1} - 9v_{2,0} + v_{2,-1} - \\ &\quad 9v_{1,2} + 81v_{1,1} + 81v_{1,0} - 9v_{1,-1} - 9v_{0,2} + \\ &\quad 81v_{0,1} - 9v_{0,-1} + v_{-1,2} - 9v_{-1,1} - 9v_{-1,0} + v_{-1,-1}) \end{aligned}$$

where $v_{k,l} := p_{i+k,j+l} - p_{i,j}$ is the vector pointing from $p_{i,j}$ to $p_{i+k,j+l}$, and $(Sp_{i,j})_{i,j\in\mathbb{Z}}$ means the grid function produced from $(p_{i,j})_{i,j\in\mathbb{Z}}$ via subdivision.

The rules that define the usual Kobbelt scheme are all of the form

$$Sp_{2i+k,2j+l} = p_{i,j} + \sum_{r,s} \lambda_{r,s}^{k,l} v_{r,s}, \tag{1.1}$$

where the vector $\sum_{r,s} \lambda_{r,s}^{k,l} v_{r,s}$ is added to the point $p_{i,j}$. In a matrix group, this point-vector addition of a point p and a vector v is realized with the matrix-exponential function exp, as $p \cdot \exp(v)$. The vector pointing from a group element p to the group element q is given by

$$\overrightarrow{pq} = \log(p^{-1}q). \tag{1.2}$$

We can thus define an analogue of the Kobbelt scheme which works in matrix groups as follows: for the rule given by (1.1), we define its log-exp analogue as

$$Tp_{2i+k,2j+l} = p_{i,j} \cdot \exp\big(\sum_{r,s} \lambda_{r,s}^{k,l} v_{r,s}\big), \quad v_{r,s} = \log(p_{i,j}^{-1} p_{i+u,j+v}).$$

To what extent does this nonlinear scheme share the properties of the Kobbelt scheme? It is certainly interpolatory, but does it converge? And if so, does it produce smooth limit functions? As we will see shortly, the answer to both questions is yes.

1.2. Other ways of constructing nonlinear analogues of linear schemes. Aside from the log-exp approach there are two other nonlinear variants that we want to study. They both rely on the observation in [12] that every affinely invariant scheme can be expressed in terms of the affine averaging operator

$$\mathrm{av}_\lambda(x, y) = \lambda x + (1 - \lambda)y, \quad \lambda \in \mathbb{R}.$$

In Euclidean space $\mathbb{R}^n$, the shortest path from a point x to another point y is described by the curve $c(t) := tx + (1 - t)y$. We see that $\mathrm{av}_\lambda(x, y)$ is given by $c(\lambda)$. This can be generalized to surfaces in $\mathbb{R}^n$, and Riemannian manifolds as well: in such spaces geodesics play the role of straight lines, and if we define the *geodesic average* $\text{g-av}_\lambda(x, y) := c(\lambda)$, where c is the shortest geodesic with $c(0) = x$ and $c(1) = y$, we can replace every ocurrence of the av_λ-operator by the g-av_λ-operator. This gives us an analogue of a linear scheme, that assumes its values on a given manifold. A unique shortest geodesic connecting two points need not always exist. However, it always exists locally, and if the distances between consecutive points in the grid function we want to subdivide, are sufficiently small, we won't run into any problems (cf. the discussion in [13]).

Another approach is to apply a projection operator on the averages. A projection operator for a surface $\mathcal{M} \subseteq \mathbb{R}^n$ in our setting is a smooth function $P : \mathbb{R}^n \to \mathcal{M}$ with the property that $P(x) = x$ for all $x \in \mathcal{M}$. For the nonlinear scheme we apply the operator $P \circ \mathrm{av}_\lambda(x, y)$ instead of $\mathrm{av}_\lambda(x, y)$. This defines the *projection analogue* of a linear scheme.

2. Outline of this paper. In this work we analyze nonlinear subdivision schemes that satisfy a proximity condition with a certain linear scheme and show that if the linear scheme is convergent (resp. smooth), then the nonlinear scheme is convergent (resp. smooth). In Section 3 we review some basic concepts of subdivision. Section 4 derives results of the linear theory needed for our analysis. Section 5 shows that if a scheme is in proximity with a linear convergent scheme, it is convergent. Section 6 does the same with smoothness. In Section 7 we verify that a proximity condition holds for various nonlinear analogues of linear schemes. Section 8 summarizes the results, until then valid only for surfaces and matrix groups, and extends them to abstract Lie groups and Riemannian manifolds.

3. Preliminaries. Let $(p_\alpha)_{\alpha\in\mathbb{Z}^s}$ be a bounded grid function of points $p_\alpha \in \mathbb{R}^n$ for $n \in \mathbb{N}$. In that case $(p_\alpha)_{\alpha\in\mathbb{Z}^s}$ can be viewed as an element of $(l^\infty(\mathbb{Z}^s))^n$. A subdivision scheme on a regular grid is a mapping that maps such a grid function to a new grid function $(Sp_\alpha)_{\alpha\in\mathbb{Z}^s}$ under the condition that there exists an integer $N > 1$ with the property that for all grid functions p, q with $q_\alpha = p_{\alpha+\delta_i}$, where δ_i is the i-th canonical basis vector in $\mathbb{Z}^s$, the relation $(Sq)_\alpha = (Sp)_{\alpha+N\delta_i}$ holds. N is called the dilation factor of S (typically $N = 2$). The case $s = 1$ corresponds to curve subdivison, $s = 2$ to surface subdivision as in our example in Section 1. A subdivision scheme S is called linear, if S is a linear mapping. It is easy to see that for a linear scheme, there exists a sequence $(S_\alpha)_{\alpha\in\mathbb{Z}^s}$ with the property

$$(Sp)_\alpha = \sum_{\beta\in\mathbb{Z}^s} S_{\alpha-N\beta}\, p_\beta.$$

This sequence is called the mask of S. We consider only subdivision schemes with finite mask, meaning that $S_\alpha \neq 0$ only for a finite number of α's. Thus the assumption that the grid function $(p_\alpha)_{\alpha\in\mathbb{Z}^s}$ is bounded is not actually a restriction, because a finite mask means that the subdivison operator is local.

If the elements S_α of the mask are $n \times n$ matrices, one speaks of matrix subdivision schemes.

An interpretation of this definition would be to view $(p_\alpha)_{\alpha\in\mathbb{Z}^s}$ as a discrete approximation of some continuous data defined in some domain in $\mathbb{R}^s$. The subdivision operator S maps this approximation to a finer approximation on the grid $\frac{1}{N}\mathbb{Z}^s$, S^2 maps the approximation on the even finer grid $\frac{1}{N^2}\mathbb{Z}^s$, and so on. If this sequence of approximations converges to a function defined on $\mathbb{R}^s$, the subdivision scheme is called convergent, or more precisely (see [1, Definition 2.1]):

Definition 3.1. *Let $S : (l^\infty(\mathbb{Z}^s))^n \to (l^\infty(\mathbb{Z}^s))^n$ be a subdivision scheme with dilation factor N. S is called convergent, if for any $p \in (l^\infty(\mathbb{Z}^s))^n$, there exists a continous function $S^\infty : \mathbb{R}^s \to \mathbb{R}^n$ with*

$$\lim_{k\to\infty} \max_{\alpha\in\mathbb{Z}^s} \|(S^k p)_\alpha - f(N^{-k}\alpha)\| = 0.$$

Moreover, if $f \in C^k(\mathbb{R}^s, \mathbb{R}^n)$, S is called C^k, $k \in \mathbb{N}$.

Clearly this definition is equivalent to saying that the functions $\mathcal{F}_l(S^l p)$, given by the piecewise s-multilinear Interpolation of $S^l p$ on $\frac{1}{N^l}\mathbb{Z}^s$, converge uniformly to f, as $l \to \infty$.

The piecewise s-multilinear interpolation of a grid function $p = (p_\alpha)_{\alpha\in\mathbb{Z}^s}$ is defined by

$$\mathcal{F}_l(p)(x) = \sum_{I\in\{0,1\}^s} \Big(\prod_{r:i_r=0} (1 - N^l x_r) \prod_{s:i_s=1} N^l x_s \Big) p_{\alpha+I}, \tag{3.1}$$

where $x = (x_1, \ldots, x_s) \in [\alpha_1, \alpha_1 + N^{-l}] \times [\alpha_2, \alpha_2 + N^{-l}] \times \ldots \times [\alpha_s, \alpha_s + N^{-l}]$. If p is a vector of points, then $\mathcal{F}_l$ is applied componentwise.

4. Linear Theory. We review some results from the theory of linear schemes. For further information concerning linear schemes, the reader may consult [1, 5, 17, 7]. We let $n = 1$, i.e. the points p_α are elements of $\mathbb{R}$. This is no restriction, since convergence and smoothness properties can be treated componentwise in the linear case. In the univariate setting, the (unique) forward difference operator plays an important role. On grids we look at all the forward difference operators in the different directions on the grid simultaneously. This allows us to define derived schemes as discrete analogues of the left derivatives.

Let Δ_i be the operator that maps a sequence of points, say $(p_\alpha)_{\alpha\in\mathbb{Z}^s}$, to the sequence $(p_\alpha - p_{\alpha-\delta_i})_{\alpha\in\mathbb{Z}^s}$, where δ_i denotes the i-th canonical basis vector in $\mathbb{Z}^s$.

Definition 4.1. *For $p \in l^\infty(\mathbb{Z}^s)$ we define $\mathbf{\Delta} p \in (l^\infty(\mathbb{Z}^s))^s$ as*

$$\mathbf{\Delta} p := \big(\Delta_1 p\ ,\ \Delta_2 p\ ,\ \ldots\ ,\ \Delta_s p\big)^T.$$

While linear schemes are in general defined for all input data, this is no longer true for a nonlinear subdivision scheme T, which is defined on its domain $\mathcal{D}(T) \subseteq \mathbb{R}^n$. In that case the definition of convergence and smoothness is the same, it only has to hold for grid functions $p \in \mathcal{D}(T)$.

For a subdivision scheme to converge, we need that the distance between consecutive points becomes small. We therefore define

Definition 4.2.

$$D(p) := \sup_{\alpha\in\mathbb{Z}^s} \|\mathbf{\Delta} p_\alpha\| =: \|\mathbf{\Delta} p\|_\infty, \quad \textit{and} \quad d_i(p) := \|\Delta_i p\|_\infty,$$

where $\|\mathbf{\Delta} p_\alpha\| := \max_{i=1,\dots,s} \|\Delta_i p_\alpha\|$.

The n-th derived scheme of a linear multivariate subdivision scheme S is defined by the relation

$$S^{[n]}\mathbf{\Delta}^n = N^n \mathbf{\Delta}^n S \quad \text{for } n \geq 1, \quad \text{and } S^{[0]} = S.$$

Note that for $s > 1$ the derived schemes are always matrix subdivision schemes. It is possible to characterize smoothness of a linear subdivision scheme by the decay of the derived schemes:

THEOREM 4.3. *Let S be a linear subdivision scheme. If there exists a constant $\mu_0 < 1$ with*

$$D(S^k p) \leq \mu_0^k D(p) \textit{ for all } p, \tag{4.1}$$

then S is convergent. Convergence of S implies the existence of $S^{[1]}$.

If in addition there exists a constant $\mu_1 < 1$ with

$$D((S^{[1]})^k \mathbf{\Delta} p) \leq \mu_1^k D(\mathbf{\Delta} p) \textit{ for all } p, \tag{4.2}$$

then S is C^1. The sufficient conditions (4.1) and (4.2) are almost necessary, for convergence, resp. smoothness:

THEOREM 4.4 ([5]). *Let S be a convergent linear subdivison scheme. Then $S^{[1]}$ exists and there is $l \in \mathbb{Z}^+$ such that S^l satisfies (4.1) with $\mu_{0l} = \frac{1}{N^l}\|(S^{[1]})^l|_{\mathbf{\Delta}}\| < 1$. If in addition S is C^1 and $S^{[2]}$ exists, then there exists $l \in \mathbb{Z}^+$ such that $(S^{[1]})^l$ satisfies (4.2) with $\mu_{1l} = \frac{1}{N^l}\|(S^{[2]})^l|_{\mathbf{\Delta}^2}\| < 1$.*

REMARK 4.5. *With the notation of Theorem 4.4 we may assume that there exists a constant A such that $\|(S^{[1]})^k|_{\mathbf{\Delta}}\| < A$ for all $k \in \mathbb{Z}^+$. This follows from the fact that the derived scheme $S^{[1]}$ converges to the first derivative of the limit function of S. This immediately implies that the spectral radius*

$$\rho_\infty\big((S^{[1]})|_{\mathbf{\Delta}}\big) = \liminf_{k\to\infty} \|(S^{[1]})^k|_{\mathbf{\Delta}}\|^{1/k} = 1. \tag{4.3}$$

For a linear convergent scheme S, the rate of convergence is known (compare [5, Prop. 6.7]):

LEMMA 4.6. *Let S be a linear subdivision scheme and assume that $S^{[1]}$ exists. Then there exists a constant $C > 0$ with*

$$\|\mathcal{F}_{j+1}(Sp) - \mathcal{F}_j(p)\|_\infty \leq CD(p). \tag{4.4}$$

If in addition $S^{[2]}$ exists, then there exists another constant $C > 0$ with

$$\|\mathcal{F}_{j+1}(S^{[1]}\mathbf{\Delta} p) - \mathcal{F}_j(\mathbf{\Delta} p)\|_\infty \leq CD(\mathbf{\Delta} p). \tag{4.5}$$

In the second case, $\mathcal{F}_j$ is applied componentwise.

5. Convergence. In this section we show that a nonlinear scheme is convergent if it satisfies a proximity condition with a linear and convergent scheme. Since most of the nonlinear schemes we are interested in are not defined for all input data, we have to restrict our analysis to initial data defined in certain subsets of $\mathbb{R}^n$. The following type of subsets is useful:

DEFINITION 5.1. *$\mathcal{P}_{M,\varepsilon}$ denotes the set of grid functions, whose points are contained in a subset $M \subseteq \mathbb{R}^n$, which fulfill the condition $D(p) \leq \varepsilon$.* We want to obtain convergence and smoothness properties for nonlinear schemes derived from linear ones, and we would like these properties to be directly related to the respective properties of these linar schemes. For this purpose we need a condition that ensures that the nonlinear scheme is not too far apart from the linear scheme. As we will see, the following proximity condition serves us well. The next definition is the multivariate analogue of Definition 7 in [13].

DEFINITION 5.2. *Let S, T be two subdivision schemes. We say that S, T satisfy a 0-proximity condition for the class $\mathcal{P}_{M,\varepsilon}$ with $\alpha > 1$, if there is a constant C, such that for all $p \in \mathcal{P}_{M,\varepsilon}$,*

$$\|Sp - Tp\|_\infty \leq C \cdot D(p)^\alpha. \tag{5.1}$$

From Theorem 4.4 we know that for a convergent linear scheme S there exists a contractivity constant $\mu_0 < 1$ with

$$D(S^l p) \leq \mu_0^l D(p). \tag{5.2}$$

Actually, Theorem 4.4 only assures that (5.2) holds for one of the iterates $\tilde{S} := S^k$ of S. Without loss of generality we start with $S = \tilde{S}$ and assume that (5.2) holds for S itself. The idea to prove that a nonlinear scheme T in proximity with a linear convergent scheme S is convergent, is as follows: From the proximity condition we show that if S satisfies (5.2), i.e. is convergent, then T satisfies a similar condition. After that we show that if such a condition is satisfied for a nonlinear scheme T in proximity with a linear scheme S, the scheme T is itself convergent. We make the following definition:

DEFINITION 5.3. *A subdivision scheme T is said to satisfy a convergence condition with factor $\mu_0 < 1$ and the class $\mathcal{P}_{M,\varepsilon}$, if*

$$D(T^l p) \leq \mu_0^l D(p) \tag{5.3}$$

for all initial data $p \in \mathcal{P}_{M,\varepsilon}$. Recall that T is defined for grid functions with values in $\mathcal{D}(T)$. In the course of the proof of the following central lemma, we will denote the constant from (4.4) by C'.

LEMMA 5.4. *Assume that S, T satisfy a proximity condition for all $p \in \mathcal{P}_{M,\varepsilon}$ with $\alpha > 1$, and S satisfies a convergence condition with factor $\mu_0 < 1$. Choose a subset M' of M such that there exists $\delta' > 0$ with*

$$\{x \in \mathcal{D}(T) : \inf_{y \in M'} \|x - y\| \leq \delta'\} \subseteq M. \tag{5.4}$$

Then there is is $\delta > 0$ and $\overline{\mu_0} < 1$ such that T satisfies a convergence condition with factor $\overline{\mu_0}$ for all $p \in \mathcal{P}_{M',\delta}$. By choosing δ small enough, we can achieve that $\overline{\mu_0} - \mu_0$ is arbitrarily small. The constant δ depends on the constant C from the proximity condition, on δ', on C' and on μ_0. In addition $T^l p \subseteq M$ for all $l \in \mathbb{Z}^+$. *Proof:* We choose δ small enough to satisfy the following two conditions:

$$\overline{\mu_0} := 1 + 2C\delta^{\alpha-1} < 1 \tag{5.5}$$

and

$$\delta\Big(\frac{C'}{1 - \overline{\mu_0}} + \frac{C\delta^{\alpha-1}}{1 - \overline{\mu_0}^\alpha}\Big) < \delta'. \tag{5.6}$$

Condition (5.6) will assure us that the sequence $T^l p$ stays within M. We show that for $p \in \mathcal{P}_{M',\delta}$ the norm of the difference sequence, $D(Tp)$, can be estimated by $\overline{\mu_0} D(p)$:

$$D(Tp) \leq D(Sp - Tp) + D(Sp) \leq 2\|Sp - Tp\| + D(Sp).$$

Since S is convergent with contractivity constant μ_0 and T is in proximity with S, we can deduce

$$D(Tp) \leq 2D(p)^\alpha + \mu_0 D(p) \leq D(p)(\mu_0 + 2\delta^{\alpha-1}) = \overline{\mu_0} D(p).$$

We show that if δ satisfies (5.6), then Tp is still contained in $\mathcal{P}_{M,\delta}$. Indeed, we can estimate

$$\|Tp - p\| \leq \|Sp - Tp\| + \|Sp - p\| \leq CD(p)^\alpha + C'D(p) \leq \delta\big(C' + C\delta^{\alpha-1}\big) \leq \delta',$$

by (5.6). From (5.4) it follows that $Tp \in \mathcal{P}_{M,\delta}$.

Now we use induction on l to prove that $D(T^l p) \leq \overline{\mu_0} D(p)^l$ and that $T^l p \in \mathcal{P}_{M,\delta}$ under the assumption that $T^{l-1} p \in \mathcal{P}_{M,\delta}$ and $D(T^{l-1}p) \leq \overline{\mu_0}^{l-1} D(p)$. Firstly,

$$D(T^l p) \leq D(T^l p - ST^{l-1}p) + D(ST^{l-1}p) \leq 2\|(S-T)T^{l-1}p\| + D(ST^{l-1}p).$$

By the proximity condition and the convergence condition satisfied by S, we get

$$D(T^l p) \leq 2CD(T^{l-1}p)^\alpha + \mu_0 D(T^{l-1}p) \leq \overline{\mu_0} D(T^{l-1}p) \leq \overline{\mu_0}^l D(p).$$

It remains to show that $T^l p \in \mathcal{P}_{M,\delta}$ (for initial data $p \in \mathcal{P}_{M',\delta}$).

$$\|T^l p - p\| \leq \sum_{j=1}^{l} \|T^j p - T^{j-1}p\| \leq \sum_{j=1}^{l} \|(T-S)T^{j-1}p\| + \|ST^{j-1}p - T^{j-1}p\|$$

$$\leq \sum_{j=1}^{l} CD(T^{j-1}p)^\alpha + C'D(T^{j-1}) \leq \sum_{j=0}^{l-1} C\overline{\mu_0}^{j\alpha} D(p)^\alpha + C'\overline{\mu_0}^j D(p) \leq \delta'$$

by (5.6). From the assumption (5.4) it follows that $T^l p \in \mathcal{P}_{M,\delta}$ and the induction is complete.

The statement regarding the difference $\overline{\mu_0} - \mu_0$ is immediate. □

Proposition 5.5. *Under the assumptions of Lemma 5.4, the subdivision scheme T is convergent for all initial data $p \in \mathcal{P}_{M',\delta}$.* *Proof:* We have

$$\|T^{j+1}p - T^j p\|_\infty \leq \|\mathcal{F}_{j+1}(T^{j+1}p) - \mathcal{F}_{j+1}(ST^j p)\|_\infty$$

$$+\|\mathcal{F}_{j+1}(ST^j p) - \mathcal{F}_j(T^j p)\|_\infty \leq C \cdot D(T^j p)^\alpha + C' \cdot D(T^j p),$$

by the proximity condition and by (4.4). By Proposition 5.4, this expression is bounded by a factor times μ^j with $\mu < 1$. It follows that $T^j p$ is a Cauchy sequence with respect to the sup norm. □

6. Smoothness. In this section we show that a nonlinear scheme that is in proximity with a linear and smooth scheme is itself smooth. To facilitate our analysis, we assume that the second derived scheme exists, which is not really a restriction (see [5]). We begin with a sufficient condition for a possibly nonlinear subdivision scheme to converge to a smooth limit (see [7] for a proof):

LEMMA 6.1. *Let T be a subdivision scheme and $p^l := T^l p$. Assume that $\mathcal{F}_l(p^l)$ converges to a continuous function $f = f(x_1, \dots, x_s)$ in the sup norm. Let*

$$g_l^j := \mathcal{F}_l(N^l \Delta_j p^l) \qquad (j = 1, \dots, s).$$

If g_l^j is a Cauchy sequence and $\lim_{l\to\infty} d_j(N^l \Delta_j p^l) = 0$, then f is C^1 in the variable x_j.

DEFINITION 6.2. *A subdivision scheme T is said to satisfy a mixed smoothness condition, if T satisfies a convergence condition, and there is $\mu_1 < 1$ such that*

$$D(N\boldsymbol{\Delta}T^l p) \le \mu_1^l P_1(l) D(p) \text{ for all } l, p, \tag{6.1}$$

where P_1 is a linear function with positive coefficients.

LEMMA 6.3. *Suppose that S, T satisfy a proximity condition for $p \in \mathcal{P}_{M,\varepsilon}$ with $\alpha > 1$, and that S satisfies a smoothness condition with factors μ_0, μ_1, such that*

$$\mu_0 < \mu_0^* = N^{-1/\alpha}, \quad \mu_1 < 1. \tag{6.2}$$

Then there is $\delta > 0$ such that T satisfies a mixed smoothness condition with factors $\overline{\mu_0}, \overline{\mu_1}$ which also satisfy (6.2), for all $p \in \mathcal{P}_{M,\delta}$.

Proof. By Proposition 5.4 we know that there exists $\delta > 0$ and a $\overline{\mu_0} < 1$ so that

$$D(T^l p) \le \overline{\mu_0}^l D(p) \text{ for } D(p) < \delta.$$

We want to show that there is $\overline{\mu_1} < 1$ and a linear polynomial with positive coefficients, $\overline{P_1}$, such that $D(N^l \boldsymbol{\Delta} T^l p) \le \overline{\mu_1}^l \overline{P_1}(l) D(p)$ for all p with $D(p) < \delta$. We write $d_l := D(N^l \boldsymbol{\Delta} T^l p)$ and $q := T^{l-1} p$. Next we proceed just as in the proof of Theorem 5.4 and make use of the fact that $\|\boldsymbol{\Delta} Sq - \boldsymbol{\Delta} Tq\|_\infty \le 2\|Sq - Tq\|_\infty$. This follows from the triangle inequality.

$$\begin{aligned} d_l = D(N^l \boldsymbol{\Delta} Tq) &\le \big(D(N^l \boldsymbol{\Delta} Sq) + 2N^l \|\boldsymbol{\Delta} Sq - \boldsymbol{\Delta} Tq\|_\infty\big) \\ &\le D(N^l \boldsymbol{\Delta} Sq) + 4N^l \|Sq - Tq\|_\infty \\ &\le \mu_1 D(N^{l-1} \boldsymbol{\Delta} q) + 4N^l C D(q)^\alpha, \end{aligned} \tag{6.3}$$

because of the smoothness condition satisfied by S and the proximity condition. We can estimate the last summand as follows:

$$4N^l C D(q)^\alpha \le 4N^l C(\overline{\mu_0}^{l-1} D(p))^\alpha = 4NC(N\overline{\mu_0}^\alpha)^{l-1} D(p)^\alpha.$$

Because of Proposition 5.4, (6.2) and the fact that we can make $D(p)$ arbitrarily small, we can choose $\overline{\mu_0} := \overline{\mu_0} \le N^{-1/\alpha} < 1$, and thus

$$4CN^l D(q)^\alpha \le \tilde{\mu_1}^{l-1} D(p)^\alpha P_0$$

with a positive constant P_0. Substituting into (6.3) yields

$$d_l \le d_{l-1}\mu_1 + P_0 D(p)^\alpha \sum_{j=0}^{l-1} \mu_1^{l-j-1} \tilde{\mu_1}^j.$$

Now we define $\overline{\mu_1} := \max\{\mu_1, \tilde{\mu_1}\}$ and get

$$d_l \leq \overline{\mu_1}^l d_0 + D(p)^\alpha \overline{\mu_1}^{l-1} l P_0.$$

There is a constant $C' > 0$ with $\frac{1}{\delta^{\alpha-1}C'} < \overline{\mu_1}$, so that we have

$$D(p)^\alpha \overline{\mu_1}^{l-1} \leq D(p) C' \overline{\mu_1}^l.$$

This shows that $d_l \leq \overline{\mu_1}^l \left(D(\mathbf{\Delta} p) + C'D(p)lP_0\right) \leq \overline{\mu_1}^l (2 + C'lP_0)D(p)$, and the proof is complete □ Condition (6.2) is automatically fulfilled if the second derived scheme of S exists.

Lemma 6.4. *Assume that S produces C^1 limit functions. If the derived schemes $S^{[1]}$ and $S^{[2]}$ exist, then (6.2) holds for an iterate of S.* *Proof:* For the iterated schemes S^k condition (6.2) reads $\mu_{0k} \leq N^{-k/\alpha}$, where μ_{0k} is the first contractivity constant of S^k given by $\frac{\|(S^{[1]})^k|_{\mathbf{\Delta}}\|}{N^k}$. Thus the statement of the lemma is equivalent to

$$\rho_\infty(\|S^{[1]}|_{\mathbf{\Delta}}\|) < N^{1-1/\alpha}. \tag{6.4}$$

By Remark 4.5 the spectral radius of the first derived scheme is equal to 1 and thus (6.4) is satisfied for every $\alpha > 1$. □

Now we can show that a nonlinear scheme T in proximity with a linear C^1 scheme that satisfies (6.2), is itself C^1.

Proposition 6.5. *Let S be linear, of finite mask with C^1 limits such that $S^{[1]}$ and $S^{[2]}$ exist. Then the limit function $\lim_{l\to\infty} \mathcal{F}_l(T^l p)(x_1, \ldots, x_s)$ is C^1.* *Proof:* First of all we have $d_i(N^l \Delta_i T^l p) \to 0$ by Proposition 6.3. Since we assume the affine invariance of the derived scheme $S^{[1]}$, By (4.5)

$$\|\mathcal{F}_{j+1}(S^{[1]}\mathbf{\Delta} p) - \mathcal{F}_j(\mathbf{\Delta} p)\|_\infty \leq C \cdot D(\mathbf{\Delta} p),$$

with a constant $C > 0$. Note that the smoothness condition is nothing but a convergence condition for the derived scheme $S^{[1]}$. Then

$$\begin{aligned}
&\|\mathcal{F}_{l+1}(N^{l+1}\mathbf{\Delta} T^{l+1}p) - \mathcal{F}_l(N^l \mathbf{\Delta} T^l p)\|_\infty \\
&\leq \|\mathcal{F}_{l+1}(N\mathbf{\Delta} T - S^{[1]}\mathbf{\Delta})N^l T^l p\|_\infty + \|(\mathcal{F}_{l+1}S^{[1]} - \mathcal{F}_l)\mathbf{\Delta} N^l T^l p\|_\infty \\
&\leq (2N)\|(T - S)(N^l T^l p)\|_\infty + CD(\mathbf{\Delta} N^l T^l p) \\
&\leq (2N)C'N^l(\mu_0 D(p))^{\alpha l} + C\mu_1^l P_1(l) D(p) =: \beta_l.
\end{aligned}$$

By Lemma 6.4 we may assume $\mu_0^\alpha N < 1$, and $\sum_l \beta_l < \infty$. Therefore the sequence $\mathcal{F}_l(N^l \Delta_i T^l p)$ is a Cauchy sequence, so we can apply Lemma 6.1 to complete the proof. □

7. Verification of the Proximity Conditions. In order to show that a proximity condition actually holds between an affinely invariant subdivision scheme and the perturbations that we want to study, we use a generalization of Lemma 5 in [13].

Lemma 7.1. *Let S be a linear, affinely invariant subdivision scheme on $\mathbb{Z}^s$. Then S is expressible by the repeated application of the affine averaging operator given by*

$$av_\lambda(x, y) = \lambda x + (1 - \lambda)y, \quad \lambda \in [0, 1].$$

Assume that $(f_\lambda(x, y))_{\lambda\in[0,1]}$ is a family of functions from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, satisfying

$$\|av_\lambda(x, y) - f_\lambda(x, y)\| \leq C \cdot \|x - y\|^2, \tag{7.1}$$

for a constant $C > 0$. If T is defined by exchanging the affine averaging operators av_λ by f_λ, then S and T satisfy a proximity condition of order $\alpha = 2$ in the sense of Definition 5.2 for the class $\mathcal{P}_{M,\varepsilon}$.

Proof. The fact that S is expressible via the affine averaging operator is just Theorem 1 in [13]. For the second assertion we perform induction on the number of iterations of averaging operators. For a subdivision scheme with dilation factor N we have N^s different rules, for each of them we have to verify

$$\|Sp_{\alpha+N\beta} - Tp_{\alpha+N\beta}\| \leq C \cdot D(p)^2. \tag{7.2}$$

Let m be the number of iterations of the averaging operator for the rule associated with $\beta \in \mathbb{Z}^s$. For $m = 1$ we get

$$Sp_{\alpha+N\beta} = \mathrm{av}_{\lambda_\beta}(p_{\alpha+r_\beta}, p_{\alpha+s_\beta}).$$

Then, by (7.1) there exists a constant C such that (7.2) holds true. Now we perform the induction step. Assume that our assertion is correct for m. Let y and z be points obtained from at most m iterations of affine averaging operators, and y', z' the corresponding points obtained by replacing the averaging operators by f. By the induction hypothesis,

$$\|y - y'\|,\ \|z - z'\| \leq C' \cdot D(p)^2 \text{ with a constant } C' > 0. \tag{7.3}$$

Clearly we also have

$$\|y - z\| \leq C \cdot D(p) \tag{7.4}$$

with a constant $C > 0$. Let $x := \mathrm{av}_\lambda(y, z)$, $x' := f(y', z')$, and $x'' := \mathrm{av}_\lambda(y', z')$. We need to show that there exists a constant $\tilde{C} > 0$ with

$$\|x - x'\| \leq \tilde{C} \cdot D(p)^2. \tag{7.5}$$

We estimate

$$\|x - x'\| \leq \|x' - x''\| + \|x - x''\| \leq C'' \cdot \|y' - z'\| + \|\mathrm{av}_\lambda(y - y', z - z')\|$$

$$\leq C'' \cdot (\|y - y'\| + \|y - z\| + \|z - z'\|)^2 + C''' \cdot \max(\|y - y'\|, \|z - z'\|).$$

Because of (7.3) and (7.4) this implies (7.5). □

7.1. Geodesic averaging. For a surface embedded in the Euclidean space $\mathbb{R}^n$ we construct the geodesic analogue of a linear affinely invariant subdivision scheme S by replacing the affine averaging operators av_λ by the geodesic averaging operators g-av_λ, where $\text{g-av}_\lambda(x, y) := c(\lambda)$ with c the shortest geodesic with $c(0) = x$ and $c(1) = y$. Lemma 4 in [13] states that (7.1) is fulfilled with $f_\lambda = \text{g-av}_\lambda$, and therefore, by Lemma 7.1, a proximity condition holds between a linear scheme and its geodesic analogue.

7.2. Projection analogue. The second possibility of perturbing a linear scheme that is studied in this paper is obtained by applying a projection operator P to the subdivided points, i.e. we set $f_\lambda := P \circ \mathrm{av}_\lambda$. P is only required to be smooth and leave the points of the manifold fixed. In [13] it has been shown that (7.1) holds for this class of schemes, therefore a proximity condition holds between S and the perturbed scheme.

7.3. Log-exp scheme. We go on to prove that a proximity condition holds between a linear scheme S and its log-exp analogue T. In [14] a different log-exp analogue is analyzed and a proximity condition that ensures C^2 smoothness is proven. However, this different log-exp analogue is not capable of being generalized to the multivariate setting.

PROPOSITION 7.2. *Let S be a linear subdivision scheme with finite mask and T its G-valued log-exp analogue for a matrix group G. Then for every bounded subset $M \subseteq G$ such that $exp|_M$ is diffeomorphic, S and T satisfy a proximity condition for the class $\mathcal{P}_{M,\varepsilon}$ for initial data p with $D(p)$ small enough.*

Proof. The matrix exponential function is given by

$$\exp(v) = I + v + \frac{v^2}{2} + \frac{v^3}{3!} \dots,$$

where $v \in \mathbb{R}^{m\times m}$. Since exp is a diffeomorphism in a neighbourhood of the identity matrix I, we can choose $\delta_1 > 0$ such that $\|\exp(v) - I\| < \delta_1$ implies

$$\alpha_1 \|\exp(v) - I\| \leq \|v\| \leq \alpha_2 \|\exp(v) - I\|, \quad \text{for some } \alpha_1,\ \alpha_2 > 0, \tag{7.6}$$

where $\|v\| := \sqrt{tr(v^T v)}$, the Frobenius norm.

Since we are operating in a bounded subset of G, we can assume that $\|p_\alpha\| \leq \gamma_1$, $\|p_\alpha^{-1}\| \leq \gamma_2$ for some $\gamma_1, \gamma_2 > 0$.

As explained in Section 1 the construction of new points in a linear subdivision scheme S and its log-exp analogue T has the form

$$\begin{array}{rcll} s &=& p + \sum_\beta \lambda_{\beta \in F}(q_\beta - p) & \text{(linear)}, \\ t &=& p \cdot \exp\left(\sum_{\beta\in F} \lambda_\beta v_\beta\right) & \text{(nonlinear)}, \end{array}$$

where $q_\beta = p \cdot \exp(v_\beta)$ and F is a finite subset of $\mathbb{Z}^s$. We show that

$$\|s - t\| \leq C \cdot \max_{\beta\in F} \|q_\beta - p\|^2 \quad \text{for a } C > 0, \tag{7.7}$$

which is clearly equivalent to the required proximity condition.

$$\begin{aligned} \|s - t\| &= \|p \cdot \big(I + \sum_{\beta\in F} \lambda_\beta(\exp(v_\beta) - I) - \exp\big(\sum_{\beta\in F} \lambda_\beta v_\beta\big)\big)\| \\ &\leq \gamma_1 \| \sum_{\beta\in F} \lambda_\beta \big(\frac{v_\beta^2}{2} + \frac{v_\beta^3}{3!} + \dots\big) - \frac{1}{2}\big(\sum_{\beta\in F} \lambda_\beta v_\beta\big)^2 - \frac{1}{3!}\big(\sum_{\beta\in F} \lambda_\beta v_\beta\big)^3 - \dots \| \\ &\leq C_1 \max_{\beta\in F} \|v_\beta\|^2. \end{aligned} \tag{7.8}$$

By (7.6) we can estimate

$$\begin{aligned} \|v_\beta\| &\leq \alpha_2 \|\exp(v_\beta) - I\| = \alpha_2 \|p^{-1} q_\beta - I\| \\ &\leq \alpha_2 \|p^{-1}\| \|q_\beta - p\| \leq \alpha_2 \gamma_2 \|q_\beta - p\|. \end{aligned} \tag{7.9}$$

If we combine (7.9) with (7.8), we get the proximity condition we want to establish. □

8. Results. In the previous section we verified that a proximity holds between a linear subdivision scheme and the nonlinear analogues we want to study. We combine these results with the inequalities obtained in Sections 5 and 6 and are able to state some theorems:

THEOREM 8.1. *Let S be a linear subdivision scheme with finite mask defined on $\mathbb{Z}^s$, $s \geq 1$. If S is convergent and produces continuous limit functions then, for initial data p with $D(p)$ sufficiently small, the following holds:*

(i) If $\mathcal{M} \subseteq \mathbb{R}^n$ is a smooth surface and T is the geodesic analoguous subdivision scheme constructed from S, then T is convergent.

(ii) If T is the projection analogue of S obtained from a smooth projection mapping $P : \mathbb{R}^n \to \mathcal{M}$ with $P(x) = x$ for all $x \in \mathcal{M}$, then T is convergent.

(iii) If $(G, \cdot)$ is a matrix group and T is the G-valued log-exp version of S, then T is convergent.

Proof. By the results of Section 7, a proximity condition is satisfied between T and S, therefore, by Proposition 5.5, the scheme T produces continuous limit functions. □

The assumption that $D(p)$ is sufficiently small does not mean that there exists a global constant $\delta > 0$ such that $T^l p$ converges for all initial data with $D(p)$ small enough. From what we have proved such a constant in general just exists locally. This implies that if the surface resp. Lie group is compact, there exists such a global constant. A global δ also exists in the following situations (compare [13] where the univariate situation is discussed):

- For the geodesic analogue if the normal curvatures of the surface are bounded,
- for the projection analogue if the gradient and second derivative of P are bounded,
- in general if p is contained in a compact set.

For certain special cases, for example the geodesic analogue of the chaikin algorithm, it is possible to show that the nonlinear analogue is convergent for all initial data [16].

THEOREM 8.2. *Let S be a linear subdivision scheme with finite mask defined on $\mathbb{Z}^s$, $s \geq 1$ that produces C^1 limit functions. Assume that the first and second derived schemes $S^{[1]}$, and $S^{[2]}$ exist. Then for all initial data p, the following holds:*

(i) If $\mathcal{M} \subseteq \mathbb{R}^n$ is a smooth surface and T is the geodesic analoguous subdivision scheme constructed from S, then T is C^1 if $T^l p$ converges.

(ii) If T is the projection analogue of S obtained from a smooth projection mapping $P : \mathbb{R}^n \to \mathcal{M}$ with $P(x) = x$ for all $x \in \mathcal{M}$, then T is C^1 if $T^l p$ converges.

(iii) If $(G, \cdot)$ is a matrix group and T is the G-valued log-exp version of S, then T is C^1 if $T^l p$ converges.

Proof. This is a direct consequence of the proximity condition and Proposition 6.5. □

Up to this point we have only considered surfaces embedded in the Euclidean space for the geodesic approach. This assumption can be dropped since, by Nash's global embedding theorem [9], any abstract Riemannian manifold can be smoothly embedded in some Euclidean space $\mathbb{R}^n$. Therefore the following theorem is true:

THEOREM 8.3. *Both Theorem 8.1 $(i) - (ii)$ and Theorem 8.2 $(i) - (ii)$ also hold if $\mathcal{M}$ is an abstract Riemannian manifold.*

In a similar manner one can define the log-exp analogue for a general abstract Lie group $(G, \cdot)$ with Lie algebra $\mathfrak{g}$: If a subdivision rule is given by $p_{i,j} + \sum_{r,s} \lambda_{r,s} v_{r,s}$, as in (1.1), we define the associated log-exp rule by

$$p_{i,j} \cdot \exp\big(\sum_{r,s} \lambda_{r,s} v_{r,s}\big), \text{ where } v_{r,s} := \log(p_{i,j}^{-1} \cdot p_{i+r,j+s}) \in \mathfrak{g}.$$

The mapping "exp" is the exponential mappping of G. The log function, which is the inverse of the exponential mapping is not defined globally, but only for a certain neighbourhood of 0 in $\mathfrak{g}$. Since we are only dealing with schemes with finite mask, we can restrict ourselves to a small neighbourhood of the identity element e of G. By Ado's theorem [11] every Lie group is locally isomorphic to a local matrix group, and therefore, for $D(p)$ small enough, the previous results also apply to general Lie groups:

THEOREM 8.4. *Theorem 8.1 (iii) and Theorem 8.2 (iii) also holds if $(G,\cdot)$ is an abstract Lie group.*

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