

Secant and Cosecant Sums and Bernoulli-Nörlund Polynomials

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SECANT AND COSECANT SUMS AND BERNOULLI-NÖRLUND POLYNOMIALS

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ABSTRACT. We give explicit formulæ for sums of even powers of secant and cosecant values in terms of Bernoulli numbers and central factorial numbers.

1. INTRODUCTION

We derive explicit formulæ for the *secant sum*

$$S_{2m}(N) := \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{\cos^{2m} \frac{k\pi}{N}}$$

and the *cosecant sum*

$$C_{2m}(N) := \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{\sin^{2m} \frac{k\pi}{N}}.$$

This research is inspired by the paper [2], where such formulæ were given for $m \leq 6$. Our approach, which uses contour integrals and residues, produces such formulæ quite effortlessly for any m . The main contribution of the present paper is the identification of the occurring coefficients as “classical” combinatorial quantities such as *central factorial numbers* and *Bernoulli numbers*.

2. CONTOUR INTEGRALS AND RESIDUES

We consider the *secant sum* first and start with the contour integral

$$\frac{1}{2\pi i} \oint_{R_T} \frac{1}{\cos^{2m} \pi z} \pi N \cot(\pi N z) dz, \quad (1)$$

where R_T is the rectangle with corners $-\frac{1}{2N} \pm iT$, $1 - \frac{1}{2N} \pm iT$. By periodicity of the integrand, the integrals along the vertical lines cancel. Furthermore, the integrals along the horizontal lines tend to 0 when $T \rightarrow \infty$, since \cot remains bounded and \cos tends to infinity exponentially.

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Thus we have

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \oint_{R_T} \frac{1}{\cos^{2m} \pi z} \pi N \cot(\pi N z) dz \\ &= 2 \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{\cos^{2m} \frac{k\pi}{N}} + 1 + \operatorname{Res}_{z=\frac{1}{2}} \frac{1}{\cos^{2m} \pi z} \pi N \cot(\pi N z) \end{aligned}$$

by the residue theorem. From this we derive

$$S_{2m}(N) = \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{\cos^{2m} \frac{k\pi}{N}} = -\frac{1}{2} - \frac{1}{2} \operatorname{Res}_{z=\frac{1}{2}} \frac{1}{\cos^{2m} \pi z} \pi N \cot(\pi N z). \quad (2)$$

In [4] the Bernoulli-Nörlund polynomials are introduced by the relation

$$\frac{\omega_1 \cdots \omega_k t^k e^{xt}}{(e^{\omega_1 t} - 1) \cdots (e^{\omega_k t} - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(k)}(x; \omega_1, \dots, \omega_k). \quad (3)$$

We specialise $\omega_1 = \cdots = \omega_k = 2i$, $x = ki$, and $t = \pi z$ to obtain

$$\left(\frac{\pi z}{\sin \pi z} \right)^k = \sum_{n=0}^{\infty} \frac{(\pi z)^n}{n!} B_n^{(k)}(ki; 2i, \dots, 2i).$$

Writing $P_n^{(k)} = i^n B_n^{(k)}(ki; 2i, \dots, 2i)$ and observing that $P_{2n+1}^{(k)} = 0$ we have

$$\frac{1}{\sin^k \pi z} = \sum_{n=0}^{\infty} \frac{(\pi z)^{2n-k}}{(2n)!} (-1)^n P_{2n}^{(k)}. \quad (4)$$

We have

$$\operatorname{Res}_{z=\frac{1}{2}} \frac{1}{\cos^{2m} \pi z} \pi N \cot(\pi N z) = \operatorname{Res}_{z=0} \frac{1}{\sin^{2m} \pi z} \pi N \cot(\pi N z + \frac{N}{2}\pi).$$

Notice that

$$\cot(\pi N z + \frac{N}{2}\pi) = \begin{cases} \cot(\pi N z) & \text{if } N \text{ is even,} \\ -\tan(\pi N z) & \text{if } N \text{ is odd.} \end{cases}$$

Thus it is natural to distinguish two cases according to the parity of N .

From [3] we have

$$\begin{aligned} \pi \cot \pi z &= \sum_{n=0}^{\infty} \frac{\pi^{2n} z^{2n-1}}{(2n)!} (-1)^n 4^n B_{2n}, \\ \pi \tan \pi z &= \sum_{n=1}^{\infty} \frac{\pi^{2n} z^{2n-1}}{(2n)!} (-1)^{n-1} 4^n (4^n - 1) B_{2n}. \end{aligned} \quad (5)$$

Then for even N we have

$$\begin{aligned}
& \text{Res}_{z=\frac{1}{2}} \frac{1}{\cos^{2m} \pi z} \pi N \cot(\pi N z) \\
&= [z^{-1}] \sum_{\ell=0}^{\infty} \frac{(\pi z)^{2\ell-2m}}{(2\ell)!} (-1)^\ell P_{2\ell}^{(2m)} \pi N \sum_{n=0}^{\infty} \frac{(N\pi z)^{2n-1}}{(2n)!} (-1)^n 4^n B_{2n} \\
&= \frac{(-1)^m}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} P_{2(m-n)}^{(2m)} B_{2n} (2N)^{2n}
\end{aligned}$$

and for odd N

$$\begin{aligned}
& \text{Res}_{z=\frac{1}{2}} \frac{1}{\cos^{2m} \pi z} \pi N \cot(\pi N z) \\
&= -[z^{-1}] \sum_{\ell=0}^{\infty} \frac{(\pi z)^{2\ell-2m}}{(2\ell)!} (-1)^\ell P_{2\ell}^{(2m)} \pi N \sum_{n=1}^{\infty} \frac{(\pi N z)^{2n-1}}{(2n)!} (-1)^{n-1} 4^n (4^n - 1) B_{2n} \\
&= \frac{(-1)^m}{(2m)!} \sum_{n=1}^m \binom{2m}{2n} P_{2(m-n)}^{(2m)} B_{2n} (4^n - 1) (2N)^{2n}.
\end{aligned}$$

Summing up, we have for even N

$$S_{2m}(N) = -\frac{1}{2} + \frac{(-1)^{m-1}}{2(2m)!} \sum_{n=0}^m \binom{2m}{2n} P_{2(m-n)}^{(2m)} B_{2n} (2N)^{2n} \quad (6)$$

and for odd N

$$S_{2m}(N) = -\frac{1}{2} + \frac{(-1)^{m-1}}{2(2m)!} \sum_{n=0}^m \binom{2m}{2n} P_{2(m-n)}^{(2m)} B_{2n} (4^n - 1) (2N)^{2n}. \quad (7)$$

Equation (2) gives us for even N :

$$\begin{aligned}
m=1 : & \quad \frac{1}{6} N^2 - \frac{2}{3} \\
m=2 : & \quad \frac{1}{90} N^4 + \frac{1}{9} N^2 - \frac{28}{45} \\
m=3 : & \quad \frac{1}{945} N^6 + \frac{1}{90} N^4 + \frac{4}{45} N^2 - \frac{568}{945} \\
m=4 : & \quad \frac{1}{9450} N^8 + \frac{4}{2835} N^6 + \frac{7}{675} N^4 + \frac{8}{105} N^2 - \frac{8336}{14175} \\
m=5 : & \quad \frac{1}{93555} N^{10} + \frac{1}{5670} N^8 + \frac{13}{8505} N^6 + \frac{82}{8505} N^4 + \frac{64}{945} N^2 - \frac{54176}{93555} \\
m=6 : & \quad \frac{691}{638512875} N^{12} + \frac{2}{93555} N^{10} + \frac{31}{141750} N^8 + \frac{278}{178605} N^6 + \frac{1916}{212625} N^4 + \frac{128}{2079} N^2 - \frac{365470016}{638512875} \\
m=7 : & \quad \frac{2}{18243225} N^{14} + \frac{691}{273648375} N^{12} + \frac{2}{66825} N^{10} + \frac{311}{1275750} N^8 + \frac{592}{382725} N^6 + \frac{944}{111375} N^4 \\
& \quad + \frac{512}{9009} N^2 - \frac{155194496}{273648375} \\
m=8 : & \quad \frac{3617}{325641566250} N^{16} + \frac{16}{54729675} N^{14} + \frac{113324}{28733079375} N^{12} + \frac{1072}{29469825} N^{10} + \frac{2473}{9568125} N^8 \\
& \quad + \frac{134432}{88409475} N^6 + \frac{8533792}{1064188125} N^4 + \frac{1024}{19305} N^2 - \frac{274946646272}{488462349375}
\end{aligned}$$

Equation (2) gives us for odd N :

$$\begin{aligned}
m = 1 : & \quad \frac{1}{2}N^2 - \frac{1}{2} \\
m = 2 : & \quad \frac{1}{6}N^4 + \frac{1}{3}N^2 - \frac{1}{2} \\
m = 3 : & \quad \frac{1}{15}N^6 + \frac{1}{6}N^4 + \frac{4}{15}N^2 - \frac{1}{2} \\
m = 4 : & \quad \frac{17}{630}N^8 + \frac{4}{45}N^6 + \frac{7}{45}N^4 + \frac{8}{35}N^2 - \frac{1}{2} \\
m = 5 : & \quad \frac{31}{2835}N^{10} + \frac{17}{378}N^8 + \frac{13}{135}N^6 + \frac{82}{567}N^4 + \frac{64}{315}N^2 - \frac{1}{2} \\
m = 6 : & \quad \frac{691}{155925}N^{12} + \frac{62}{2835}N^{10} + \frac{527}{9450}N^8 + \frac{278}{2835}N^6 + \frac{1916}{14175}N^4 + \frac{128}{693}N^2 - \frac{1}{2} \\
m = 7 : & \quad \frac{10922}{6081075}N^{14} + \frac{691}{66825}N^{12} + \frac{62}{2025}N^{10} + \frac{5287}{85050}N^8 + \frac{592}{6075}N^6 + \frac{944}{7425}N^4 + \frac{512}{3003}N^2 - \frac{1}{2} \\
m = 8 : & \quad \frac{929569}{1277025750}N^{16} + \frac{87376}{18243225}N^{14} + \frac{113324}{7016625}N^{12} + \frac{33232}{893025}N^{10} + \frac{42041}{637875}N^8 \\
& \quad + \frac{134432}{1403325}N^6 + \frac{8533792}{70945875}N^4 + \frac{1024}{6435}N^2 - \frac{1}{2}
\end{aligned}$$

For the *cosecant* sum, we start with the contour integral

$$\frac{1}{2\pi i} \oint_{R_T} \frac{1}{\sin^{2m} \pi z} \pi N \cot(\pi N z) dz, \quad (8)$$

which is again zero and, by summing residues, leads to the equation

$$0 = \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{\sin^{2m} \frac{k\pi}{N}} + \frac{1}{2} \operatorname{Res}_{z=0} \frac{1}{\sin^{2m} \pi z} \pi N \cot(\pi N z) + \frac{1 + (-1)^N}{4}.$$

We observe that

$$\sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{\sin^{2m} \frac{k\pi}{N}} + \frac{1 + (-1)^N}{4}$$

equals the residue that we already calculated for $S_{2m}(N)$ and N even. Thus we have

$$C_{2m}(N) = \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{\sin^{2m} \frac{k\pi}{N}} = \frac{(-1)^m}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} P_{2(m-n)}^{(2m)} B_{2n}(2N)^{2n} - \frac{1 + (-1)^N}{4}. \quad (9)$$

3. COMPUTING $P_{2n}^{(2m)}$

In this section we want to have a closer look at the Laurent series expansion of $\sin^{-2m} \pi z$. Our approach is somewhat similar to the one used in [1].

We start with the expansion (5). Differentiating yields

$$\frac{1}{\sin^2 \pi z} = \sum_{n=0}^{\infty} \frac{(\pi z)^{2n-2}}{(2n)!} (2n-1)(-1)^{n-1} 4^n B_{2n}.$$

This gives

$$P_{2n}^{(2)} = -(2n-1)4^n B_{2n}. \quad (10)$$

Differentiating $\sin^{-2m} \pi z$ twice yields

$$\frac{d^2}{dz^2} \frac{1}{\sin^{2m} \pi z} = 2m(2m+1)\pi^2 \frac{1}{\sin^{2m+2} \pi z} - 4m^2 \pi^2 \frac{1}{\sin^{2m} \pi z}. \quad (11)$$

We now write

$$\frac{1}{\sin^{2m} \pi z} = H_{2m}(z) + R_{2m}(z) = \frac{1}{(2m-1)!} \sum_{\ell=1}^m \frac{(2\ell-1)! b_{2\ell}^{(2m)} 4^{m-\ell}}{(\pi z)^{2\ell}} + R_{2m}(z), \quad (12)$$

where H_{2m} is the principal part around $z = 0$ and R_{2m} denotes the regular part. Since differentiation preserves principal and regular parts, (11) gives

$$H_{2m}''(z) = \pi^2 2m(2m+1) H_{2m+2}(z) - 4m^2 \pi^2 H_{2m}(z), \quad (13)$$

which gives the recursion (setting $b_0^{(2m)} = b_{2m+2}^{(2m)} = 0$ and $b_2^{(2)} = 1$)

$$b_{2\ell}^{(2m+2)} = m^2 b_{2\ell}^{(2m)} + b_{2\ell-2}^{(2m)} \text{ for } 1 \leq \ell \leq m+1. \quad (14)$$

This recursion shows that the numbers $b_{2\ell}^{(2m)}$ are given by

$$\sum_{\ell=0}^m b_{2\ell}^{(2m)} x^{2\ell} = \prod_{k=0}^{m-1} (x^2 + k^2). \quad (15)$$

Thus they are closely related to the *central factorial* numbers $t(n, k)$ studied in [5, p. 213]:

$$x \prod_{k=1}^{m-1} (x^2 - k^2) = \sum_{k=0}^{2m} t(2m, 2k+1) x^{2k+1}$$

and a similar expression for odd first argument. This gives $b_{2\ell}^{(2m)} = (-1)^{\ell+m} t(2m, 2\ell)$. We notice that the polynomials in (15) appear *mutatis mutandis* in [2] as differential operators. These operators are used to model the recursion (13).

In Table 1 we computed the values $b_{2k}^{(2m)}$ for small values of m .

$b_k^{(m)}$	$k=2$	4	6	8	10	12	14	16	18
$m=2$	1								
4	1	1							
6	4	5	1						
8	36	49	14	1					
10	576	820	273	30	1				
12	14400	21076	7645	1023	55	1			
14	518400	773136	296296	44473	3003	91	1		
16	25401600	38402064	15291640	2475473	191620	7462	140	1	
18	1625702400	2483133696	1017067024	173721912	14739153	669188	16422	204	1

TABLE 1. Table of $b_k^{(m)}$ for small values of m (compare with [5, Table 6.1, p. 217])

We now consider the Mittag-Leffler expansion

$$\frac{1}{\sin^{2m} \pi z} = \sum_{n \in \mathbb{Z}} H_{2m}(z+n) = H_{2m}(z) + \sum_{n=1}^{\infty} (H_{2m}(z+n) + H_{2m}(z-n)). \quad (16)$$

Expanding the last sum into a power series and using (12) yields

$$\frac{1}{\sin^{2m} \pi z} = H_{2m}(z) + \frac{4^m}{(2m-1)!} \sum_{k=0}^{\infty} \frac{(\pi z)^{2k}}{(2k)!} 4^k (-1)^k \sum_{\ell=1}^m (-1)^{\ell-1} \frac{1}{2\ell+2k} b_{2\ell}^{(2m)} B_{2\ell+2k},$$

where we have used $\zeta(2k) = (-1)^{k-1} \frac{2^{2k-1} \pi^{2k}}{(2k)!} B_{2k}$. This gives

$$P_{2k}^{(2m)} = \begin{cases} 2m \binom{2k}{2m} 4^k \sum_{\ell=0}^{m-1} (-1)^{\ell-1} \frac{1}{2k-2\ell} b_{2m-2\ell}^{(2m)} B_{2k-2\ell} & \text{for } k \geq m, \\ (-1)^k 4^k b_{2m-2k}^{(2m)} / \binom{2m-1}{2k} & \text{for } 0 \leq k \leq m-1. \end{cases} \quad (17)$$

Inserting this into (6) and (7) yields for even N

$$S_{2m}(N) = \frac{4^{m-1}}{(2m-1)!} \sum_{\ell=1}^m \frac{(-1)^{\ell-1}}{\ell} b_{2\ell}^{(2m)} B_{2\ell} N^{2\ell} - \frac{4^{m-1}}{(2m-1)!} \sum_{\ell=1}^m \frac{(-1)^{\ell-1}}{\ell} b_{2\ell}^{(2m)} B_{2\ell} - \frac{1}{2} \quad (18)$$

and for odd N

$$S_{2m}(N) = \frac{4^{m-1}}{(2m-1)!} \sum_{\ell=1}^m \frac{(-1)^{\ell-1}}{\ell} b_{2\ell}^{(2m)} B_{2\ell} (4^\ell - 1) N^{2\ell} - \frac{1}{2} \quad (19)$$

Similarly, we obtain

$$C_{2m}(N) = \frac{4^{m-1}}{(2m-1)!} \sum_{\ell=1}^m \frac{(-1)^{\ell-1}}{\ell} b_{2\ell}^{(2m)} B_{2\ell} N^{2\ell} - \frac{4^{m-1}}{(2m-1)!} \sum_{\ell=1}^m \frac{(-1)^{\ell-1}}{\ell} b_{2\ell}^{(2m)} B_{2\ell} - \frac{1 + (-1)^N}{4}. \quad (20)$$

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