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The inverse Fermat-Weber problem

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The inverse Fermat-Weber problem

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Abstract

Given n points in the plane with nonnegative weights, the inverse Fermat-Weber problem consists in changing the weights at minimum cost such that a prespecified point in the plane becomes the Euclidean 1-median. The cost is proportional to the increase or decrease of the corresponding weight. In case that the prespecified point does not coincide with one of the given n points, the inverse Fermat-Weber problem can be formulated as linear program. We derive a purely combinatorial algorithm which solves the inverse Fermat-Weber problem with unit cost in $O(n \log n)$ time. If the prespecified point coincides with one of the given n points, it is shown that the corresponding inverse problem can be written as convex problem and hence is solvable in polynomial time to any fixed precision.

1 Inverse and reverse location problems

In recent years inverse and reverse optimization problems found an increased interest. In a *reverse optimization problem*, we are given a budget for modifying parameters of the problem. The goal is to modify parameters of the problem such that an objective function attains its best possible value subject to the given budget. The *inverse optimization problem* consists in changing parameters of the problem at minimum cost such that a prespecified solution becomes optimal. In one of the first papers on this subject, Burton and

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Toint [6] considered the inverse shortest path problem with an interesting application to geological sciences. For a network, they changed the edge lengths as little as possible such that a given path becomes a shortest path. A survey on inverse optimization has been compiled by Heuberger [13].

In the context of location problems Berman, Ingco and Odoni [1] studied the reverse 1-median problem in a network. Zhang, Liu and Ma [24] considered the reverse center location problem on a tree where all vertices have equal weight. For trees with n vertices they derived an algorithm with $O(n \log n)$ time complexity. On the other hand, Burkard, Gassner and Hatzl [2] proved that the reverse 1-median problem is $\mathcal{NP}$-hard on general graphs but can be solved in linear time on a cycle. In [3] the same authors suggest an $O(n \log n)$ time algorithm for the reverse 2-median problem on trees and the reverse 1-median problem on unicycle graphs.

Cai, Yang and Zhang [7] considered an inverse center location problem in networks where the weights of the vertices should be changed within given bounds such that a given vertex becomes the 1-center. They showed that even though the 1-center problem in networks is polynomially solvable, the inverse 1-center problem is $\mathcal{NP}$-hard. Recently, inverse p-median problem has been investigated by Burkard, Pleschiutschnig and Zhang [4, 5]. They showed that the discrete inverse p-median problem with real weights can be solved in polynomial time provided p is fixed and not an input parameter. They developed a greedy-like algorithm for the inverse 1-median problem in trees with positive weights with $O(n \log n)$ time complexity. They also presented a greedy-like $O(n \log n)$ time algorithm for the inverse 1-median problem in the plane provided the distances between the points are measured in the Manhattan or maximum metric. For the inverse 1-median problem on a cycle they observed that the problem can be formulated as a linear program with bounded variables and a special structure of the constraint matrix: the columns of the constraint matrix in the linear program can be partitioned into two classes in which they are monotonically decreasing. This allows to solve the problem in $O(n^2)$ time. The inverse 1-maxian problem on a tree with variable edge lengths was solved by Gassner [10].

In this paper we investigate the inverse Euclidean 1-median problem in the plane. The paper is organized as follows: In Section 2 we recall basic properties of the classical Fermat-Weber problem and we state the inverse Fermat-Weber problem. In Section 3 we derive an optimality condition for the case that P_0 is different from the given points P_j, $j = 1, 2, ..., n$. The main result is a purely combinatorial algorithm for unit cost coefficients. The algorithm is developed in Subsection 3.1, numerical example is given in Subsection 3.2 and in Subsection 3.3 we show that our algorithm does not work for general cost coefficients. Finally, in Section 4 we consider the inverse Fermat-Weber problem if the prespecified point coincides with a given point. We show that in case of unit costs the inverse Fermat-Weber problem can be written as convex problem and hence is solvable in polynomial time to any precision.

2 The Fermat-Weber problem and its inverse

Given n points P_i, $i = 1, 2, ..., n$, in the Euclidean space $\mathbb{R}^d$, together with nonnegative weights w_i, $i = 1, 2, ..., n$, the classical Fermat-Weber problem (Euclidean 1-median problem) asks for a point P_0 in $\mathbb{R}^d$ that minimizes the sum of the (weighted) Euclidean distances to P_i, $i = 1, 2, ..., n$:

$$\min_{P_0} \sum_{i=1}^{n} w_i d(P_0, P_i) \tag{1}$$

Fermat noted on the margin of his treatise on maxima and minima the question to find a point whose distance from three given points is minimum. This Euclidean 1-median problem on three points in the plane was first solved by Torricelli early in the 17th century. Other geometric solution techniques were subsequently found by Cavalieri and Simpson. For a history of this problem consult the nice paper of Krarup and Vajda [14]. At the beginning of the 20th century the German economist Weber wrote a fundamental article *Über den Standort von Industrien* in which he introduced the weighted version of Fermat's problem for n points in the Euclidean plane. For this reason the Euclidean 1-median problem in the plane is nowadays called Fermat-Weber problem.

The problem is easy to solve on the real line in $O(n)$ time. In two or more dimensions, however, the exact location of the Euclidean 1-median is difficult. Indeed, neither a polynomial-time algorithm is known, nor the problem has been shown to be $\mathcal{NP}$-hard [12]. The most common approach for solving the Fermat-Weber problem in the plane is Weiszfeld's fixed point algorithm [22], an iterative procedure that converges under special assumptions to the Euclidean 1-median. As it turned out that Weiszfeld's algorithm does not yield in general the optimal solution, the method has been analyzed by several authors including Miehle [18], Kuhn and Kuenne [15], Cooper [8], Ostresh [19], and recently Rautenbach et al.[21]. An survey on the history and solution approaches for the Euclidean 1-median problem can be found in [9, 23].

In the following we consider the weighted Fermat-Weber problem (1-median problem) in $\mathbb{R}^2$. Given distinct points P_i in $\mathbb{R}^2$ with positive weights w_i we want to find a point P_0 that minimizes the weighted sum of Euclidean distances from P_0 to the given points:

$$f(P_0) = \sum_{i=1}^{n} w_i d(P_0, P_i).$$

The points are assumed to be not collinear. In order to state necessary and sufficient conditions for a point P_0 to be an optimal location we start with the definition of the resultant force in P_0.

Definition 2.1. *If $P_0 \neq P_i$ for all $i = 1, 2, ..., n$ the resultant force $R(P_0)$ at P_0 is given by:*

$$R(P_0) := \sum_{i=1}^{n} \frac{w_i}{d(P_i, P_0)}(P_i - P_0);$$

If $P_0 = P_j$ for some $j = 1, 2, ..., n$, we have

$$R(P_0) := \max(||R_j|| - w_j, 0)\frac{R_j}{||R_j||}$$

where

$$R_j := \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{w_i}{d(P_i, P_j)}(P_i - P_j).$$

Thus, for $P_0 = P_j$, we have $R(P_j) = 0$, if $w_j \geq ||R_j||$. Otherwise, there is a resultant vector in the direction of R_j with length $||R_j|| - w_j$. This definition leads to the following optimality criterion

Theorem 2.2. *The point P_0 is a solution of the Fermat-Weber problem if and only if*

$$R(P_0) = 0.$$

For a proof, see e.g., Kuhn [16]. As a consequence of this condition we get

Theorem 2.3. *If the point P_0 is an optimal solution of the Fermat-Weber problem, then P_0 lies in the convex hull of the points P_i, $i = 1, 2, ..., n$.*

Now we introduce the *inverse Fermat-Weber problem in the plane.* Let $n + 1$ points $P_i = (x_i, y_i)$, $i = 0, 1, 2, ..., n$, with some positive weight w_i be given. We want to modify the vertex weights at minimum cost such that P_0 becomes the Euclidean 1-median. Suppose that we incur the nonnegative cost c_i, if the weight w_i is increased or decreased. In order to guarantee a finite solution we assume that the changed vertex weights w^* must obey the bounds $\underline{w}_i \le w_i^* \le \overline{w}_i$ for all $i = 0, 1, 2, ..., n$. Let p_i denote the amount by which the weight w_i is increased. Similarly, let q_i denote the amount by which the weight w_i is decreased. Thus the inverse Fermat-Weber problem can be expressed as follows:

Find new vertex weights w_i^*, $i = 0, 1, ..., n$, such that the point P_0 is a Euclidean 1-median with respect to the new weights w_i^*, the new weights lie within the given bounds $\underline{w}_i \le w_i^* \le \overline{w}_i$ for all $i = 0, 1, ..., n$, and the total cost

$$\sum_{i=1}^{n} c_i(p_i + q_i)$$

for changing the weights becomes minimum.

If point P_0 does not belong to the given set of weighted points in the plane, then the task is to change the weights of $P_1, \dots, P_n$ at minimum cost such that the weights satisfy the bound constraints and P_0 becomes 1-median with respect to the new weights.

Whereas the classical Fermat-Weber problem is computationally hard, we shall show in the following that its inverse version is polynomially solvable.

3 The inverse Fermat-Weber problem for $P_0 \neq P_j$

In the following we assume that P_0 lies in the interior of the convex hull of the points P_i, $i = 1, 2, ..., n$, and $P_0 \neq P_i$ for all $i = 1, 2, ..., n$. The case that P_0 lies on the boundary of the convex region and $P_0 \neq P_i$ reduces to an inverse 1-median problem on a line which can be solved by the method developed in Burkard, Pleschiutschnig and Zhang [5]. The case that P_0 coincides with a given point P_j will be treated in Section 4. Under the assumptions above the optimality criterion of Theorem 2.2 states that the weighted sum of *directions* from P_0 to P_i must vanish. Thus we can reduce the given problem to an inverse Fermat-Weber problem where all points P_i, $i = 1, 2, ..., n$, lie on the unit circle with center $P_0 = (0, 0)$. We choose P_0 as origin of our coordinate system and we replace the coordinates (x_i, y_i) of a given point P_i by

$$\hat{x}_i := \frac{x_i}{d(P_i, P_0)} \quad \text{and} \quad \hat{y}_i := \frac{y_i}{d(P_i, P_0)}. \tag{2}$$

Note that possibly two or more different points may coincide on the unit cycle. By Theorem (2.2) point $P_0 = (0, 0)$ is a Euclidean 1-median for points on the unit circle if and only if

$$\begin{aligned} R_x(w) &:= \sum_{i=1}^{n} w_i \hat{x}_i = 0, && (3)\\ R_y(w) &:= \sum_{i=1}^{n} w_i \hat{y}_i = 0. && (4) \end{aligned}$$

Thus, the inverse Fermat-Weber problem can be stated as the following linear program with $2n$ bounded variables and two equality constraints as was already pointed out by Plastria [20]:

$$\begin{array}{lllll} \min & \sum_{i=1}^{n} c_i(p_i + q_i) & & & \\ \text{s.t.} & \sum_{i=1}^{n} (w_i + (p_i - q_i))\hat{x}_i & = & 0 & \\ & \sum_{i=1}^{n} (w_i + (p_i - q_i))\hat{y}_i & = & 0 & \\ & p_i & \leq & \overline{w}_i - w_i & \text{for } i = 1, 2, ..., n, \\ & q_i & \leq & w_i - \underline{w}_i & \text{for } i = 1, 2, ..., n, \\ & p_i, q_i & \geq & 0 & \text{for } i = 1, 2, ..., n. \end{array}$$

Therefore it can be solved in linear time due to Megiddo and Tamir [17]. Moreover, the above formulation as a linear program shows immediately the following Lemma.

Proposition 3.1. *If P_0 lies in the interior of the convex hull of the points P_i, $i = 1, 2, ..., n$, and the given bounds of the weights allow a feasible solution, then there exists always an optimal solution of the inverse Fermat-Weber problem where at most two modified weights lie strictly between their lower and their upper bound.*

Due to the positive cost coefficients c_i, $i = 1, 2, ..., n$, we get for any optimal solution (p^*, q^*) the orthogonality condition

$$p_i^* q_i^* = 0 \quad \text{for all} \quad i = 1, 2, ..., n. \tag{5}$$

For the sake of a simple notation we shall call in the following the coordinates of the points P_i on the unit cycle just (x_i, y_i).

Since the Euclidean distance is invariant with respect to rotation and reflection, we can always assume that

$$R_x(w) = 0 \quad \text{and} \quad R_y(w) \leq 0. \tag{6}$$

If $R_y(w) = 0$, then the weights w_i, $i = 1, 2, ..., n$, provide an optimal solution. Therefore we assume in the following $R_y(w) < 0$. We call $|R_y(w)|$ the *optimality gap* $G(w)$. In the following section we derive a purely combinatorial greedy-type algorithm for the inverse Fermat-Weber problem with unit cost which keeps in every step $R_x(w) = 0$ and reaches after at most n steps an optimal solution, i.e., $R_y(w) = 0$, if the problem is feasible.

3.1 A greedy algorithm for the inverse Fermat-Weber problem with unit cost

The greedy algorithm is based on a sequence of weight changes for points. If by chance one of the given points coincides with $A := (0, 1)$ or $B := (0, -1)$, then we can decrease $G(w)$ by changing the weight of this point without violating $R_x(w) = 0$. If A is a given point, then the weight w_A of A is increased by $\min(\overline{w}_A - w_A, G(w))$. This yields a new optimality gap $G(w)$. Thereafter, if $G(w)$ is still positive, the weight of B is decreased to $\min(w_B - \underline{w}_B, G(w))$.

In the following we shall always assume that $G(w) > 0$ and that the weights of A and B are on their upper and lower bound, provided these points belong to the given points.

In order to reduce the optimality gap we simultaneously change the weights of two points, say point P_s and point P_t. If we want to decrease the optimality gap by δ, the weight change δ_s of point P_s and δ_t of point P_t have to fulfill according to (3) and (4):

$$\begin{aligned} x_s\delta_s + x_t\delta_t &= 0, & (7) \\ y_s\delta_s + y_t\delta_t &= \delta. & (8) \end{aligned}$$

If $\frac{y_s}{x_s} = \frac{y_t}{x_t}$ then the system (7) and (8) is only solvable for $\delta = 0$, i.e., the optimality gap can not be decreased by simultaneously changing the weights of P_s and P_t while the optimality condition in x-direction remains satisfied. Thus, we assume $\frac{y_s}{x_s} \neq \frac{y_t}{x_t}$. Then Cramer's rule yields

$$\begin{aligned} \delta_s &= -\frac{x_t}{x_s y_t - x_t y_s}\delta, & (9) \\ \delta_t &= \frac{x_s}{x_s y_t - x_t y_s}\delta, & (10) \end{aligned}$$

δ is to be chosen as large as possible such that $\delta \leq G(w)$ and the weight bounds for w_s and w_t are fulfilled:

$$\underline{w}_s \leq w_s - \frac{x_t}{x_s y_t - x_t y_s}\delta \leq \overline{w}_s, \tag{11}$$

$$\underline{w}_t \leq w_t + \frac{x_s}{x_s y_t - x_t y_s}\delta \leq \overline{w}_t. \tag{12}$$

The maximal possible value of δ is called the augmentation value δ_{st}. If $\delta_{st} > 0$, we call (P_s, P_t) an *augmenting pair*. The cost of an augmentation δ by the pair (P_s, P_t) is given by $|\delta_s| + |\delta_t|$. We can evaluate the efficiency of the weight change incurred by the augmenting pair (P_s, P_t) by defining the *efficiency* e_{st} as fraction of the gain in closing the optimality gap divided by costs:

$$e_{st} := \frac{\delta}{|\delta_s| + |\delta_t|}.$$

A simple calculation yields

$$e_{st} = \frac{|x_s y_t - x_t y_s|}{|x_s| + |x_t|}. \tag{13}$$

The fact that a pair with $\frac{y_s}{x_s} = \frac{y_t}{x_t}$ is not able to reduce the optimality gap is reflected by $e_{st} = 0$ if $\frac{y_s}{x_s} = \frac{y_t}{x_t}$.

An augmenting pair with maximum efficiency is called *maximal augmenting pair*. For each point $P_i = (x_i, y_i)$ let $\alpha_i = \frac{y_i}{x_i}$ denote the *slope* of P_i. In the following lemma the efficiency is investigated in more detail.

Lemma 3.2. *Let* $\alpha_k = \frac{y_k}{x_k}$ *for* $k = s, t$. *Then*

$$e_{st} = \left| \frac{\alpha_s - \alpha_t}{\sqrt{1+\alpha_s^2} + \sqrt{1+\alpha_t^2}} \right|. \tag{14}$$

Proof. Since $x_k^2 + y_k^2 = 1$, we get $|x_k| = \frac{1}{1+\alpha_k^2}$ for $k = s, t$. This leads to the formulation of e_{st} as stated in the lemma. $\square$

Assume that $\alpha_s \geq \alpha_t$ and α_t is fixed. Then e_{st} can be considered as function of α_s with derivative

$$\left(\frac{1}{\sqrt{1+\alpha_s^2}+\sqrt{1+\alpha_t^2}}\right)^2 \left(\sqrt{1+\alpha_s^2}+\sqrt{1+\alpha_t^2}-(\alpha_s-\alpha_t)\frac{\alpha_s}{\sqrt{1+\alpha_s^2}}\right)$$

The fact that $\sqrt{1+\alpha_s^2}\sqrt{1+\alpha_t^2} > |\alpha_s\alpha_t|$ holds implies that the derivative is positive and hence e_{st} is monotonically increasing in α_s if α_t is fixed. Due to symmetry it follows that e_{st} is monotonically decreasing in α_t and hence e_{st} gets larger for increasing $\alpha_s - \alpha_t = |\alpha_s - \alpha_t|$. An analogue argument leads to the same result for $\alpha_s \leq \alpha_t$. Hence, we immediately get the following corollary:

Corollary 3.3. *Let (P_s, P_t) be a maximal augmenting pair then*

$$\left|\frac{y_s}{x_s} - \frac{y_t}{x_t}\right| \geq \left|\frac{y_s}{x_s} - \frac{y_j}{x_j}\right|$$

holds for every point P_j.

The next lemma shows that as long as the weights are not optimal there exists an augmenting pair.

Lemma 3.4. *Let (p^*, q^*) be an optimal solution. Then (p^*, q^*) can be decomposed into a sequence of augmenting pair transformations.*

Proof Let (p^*, q^*) be an optimal solution. Since $R_x(w) = 0$ and

$$\sum_{i=1}^{n} x_i(w_i + p_i^* - q_i^*) = \sum_{i=1}^{n} x_i(p_i^* - q_i^*) = 0$$

there exist two points P_s and P_t with $p_k^* - q_k^* \neq 0$ for $k = s, t$ and

$$\operatorname{sgn}(x_s(p_s^* - q_s^*)) \neq \operatorname{sgn}(x_t(p_t^* - q_t^*)).$$

Then there exist two values δ_s and δ_t such that

$$\begin{aligned} x_s\delta_s + x_t\delta_t &= 0, \\ y_s\delta_s + y_t\delta_t &= \delta. \end{aligned}$$

If $p_k^* > 0$ then $0 < \delta_k \leq p_k^*$ and if $q_k^* > 0$ then $-q_k^* \leq \delta_k < 0$ holds for $k = s, t$. Now choose δ_s and δ_t such that $|\delta|$ is maximal and reduce (p^*, q^*) by (δ_s, δ_t). The procedure described above can again be applied to the reduced solution. Hence, (p^*, q^*) can be reached by a finite sequence $(P_{i_1}, P_{j_1}, \delta_1), \ldots, (P_{i_r}, P_{j_r}, \delta_r)$ of modifications. Observe that if (p^*, q^*) is feasible then

$$G(w^*) = \sum_{k=1}^{r} \delta_r = 0.$$

There exists at least one pair (P_{i_k}, P_{j_k}) with $\delta_k > 0$ and hence it is an augmenting pair. Moreover, if there exists a pair $(P_{i_k}, P_{j_k}, \delta_k)$ with $\delta_k < 0$ then one may simultaneously reduce the modification of this pair and the modification of an augmenting pair which yields a feasible solution with less cost and hence leads to a contradiction to the optimality of (p^*, q^*). Hence, every decomposition consists of only augmenting pairs. □

We can always use a sequence of maximal augmenting pairs with maximal augmentation values in order to obtain an optimal solution due to the following lemma.

Lemma 3.5. *If the problem is feasible and (P_s, P_t) is a maximal augmenting pair with maximal augmentation value $\delta_{st} > 0$ then there exists an optimal solution (p^*, q^*) that can be decomposed into a sequence of augmenting pair modifications and (P_s, P_t, δ) is contained in this sequence with $\delta_{st} \leq \delta$.*

Proof Let (p^*, q^*) be an optimal solution with cost c^* such that the value of augmentation δ of (P_s, P_t) is maximal. Assume that $0 \leq \delta < \delta_{st}$. The idea is to define a new optimal solution with higher augmentation value of (P_s, P_t).

If in the optimal solution neither the optimal weight of P_s nor the optimal weight of P_t attains one of its lower or upper bound, then a new solution is obtained by choosing any arbitrary augmenting pair (P_i, P_j) of (p^*, q^*) and increasing the augmentation of (P_s, P_t) and decreasing the augmentation of (P_i, P_j) by $\epsilon > 0$. Clearly, the new solution is again feasible. Since $e_{st} \geq e_{ij}$ the new cost are smaller than c^* in contradiction that c^* is the minimum cost.

If both weight w_s^* and w_t^* are equal to their lower or upper bound, then $\delta = \delta_{st}$ holds, which contradicts our assumption $0 \leq \delta < \delta_{st}$.

Hence, there exists at most one point, say P_s, whose optimal weight is equal to its lower or upper bound. Since $\delta < \delta_{st}$ there exists at least one augmenting pair (P_s, P_j) with $j \neq t$. A new solution is obtained by increasing the augmentation of (P_s, P_t) and decreasing the augmentation of (P_s, P_j) by $\epsilon > 0$. The bound constraints for P_t and P_j are satisfied because $\underline{w}_t < w_t^* < \overline{w}_t$ holds and the modification of P_j was decreased. It remains to show that the bound constraint of P_s is fulfilled. Recall that (P_s, P_t) is a maximal augmenting pair and hence due to Corollary 3.3 we have

$$\left|\frac{y_t}{x_t} - \frac{y_s}{x_s}\right| \geq \left|\frac{y_{j_k}}{x_{j_k}} - \frac{y_s}{x_s}\right|.$$

Simple calculations yield

$$\left|\frac{x_t}{x_s y_t - x_t y_s}\right| \leq \left|\frac{x_{j_k}}{x_s y_{j_k} - x_{j_k} y_s}\right|.$$

Observe that the ϵ-modification increases the weight modification of P_s by

$$\left|\frac{x_t}{x_s y_t - x_t y_s}\epsilon\right|$$

and simultaneously decreases it by

$$\left|\frac{x_{j_k}}{x_s y_{j_k} - x_{j_k} y_s}\epsilon\right|.$$

Thus, the new modified sequence leads to a feasible solution. Since the efficiency of pair (P_s, P_t) is maximal, we get again a new solution with cost at most c^* but higher augmentation value of pair (P_s, P_t) in contradiction to the maximality of (p^*, q^*). □

For solving the inverse Fermat-Weber problem we shall apply now the following Algorithm 1.

Theorem 3.6. *If due to the weight bounds the inverse Fermat-Weber problem is solvable, then the above greedy algorithm determines an optimal solution.*

Proof According to Lemma 3.3 a pair has maximum efficiency if and only if one point of the pair has a maximum slope and the other point has a minimum slope. Hence, in each iteration the algorithm chooses a pair of maximum efficiency and performs an augmentation of maximal value.

Let w be the current weight vector. Due to Lemma 3.5 there exists an optimal solution that contains a maximal augmentation using a maximal augmenting pair (P_s, P_t). If the weight of P_s or P_t reaches its upper or lower bound then an optimal weight modification of this point is reached. Therefore, it can be fixed and its weight is not changeable any more.

Let $\tilde{w}$ be the weight after changing the weights of P_s and P_t by δ_s and δ_t, respectively. Let $(\tilde{p}, \tilde{q})$ be an optimal solution with respect to $\tilde{w}$. Since there

Algorithm 1 A greedy algorithm for solving the inverse Fermat-Weber problem

Rotate the coordinate system such that $R_x(w) = 0$.
Label all points as `free`.

while $G(w) > 0$ **do**
  Let P_s have maximal slope and P_t minimal slope among the free vertices. (If the maximal (minimal) slope does not uniquely correspond to one vertex then, if possible, choose that free point with maximal (minimal) slope that has already been considered in the previous iteration.)
  Compute the maximum value of δ according to (11), (12) and $\delta \leq G(w)$, change the weights of P_s and P_t according to (9) and (10) and update the gap.
  If a point reaches its upper or lower bound, label it as `fixed`.
  If all points are `fixed` and $G(w) > 0$ then the problem is infeasible.
end while

exists an optimal solution with respect to the original weight that contains a maximal augmentation using a maximal augmenting pair, $(\tilde{p}, \tilde{q})$ together with δ_s and δ_t yields an optimal solution of the original instance. □

Moreover, we get

Proposition 3.7. *The algorithm terminates after at most n weight exchanges. It yields a solution where at most two of the changed weights lie strictly between their lower and upper bound. The overall running time of the algorithm is $O(n \log n)$.*

Proof In each step of the algorithm at least one vertex reaches its upper or lower bound and is then fixed and not considered any more. Therefore, there are at most n weight exchanges.

If the weight of a vertex is changed then the point has maximum or minimum slope among the free vertices. If the weight is only partially changed then this point is still free in the next iteration and hence has maximum or minimum slope among the new set of free vertices. According to the algorithm this point is part of the following augmentation. Hence, there are at most two vertices (those in the last augmentation step) whose weight is changed but lies strictly between lower and upper bound.

Since the vertices can be sorted according to their slopes and each of the $O(n)$ augmentation steps takes linear time, Algorithm 1 runs in $O(n \log n)$ time. □

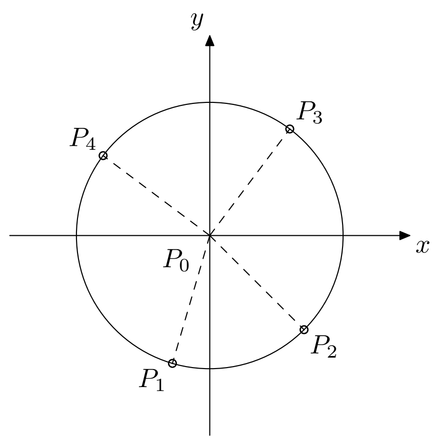

Figure 1: Vertices for an instance of the inverse Fermat-Weber Problem.

i	w_i	$\underline{w}_i$	$\overline{w}_i$	$P_i = (x_i, y_i)$	slope s_i
1	$\frac{50}{7}$	5	8	$P_1 = \left(-\frac{7}{25}, -\frac{24}{25}\right)$	$\frac{24}{7}$
2	$2\sqrt{2}$	1	3	$P_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	-1
3	4	3	5	$P_3 = \left(\frac{3}{5}, \frac{4}{5}\right)$	$\frac{4}{3}$
4	3	3	4	$P_4 = \left(-\frac{4}{5}, \frac{3}{5}\right)$	$-\frac{3}{4}$

Table 1: Parameter set of the considered instance.

3.2 Numerical example

In this subsection Algorithm 1 is applied to the following example given in Figure 1 and Table 1. Observe that $R_x(w) = 0$, $R_y(w) = -\frac{27}{7}$ and $G(w) = \frac{27}{7}$.

- $P_s = P_1$ and $P_t = P_2$ because $s_1 = \max\{s_i \mid i = 1, \ldots, 4\}$ and $s_2 = \min\{s_i \mid i = 1, \ldots, 4\}$:

$$\delta_1 = -\frac{25}{31}\delta, \quad \delta_2 = -\frac{7\sqrt{2}}{31}\delta$$

$$\delta = \min\left\{\frac{31}{25}(w_1 - \underline{w}_1), \frac{31}{7\sqrt{2}}(w_2 - \underline{w}_2), G(w)\right\} = \frac{93}{35}$$

The weight of P_1 is decreased by $\frac{25}{31} \cdot \frac{93}{35} = \frac{15}{7}$ and the weight of P_2 is decreased by $\frac{25}{31} \cdot \frac{7\sqrt{2}}{31} = \frac{3}{5}\sqrt{2}$. Hence, the new weights are of the form

$$w_1 = 5, \quad w_2 = \frac{7}{5}\sqrt{2}, \quad w_3 = 3, \quad w_4 = 3$$

with new gap $G(w) = \frac{6}{5}$. Vertex P_1 is fixed because its weight reached its lower bound.

- $P_s = P_3$ and $P_t = P_2$ because $s_3 = \max\{s_i \mid i = 2,3,4\}$ and $s_2 = \min\{s_i \mid i = 2,3,4\}$ (moreover, P_2 is partially modified):

  $$\delta_2 = -\frac{3\sqrt{2}}{7}\delta, \quad \delta_3 = \frac{5}{7}\delta$$

  $$\delta = \min\left\{\frac{7}{3\sqrt{2}}(w_2 - \underline{w}_2), \frac{7}{5}(\overline{w}_3 - w_4), G(w)\right\} = \frac{6}{5}$$

  The weight of P_2 is decreased by $\frac{6}{5} \cdot \frac{3\sqrt{2}}{7} = \frac{18}{35}\sqrt{2}$ and the weight of P_3 is increased by $\frac{6}{5} \cdot \frac{5}{7} = \frac{6}{7}$. Hence, the new weights are of the form

  $$w_1 = 5, \quad w_2 = \frac{31}{35}\sqrt{2}, \quad w_3 = \frac{34}{7}, \quad w_4 = 3$$

  with new gap $G(w) = 0$. Therefore, we have reached an optimal solution.

Observe that there are two vertices, P_1 and P_4, whose weight is changed but does not coincide with the upper or lower bound.

3.3 Remark on general cost coefficients

Algorithm 1 is developed for the unit cost model. By definition the efficiency of a pair (P_s, P_t) is the fraction of the gain in closing the optimality gap divided by the costs. In case of general cost coefficients this would imply

$$e_{st} := \frac{|x_s y_t - x_t y_s|}{c_t|x_s| + c_s|x_t|}.$$

However, the following example demonstrated that successively choosing maximal augmenting pairs does in general not yield an optimal solution.

Consider the instance of the Inverse Fermat-Weber problem given in Figure 2 and Table 2.

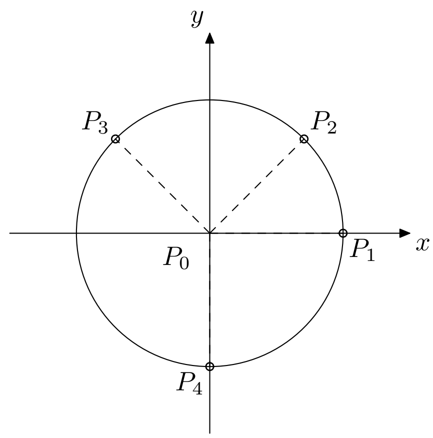

Figure 2: Vertices for an instance of the Inverse Fermat-Weber Problem.

i	w_i	$\underline{w}_i$	$\overline{w}_i$	$P_i = (x_i, y_i)$	c_i
1	0	0	5	$P_1 = (1, 0)$	$\sqrt{2}$
2	0	0	5	$P_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	7
3	0	0	5	$P_3 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	1
4	$\frac{10}{\sqrt{2}}$	$\frac{10}{\sqrt{2}}$	$\frac{10}{\sqrt{2}}$	$P_4 = (0, -1)$	0

Table 2: Parameter set for an example with general cost coefficient.

Observe that the weight of P_4 is not allowed to be changed. Therefore, we are only interested in the efficiencies involving the vertices P_i for $i = 1, 2, 3$:

$$e_{1,2} = \frac{1}{8\sqrt{2}}, \quad e_{1,3} = \frac{1}{2\sqrt{2}}, \quad e_{2,3} = \frac{1}{4\sqrt{2}}.$$

The algorithm chooses the maximal augmenting pair (P_1, P_3) and increases the weight of P_1 by $\frac{5}{\sqrt{2}}$ and increases the weight of P_3 by 5. This decision is irrevocable. However, it is easy to check that the unique optimal solution of this instance is $p_1^* = 0$ and $p_2^* = p_3^* = 5$. Hence, there is no optimal solution that contains the augmentation of a maximal augmenting pair.

4 The inverse Fermat-Weber problem for $P_0 = P_j$

In this section we discuss the special case of the inverse Fermat-Weber problem if the prespecified point coincides with one of the given vertices. Assume that P_0 is a vertex and should become 1-median. According to Theorem 2.2 vertex P_0 is 1-median if and only if

$$R_x^2(w) + R_y^2(w) \leq w_0^2$$

holds, where $R_x(w) = \sum_{i=1}^n w_i x_i$ and $R_y(w) = \sum_{i=1}^n w_i y_i$. Hence, the inverse Fermat-Weber problem can be written in the following form:

$$\begin{array}{ll}
\min \displaystyle\sum_{i=1}^n c_i(p_i + q_i) \quad \text{s.t.} & \\
\left(\displaystyle\sum_{i=1}^n (w_i + (p_i - q_i))x_i\right)^2 + \left(\displaystyle\sum_{i=1}^n (w_i + (p_i - q_i))y_i\right)^2 \leq (w_0 + p_0 - q_0)^2 & \\
p_i \leq \overline{w}_i - w_i & \text{for } i = 1, 2, ..., n, \\
q_i \leq w_i - \underline{w}_i & \text{for } i = 1, 2, ..., n, \\
p_i, q_i \geq 0 & \text{for } i = 1, 2, ..., n.
\end{array}$$

Unfortunately, the problem above is in general not a convex problem since the first constraint is in general a non-convex function. However, in case of the unit-cost model it is possible to fix the decision variables (p_0, q_0) in advance.

Lemma 4.1. *There exists an optimal solution (p^*, q^*) with*

$$p_0^* = \min\left\{\overline{w}_0 - w_0, \sqrt{R_x^2(w) + R_y^2(w)} - w_0\right\}.$$

Proof If $\sqrt{R_x^2(w)+R_y^2(w)} \leq \overline{w}_0$ then $p_0 = \sqrt{R_x^2(w)+R_y^2(w)} - w_0$, $q_0 = 0$ and $p_i^* = q_i^* = 0$ for $i = 1, \ldots, n$ is a feasible solution. Therefore, $p_0^* \leq \min\{\overline{w}_0 - w_0, \sqrt{R_x^2(w)+R_y^2(w)} - w_0\}$ holds for every optimal solution.

Assume that $p_0 < \min\{\overline{w}_0 - w_0, \sqrt{R_x^2(w)+R_y^2(w)} - w_0\}$ and let (p^*, q^*) be an optimal solution such that p_0^* is maximal.

Then there exists at least one point P_i with $p_i^* - q_i^* \neq 0$. Assume that $p_i^* > 0$ (the other case can be proved in an analoguous way). Define a new solution $(\tilde{p}, \tilde{q})$ that is obtained from (p^*, q^*) by increasing p_0^* and decreasing p_i^* by $\varepsilon > 0$ such that $(\tilde{p}, \tilde{q})$ satisfied the bound constraints. Then $R_x(\tilde{w}) = R_x(w^*) - \varepsilon x_i$ and $R_y(\tilde{w}) = R_y(w^*) - \varepsilon y_i$ where $\tilde{w} = w + \tilde{p} - \tilde{q}$ and

$$\begin{aligned} R_x^2(\tilde{w}) + R_y^2(\tilde{w}) &= R_x^2(w^*) + R_y^2(w^*) - 2\varepsilon(R_x(w^*)x_i + R_y(w^*)y_i) + \varepsilon^2 \\ &\leq (w_0^*)^2 + 2\varepsilon|R_x(w^*)x_i + R_y(w^*)y_i| + \varepsilon^2 \\ &\leq (w_0^*)^2 + 2\varepsilon w_0^* + \varepsilon^2 \leq (w_0^* + \varepsilon)^2 = \tilde{w}_0^2 \end{aligned}$$

In the above chain of inequalities we use the following facts: (p^*, q^*) is an optimal solution and hence $R_x^2(w^*)+R_y^2(w^*) = (w_0^*)^2$. Moreover, $x_i^2+y_i^2 = 1$ implies $|R_x(w^*)x_i + R_y(w^*)y_i| \leq \sqrt{R_x^2(w^*)+R_y^2(w^*)}$.

We have shown that $(\tilde{p}, \tilde{q})$ is feasible and optimal but $\tilde{p}_0 > p_0^*$ which leads to a contradiction. □

Lemma 4.1 implies that the weight of P_0 can be fixed. After modifying the weight of P_0 according to Lemma 4.1 the remaining problem is convex. Hence, by using e.g., the ellipsoid method the problem can be solved in polynomial time to any fixed precision (e.g., see Grötschel et al. [11]).

Theorem 4.2. *If the prespecified point is one of the given n points, then an optimal solution (to any fixed precision) of the inverse Fermat-Weber problem with unit cost can be computed in polynomial time.*

5 Conclusion

This paper deals with an inverse approach to the classical Fermat-Weber problem. While the complexity status of the Fermat-Weber problem is still unclear, we show that its inverse can be solved in polynomial time. If the prespecified point that should become 1-median does not coincide with the given points (vertices) in the plane, then the inverse Fermat-Weber problem can be written as linear programming problem which can in principle be solved in linear time. We suggest a purely combinatorial and very simple algorithm for the unit-cost model. In case that the prespecified point is a vertex then the inverse Fermat-Weber problem can be written as convex optimization problem.

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