Algorithmen &
mathematische
Modellierung

**Forschungsschwerpunkt**

Algorithmen und mathematische Modellierung

TU Graz
Graz University of Technology

# On the solutions of a class of iterated generalized Bers-Vekua equations in Clifford analysis

Peter Berglez

Institut für Analysis und Computational Number Theory (Math A)

Report 2008-33, June 2008

# On the solutions of a class of iterated generalized Bers-Vekua equations in Clifford analysis

P. Berglez

Department of Mathematics, Graz University of Technology, A-8010 Graz, Austria

**Abstract**

We consider functions with values in the Clifford algebra $Cl_{p,q}$ which are solutions of a certain class of the iterated generalized Bers-Vekua equation $D^m w = 0$ with $Dw = \partial w + c\bar{w}$ where $\partial = \sum_{j=0}^{n} e_j \, \partial/\partial x_j$ is the generalized Cauchy-Riemann operator. We prove that any such function $w$ has a Almasi-type decomposition of the form $w = v_0 + x_0 v_1 + \ldots + x_0^{m-1} v_{m-1}$ where $x = x_0 + x_1 e_1 + \ldots x_n e_n$, and the functions $v_j, j = 0, 1, \ldots, m-1$, are solutions of the generalized Bers-Vekua equation $Dv = 0$.

**Key words:** iterated generalized Cauchy-Riemann operator; iterated generalized Bers-Vekua operator; polymonogenic functions; differential operators of Bauer-type

## 1 Introduction

Let $Cl_{p,q}$ with $p + q = n$ be the $2^n$-dimensional Clifford algebra generated by the elements $e_1, \ldots, e_n$, which obey the multiplication rules

$$e_j^2 = e_0, j = 1, \ldots, p, \; e_j^2 = -e_0, j = p+1, \ldots, n \quad \text{and} \quad e_i e_j + e_j e_i = 0 \text{ for } i < j \,,$$

where $e_0$ denotes the identity of the algebra. An arbitrary element $a$ of $Cl_{p,q}$ is given by

$$a = \sum_A \lambda_A e_A \,, \; \lambda_A \in \mathbb{R}$$

with $e_A = e_{\alpha_1} e_{\alpha_2} \ldots e_{\alpha_k}, \alpha_1, \ldots, \alpha_k \in \{1, \ldots, n\}$ and $1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq n$.
The conjugation is defined by $\bar{a} = \sum_A \lambda_A \, \bar{e}_A$ where $\bar{e}_A = \bar{e}_{\alpha_k} \ldots \bar{e}_{\alpha_1}$ and $\bar{e}_0 = e_0$, $\bar{e}_j = -e_j, j = 1, \ldots, n$, holds.
Now an element $x$ in $\mathbb{R}^{n+1}$ with the coordinates $x_0, x_1, \ldots, x_n$ is identified with

$$x = x_0 + x_1 e_1 + \ldots + x_n e_n$$

and the conjugate of $x$ is $\bar{x} = x_0 - x_1 e_1 - \ldots - x_n e_n$.

In $\mathbb{R}^{n+1}$ the generalized Cauchy-Riemann operator is defined by

$$\partial = \frac{\partial}{\partial x_0} + \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}$$

and the conjugate operator $\bar{\partial}$ is given by

$$\bar{\partial} = \frac{\partial}{\partial x_0} - \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}$$

These operators act on the space $C^1(\Omega, Cl_{p,q})$ where $\Omega$ denotes a domain in $\mathbb{R}^{n+1}$.
Using such a Clifford algebra in a multidimensional space, elliptic, hyperbolic and parabolic differential equations can be considered in an inductive way (cf. e.g. [1]).
A function $u$ is said to be (left) Clifford holomorphic in $\Omega$ if it is a solution of the differential equation

$$\partial u = 0$$

Here we consider the generalized Bers-Vekua operator $D$ defined by

$$Dv := \partial v + c\,\bar{v}$$

where $c$ is a real valued function of $x_0$.
In the algebra of complex quaternions H. Malonek [2] investigated some classes of generalized Bers-Vekua equations. B. Goldschmidt [3] presented regularity properties of generalized analytic vectors a particular subset of which is described by $Dv = 0$. In [4] a similar operator was used to describe pseudoanalytic functions in the space, whereas in a forthcoming paper of the author for $c(x_0) = k/x_0, k \in \mathbb{Z}$, a representation theorem for the solutions of $Dv = 0$ is proved using certain differential operators of Bauer-type acting on Clifford holomorphic functions.
Here we study the iterated operator $D^m$ with $m \in \mathbb{N}$ and $D^m f = D(D^{m-1} f)$ and $D^0 f = f$. There exists a close connection between the solutions of the iterated generalized Bers-Vekua equation

$$D^m w = 0 \tag{1}$$

and the solutions of $Dv = 0$. This result generalizes the representation of the solutions of the iterated generalized Cauchy-Riemann equation

$$\partial^m u = 0 \tag{2}$$

(sometimes called ($k$)-monogenic or polymonogenic functions) given in [5] (see also [6]) and further the classical Almansi theorem [7] for polyharmonic functions.
Supposing that there exists a representation of the solutions of $Dv = 0$ by means of certain differential operators as it was discussed e.g. in the forthcoming article we prove a close connection between the solutions of (1) and the solutions of (2).

## 2 An Almansi-type decomposition

**Proposition 1.** *For any real valued, continuously differentiable function $\varphi(x_0)$ of the variable $x_0$ and for each function $\tilde{w} \in C^1(\Omega, Cl_{p,q})$ we have*

*(i)* $\partial(\varphi(x_0)\,\tilde{w}) = \varphi'(x_0)\tilde{w} + \varphi(x_0)\,(\partial\tilde{w})$

*(ii)* $D(\varphi(x_0)\,\tilde{w}) = \varphi'(x_0)\tilde{w} + \varphi(x_0)\,(D\tilde{w})$

*(iii) If $u \in C^1(\Omega, Cl_{p,q})$ is a solution of $Du = 0$ then $\alpha\, u$ with $\alpha \in \mathbb{R}$ is a solution of $Du = 0$ also.*

*Proof.* Using the relation $\tilde{w}\,\varphi(x_0) = \varphi(x_0)\,\tilde{w}$ which is true for any real valued function $\varphi$ and all $\tilde{w} \in C(\Omega, Cl_{p,q})$ the assertions can be proved by direct calculation. □

To find representations for the solutions of the iterated equation (1) we first choose a new function $w_1$ as

$$w_1 := D^{m-1}w - \frac{1}{x_0}D^{m-2}w$$

which obeys the differential equation

$$Dw_1 + \frac{1}{x_0}w_1 = 0 \tag{3}$$

This can be proved immediately using proposition 1(ii) and equation (1). Further we define the functions

$$w_k := D^{m-k}w - \frac{k}{x_0}\,D^{m-k-1}w\ , \quad k = 2, 3, \ldots, m-1 \tag{4}$$

for which the relations

$$Dw_k + \frac{1}{x_0}w_k = w_{k-1}\ , \quad k = 2, 3, \ldots, m-1 \tag{5}$$

hold. With $k = m-1$ equation (4) leads to

$$w_{m-1} = Dw - \frac{m-1}{x_0}w \tag{6}$$

Setting $w_m := w$ we get from (3), (5) and (6) a system of $m$ differential equations for the functions $w_k, k = 1, \ldots, m$. With the vector $W := (w_1, \ldots, w_m)^t$ and the matrices $A = (a_{jk})$ with

$$a_{jk} = \begin{cases} 1/x_0 & \text{for} \quad 1 \le j = k \le m-1 \\ (1-m)/x_0 & \text{for} \quad j = k = m \\ -1 & \text{for} \quad j = k+1, 1 \le k \le m-1 \\ 0 & \text{else} \end{cases}$$

and $B = (b_{jk})$ with

$$b_{jk} = \begin{cases} c & \text{for} \quad 1 \leq j = k \leq m \\ 0 & \text{else} \end{cases}$$

this system can be given in matrix notation as

$$\partial W + AW + B\overline{W} = 0 \tag{7}$$

Writing $\partial W$ means the application of the operator $\partial$ to each component of $W$.
Let $P = (p_{jk})$ denote a $m \times m$-matrix the components of which are real valued, continuously differentiable functions of $x_0$. By the transformation $W = PU$ in connection with proposition 1(i) equation (7) leads to the relation

$$P(\partial U) + (P' + AP)U + PB\overline{U} = 0$$

for the new unknown vector $U = (u_1, \ldots, u_m)^t$.
The request $P' + AP = 0$ can be satisfied by the matrix $P$ with

$$p_{jk} = \begin{cases} \dfrac{x_0^{j-k-1}}{(j-k)!} & \text{for} \quad 1 \leq k \leq j, 1 \leq j \leq m-1 \\ 0 & \text{for} \quad j < k \leq m, 1 \leq j \leq m-1 \\ -\dfrac{1}{k}\,\dfrac{x_0^{m-k-1}}{(m-k-1)!} & \text{for} \quad j = m, 1 \leq k \leq m-1 \\ x_0^{m-1} & \text{for} \quad j = k = m \end{cases}$$

which is nonsingular since det$P$= 1. Thus for the vector $U$ we have the simple system

$$\partial U + c\,\overline{U} = 0$$

which states that each component $u_k$ of the vector $U$ obeys the generalized Bers-Vekua equation

$$Du_k = \partial u_k + c\bar{u}_k = 0\,, \quad k = 1, \ldots, m$$

The focal question is which form has the function $w = w_m$, the $m$-th component of the vector $W$. From $W = PU$ we have

$$w \equiv w_m = \sum_{k=1}^{m} p_{mk}\, u_k = \sum_{k=1}^{m-1} -\frac{1}{k}\,\frac{x_0^{m-k-1}}{(m-k-1)!} u_k + x_0^{m-1} u_m$$

Finally with proposition 1(iii) the function $w$ can be written in the form

$$w = \sum_{k=0}^{m-1} x_0^k\, v_k \tag{8}$$

where the $v_k$ are solutions of $Dv_k = 0$ and we have the following

**Theorem 1.** *1. Let the functions $v_k, k = 0, 1, \ldots, m-1$, be solutions of the generalized Bers-Vekua equation $Dv_k = 0$ in $\Omega$. Then the function $w$ according to (8) represents a solution of the iterated generalized Bers-Vekua equation $D^m w = 0$ in $\Omega$.*

*2. For each solution $w$ of (1) defined in $\Omega$ there exist solutions $v_k, k = 0, 1, \ldots, m-1$, of $Dv_k = 0$ in $\Omega$ such that $w$ can be written in the form (8).*

To bring out the connection between a solution $w$ of (1) and the functions $v_k$ in the representation (8) we first prove the relation

$$D^l w = \sum_{k=l}^{m-1} \frac{k!}{(k-l)!} x_0^{k-l} v_k \,, \quad l = 0, 1, \ldots, m-1 \tag{9}$$

This can be considered as a linear system of $m$ equations for the functions $v_k, k = 0, \ldots, m-1$, which has the form

$$M\, V = \hat{W} \quad \text{with} \quad V = (v_0, \ldots, v_{m-1})^t \quad \text{and} \quad \hat{W} = (w, Dw \ldots, D^{m-1} w)^t$$

The coefficient matrix $M = (m_{ik})$ with

$$m_{ik} = \begin{cases} \dfrac{(k-1)!}{(k-i)!} x_0^{k-i} & \text{for } i \le k \\ 0 & \text{for } i > k \end{cases}$$

is non singular, its inverse $M^{-1} = (\mu_{ik})$ is given by

$$\mu_{ik} = \begin{cases} \dfrac{(-1)^{i+k}}{(i-1)!(k-i)!} x_0^{k-i} & \text{for } i \le k \\ 0 & \text{for } i > k \end{cases}$$

Thus the solution of system (9) can be written as

$$v_k = \sum_{l=0}^{m-k-1} \frac{(-1)^l}{k!\, l!} x_0^l D^{k+l} w \,, \quad k = 0, 1, \ldots, m-1 \tag{10}$$

and we have the

**Lemma 1.** *For each solution $w$ of (1) in the form (8) the functions $v_k$ are determined uniquely by (10).*

In $Cl_{0,1}$ the operator $\partial$ reduces to the Cauchy-Riemann operator and eq. (1) to the iterated Bers-Vekua equation for which in [8] a corresponding representation theorem was proved. For the iterated Dirac operator $\tilde{\partial}^k$ with $\tilde{\partial} = \sum_{j=1}^n e_j \, \partial / \partial x_j$ a similar decomposition was proved in [9], whereas in [10] a unified approach to decomposing kernels of iterated operators was investigated.

## 3 Differential operators for the solutions

Now let us consider the case when the solutions of $Dv = 0$ can be represented by means of a suitable differential operator of Bauer-type acting on solutions of $\partial u = 0$. In the paper mentioned above a sufficient condition on the coefficient $c$ in $Dv = 0$ was given for the existence of such a differential operator with which Clifford holomorphic functions $g$ can be transformed into solutions $v$ of $Dv = 0$ by

$$v = \sum_{j=0}^{N} a_j(x_0)\left(g\bar{\partial}^j\right) + \sum_{j=0}^{N-1} b_j(x_0)\left(\partial^j \bar{g}\right)\ ,\ N \in \mathbb{N}$$

In particular for the coefficient $c(x_0) = N/x_0, N \in \mathbb{Z}$, such a Bauer-type differential operator exists and the coefficients $a_k$ and $b_k$ can be given in an explicite form.
Let us assume that there exists a representation for the functions $v_k$ in (8) of the form

$$v_k = \sum_{j=0}^{N} a_j(x_0)\left(g_k\bar{\partial}^j\right) + \sum_{j=0}^{N-1} b_j(x_0)\left(\partial^j \bar{g}_k\right)\ ,\ k = 0, \ldots, m-1 \tag{11}$$

with suitabel functions $g_k$ which are solutions of $\partial g_k = 0, k = 0, \ldots, m-1$. With the operator $\delta$ defined by

$$\delta u := (u\bar{\partial}) - (\partial u)\ ,\ \delta^{j+1}u := \delta(\delta^j u)\ ,\ \delta^0 u := u$$

by direct calculation we can prove the

**Proposition 2.**

*(i) For any Clifford holomorphic function $g_k$ the function $u$ according to*

$$u = \sum_{k=0}^{m-1} x_0^k g_k \tag{12}$$

*represents a solution of $\partial^m u = 0$ (see also [5]).*

*(ii) For a function $u$ given by (12) we have the relation*

$$\delta^j u = \sum_{k=0}^{m-1} x_0^k \left(g_k \bar{\partial}^j\right)$$

Now from the representation (8) with the functions $v_k$ given in (11) and with proposition 2(ii) we get the following form for the solutions of (1)

$$w = \sum_{j=0}^{N} a_j(x_0)(\delta^j u) + \sum_{j=0}^{N-1} b_j(x_0)\overline{(\delta^j u)} \tag{13}$$

**Theorem 2.** *In the case of the existence of Bauer-type operators for the representation of the solutions of the generalized Bers-Vekua equation $Dv = 0$ the solutions of the iterated generalized Bers-Vekua equation $D^m w = 0$ can be given in terms of solutions of the iterated generalized Cauchy-Riemann equation $\partial^m u = 0$ by a differential operator in the form (13).*

## References


[1] Obolashvili E. *Partial Differential Equations in Clifford Analysis*. Longman: Harlow, 1998.

[2] Malonek HR. Generalizing the (F,G)-derivative in the sense of Bers. In *Clifford Algebras and their Applications in Mathematical Physics*, Dietrich V, Habetha K, Jank G (eds). Kluwer: Dordrecht, 1998; 247–257.

[3] Goldschmidt B. Regularity properties of generalized analytic vectors in $\mathbb{R}^n$. *Mathematische Nachrichten* 1981; **103**: 245–254.

[4] Berglez P. Representation of pseudoanalytic functions in the space. In *More Progresses in Analysis*, Begehr H, Nicolosi F (eds). World Scientific Publ: Singapore, 2008; 1119–1126.

[5] Constales D, De Almeida R, Kraußhar RS. On Cauchy estimates and growth orders of entire solutions of iterated Dirac and generalized Cauchy–Riemann equations. *Mathematical Methods in the Applied Sciences* 2006; **29**: 1663–1686.

[6] Brackx F. On $(k)$-monogenic functions of a quaternion variable. *Function Theoretic Methods in Differential Equations*. Research Notes in Mathematics, vol. 8. Pitman: London, 1976; 22–44.

[7] Almansi E. Sull' integrazione dell' equazione differenziale $\Delta^{2n} = 0$. *Annali di Matematica Pura ed Applicata* (Ser. 3) 1899; **2**: 1–51.

[8] Berglez P. On the solutions of the iterated Bers-Vekua equation. In *Functional–Analytic and Complex Methods, their Interactions, and Applications to Partial Differential Equations*, Florian H. et al. (eds). World Scientific Publ: Singapore, 2001; 266–275.

[9] Malonek HR, Ren G. Almansi-type theorems in Clifford analysis. *Mathematical Methods in the Applied Sciences* 2002; **25**: 1541–1552.

[10] Ren G, Malonek HR. Decomposing kernels of iterated operators – a unified approach. *Mathematical Methods in the Applied Sciences* 2007; **30**: 1037–1047.