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A precise description of the p-adic valuation of the number of alternating sign matrices

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A PRECISE DESCRIPTION OF THE p-ADIC VALUATION OF THE NUMBER OF ALTERNATING SIGN MATRICES

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ABSTRACT. Following Sun and Moll [4], we study $v_p(T(N))$, the p-adic valuation of the counting function of the alternating sign matrices. We find an exact analytic expression for it that exhibits the fluctuating behaviour, by means of Fourier coefficients. The method is the Mellin-Perron technique, which is familiar in the analysis of the sum-of-digits function and related quantities.

1. INTRODUCTION

Sun and Moll [4] consider the counting function

$$T(N) = \prod_{j=0}^{N-1} \frac{(3j+1)!}{(N+j)!},$$

which is famous because of the enumeration of the *Alternating Sign Matrices.* Their paper provides also some historic remarks about the fascinating story of this subject.

Sun and Moll write:

> *Given an interesting sequence of integers, it is a natural question to explore the structure of their factorization into primes. This is measured by the p-adic valuation of the elements of the sequence.*

Indeed, in [4] the p-adic valuation $v_p(T(N))$ is studied, i.e., $v_p(m)$ denotes the maximum k such that p^k divides m.

Most results are, however, for $p = 2$ only.

Their key result is (Corollary 2.2):

$$v_p(T(N)) = \frac{1}{p-1}\left(\sum_{j=0}^{N-1} S_p(N+j) - \sum_{j=0}^{N-1} S_p(3j+1)\right),$$

where $S_p(k)$ is the sum-of-digits function of k to the base p.

We use an analytic approach (the Mellin-Perron technique and its extension, [2, 3]) to derive an *exact* analytic expression for this function, and this works for *all* primes. The periodicities are made fully explicit by computing the relevant Fourier coefficients.

2. Results

Now we study the function $v_p(T(N))$ more closely. The prime p is fixed throughout this paper, everything may depend on it, including implicit constants in the O-notation.

We will prove that asymptotically,

$$v_p(T(N)) = N \log_p \frac{2}{\sqrt{3}} + N\Phi(\log_p N) + O(\sqrt{N}), \tag{1}$$

where

$$\Phi(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} c_k e^{2k\pi ix}$$

is a 1-periodic and continuous function of mean 0 with Fourier coefficients

$$c_k = \frac{(1 - 2^{1+\chi_k} + 3^{\chi_k})\zeta(\chi_k)}{\chi_k(1 + \chi_k)\log p} \qquad \text{with} \qquad \chi_k = \frac{2k\pi i}{\log p}$$

for $k \in \mathbb{Z} \setminus \{0\}$.

The O-term in (1) can also be computed explicitly. Here, the cases $p \equiv 1 \pmod 3$, $p \equiv -1 \pmod 3$ and $p = 3$ differ in several aspects, thus we formulate one theorem for each case: Theorems 1, 2 and 3, respectively.

The occurring quantities are expressed in terms of the Hurwitz ζ function defined by

$$\zeta(s, \alpha) = \sum_{n > -\alpha} \frac{1}{(n + \alpha)^s},$$

where we allow arbitrary $\alpha \in \mathbb{R}$. For $0 < \alpha \leq 1$, this coincides with the usual definition. In our version, however, it is 1-periodic in the second variable, which will be useful for our purposes. The special case $\alpha \in \mathbb{Z}$ corresponds to the Riemann ζ function $\zeta(s)$.

Theorem 1. *Assume that $p \equiv 1 \pmod 3$. Then the asymptotic expansion (which is also exact) of the function $v_p(T(N))$ is given by*

$$v_p(T(N)) = N \log_p \frac{2}{\sqrt{3}} + N\Phi(\log_p N) + \Psi(N) + \frac{1}{9} \log_p N + f_0(N).$$

Here,

- *the function*

$$\Phi(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} c_k e^{2k\pi ix}$$

is 1-periodic and continuous;

- *the function*

$$\Psi(N) = \left(N + \frac{1}{3}\right)\psi_1\left(\log_p\left(N + \frac{1}{3}\right)\right) + \left(N - \frac{1}{3}\right)\psi_{-1}\left(\log_p\left(N - \frac{1}{3}\right)\right)$$
$$- N\psi_0(\log_p N),$$

  is expressed in terms of the continuous 1-periodic functions

$$\psi_j(x) = \sum_{k \in \mathbb{Z}\setminus\{0\}} d_{k,j}e^{2k\pi ix}, \qquad j \in \{-1, 0, 1\}$$

  and satisfies

$$\Psi(N) = O(N^{1/2}\log N);$$

- *the least order term is given by*

$$f_0(N) = \frac{1}{3}\log_p\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right) + \frac{g_1(N) - g_{-1}(N)}{6} - \frac{1}{9\log p} + \frac{p+1}{6(p-1)}$$

  with

$$g_j(N) = \left(1 + \frac{j}{3N}\right)N\log_p\left(1 + \frac{j}{3N}\right) \underset{N\to\infty}{\sim} \frac{j}{3\log p} + O\left(\frac{1}{N}\right), \tag{2}$$

  thus $f_0(N)$ converges to

$$\frac{1}{3}\log_p\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right) + \frac{p+1}{6(p-1)}$$

  for $N \to \infty$.

The Fourier coefficients of $\Phi(x)$, $\psi_{\pm 1}(x)$ and ψ_0 are given by

$$c_k = \frac{(1 - 2^{1+\chi_k} + 3^{\chi_k})\zeta(\chi_k)}{\chi_k(1 + \chi_k)\log p} \qquad \text{with} \qquad \chi_k = \frac{2k\pi i}{\log p},$$

$$d_{k,j} = \frac{\zeta(\chi_k, j/3)}{\chi_k(1 + \chi_k)\log p} \qquad \text{for} \qquad j \in \{\pm 1\},$$

$$d_{k,0} = d_{k,1} + d_{k,-1} = \frac{(3^{\chi_k} - 1)\zeta(\chi_k)}{\chi_k(1 + \chi_k)\log p},$$

respectively.

The case $p \equiv -1 \pmod{2}$ is similar, but the structure of $\Psi(x)$ is more complicated: it is now composed of 2-periodic functions instead of 1-periodic functions. On the other hand, the least order term is less complicated.

Theorem 2. *Assume that $p \equiv -1 \pmod{3}$. Then the asymptotic expansion (which is also exact) of the function $v_p(T(N))$ is given by*

$$v_p(T(N)) = N\log_p\frac{2}{\sqrt{3}} + N\Phi(\log_p N) + \Psi(N) + \frac{p+1}{6(p-1)}.$$

Here,

- *the function*
$$\Phi(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} c_k e^{2k\pi i x}$$
  is 1-periodic and continuous;
- *the function*
$$\Psi(N) = \left(N + \frac{1}{3}\right)\psi_1\left(\log_p\left(N + \frac{1}{3}\right)\right) + \left(N - \frac{1}{3}\right)\psi_{-1}\left(\log_p\left(N - \frac{1}{3}\right)\right)$$
$$- N\psi_0(\log_p N),$$
  is expressed in terms of the continuous 2-periodic functions
$$\psi_j(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} d_{k,j} e^{k\pi i x}, \qquad j \in \{-1, 0, 1\}$$
  and satisfies
$$\Psi(N) = O(N^{1/2} \log N).$$

The Fourier coefficients of $\Phi(x)$, $\psi_{\pm 1}(x)$ and ψ_0 are given by

$$c_k = \frac{(1 - 2^{1+\chi_k} + 3^{\chi_k})\zeta(\chi_k)}{\chi_k(1 + \chi_k)\log p} \qquad\qquad with \qquad \chi_k = \frac{2k\pi i}{\log p},$$

$$d_{k,j} = \frac{\zeta(\chi_{k/2}, j/3) + (-1)^k \zeta(\chi_{k/2}, -j/3)}{2\chi_{k/2}(1 + \chi_{k/2})\log p} \qquad\qquad for \qquad j \in \{\pm 1\},$$

$$d_{k,0} = d_{k,1} + d_{k,-1} = \begin{cases} \frac{(3^{\chi_{k/2}} - 1)\zeta(\chi_{k/2})}{\chi_{k/2}(1 + \chi_{k/2})\log p}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd,} \end{cases}$$

respectively.

Finally, for $p = 3$, the result is that the O-term in (1) can be omitted:

Theorem 3. *Then $v_3(T(N))$ is given by*

$$v_3(T(N)) = N\left(\log_3 2 - \frac{1}{2}\right) + N\Phi(\log_3 N),$$

where

$$\Phi(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} c_k e^{2k\pi i x}$$

is a 1-periodic continuous function with Fourier coefficients

$$c_k = \frac{2(1 - 2^{\chi_k})\zeta(\chi_k)}{\chi_k(1 + \chi_k)\log p} \qquad with \qquad \chi_k = \frac{2k\pi i}{\log p}.$$

As examples, Figures 1, 2 and 3 show $v_p(T(N))/N$ (gray dots) and an approximation of $\log_p \frac{2}{\sqrt{3}} + \Phi(\log_p N)$ (black line) for $p = 2$, $p = 3$ and $p = 7$, respectively. For the approximation, 400 Fourier coefficients of Φ are used. Note that for $p = 3$, the approximation is much better as the only approximation consisted in truncating the Fourier series, in contrast to the lower order terms present in the other cases.
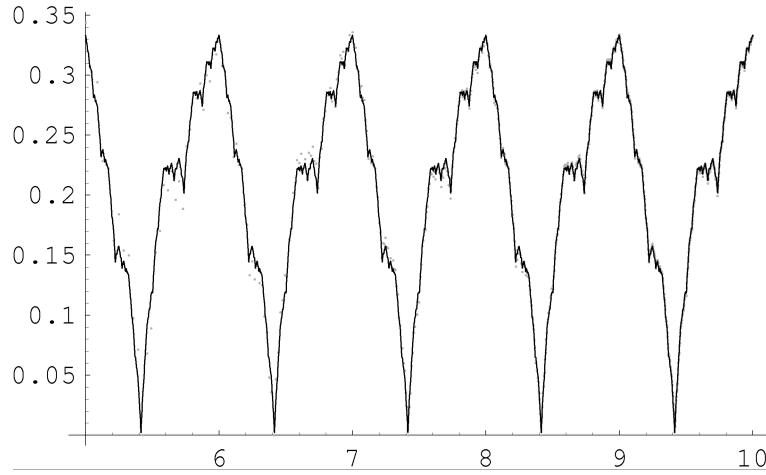
FIGURE 1. Periodic fluctuation for $p = 2$

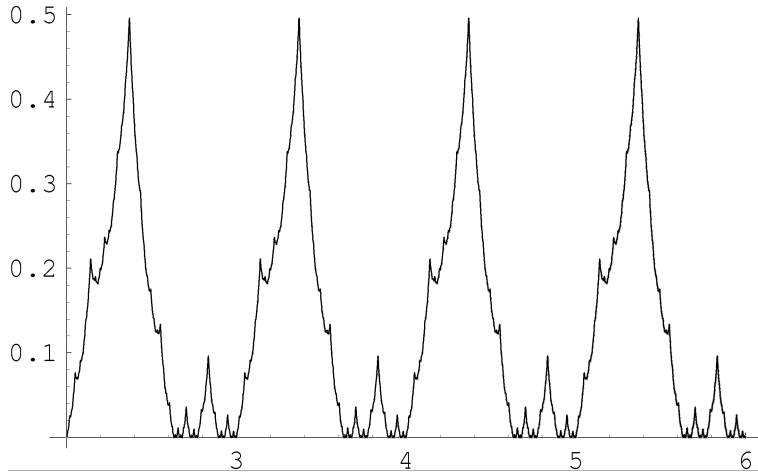FIGURE 2. Periodic fluctuation for $p = 3$

Our analysis builds on a result of Delange [1]:
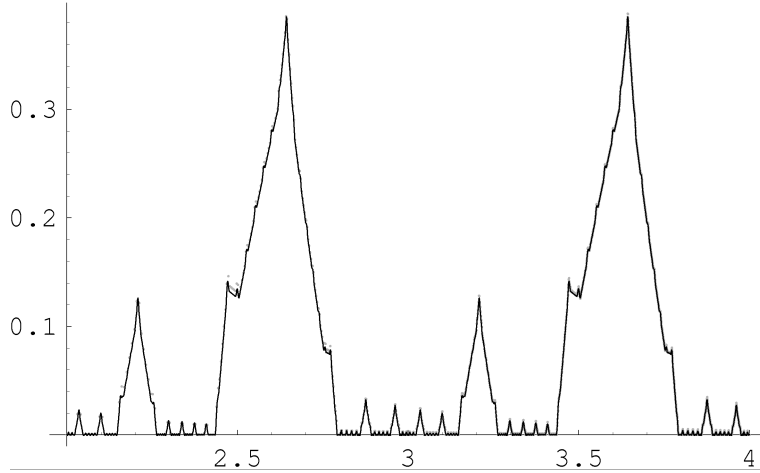
Lemma 2.1 (Delange [1]).

$$\sum_{n=0}^{N-1} S_p(n) = \frac{p-1}{2} N \log_p N + N c_0^{(1)} + N \Phi^{(1)}(\log_p N),$$

with

$$c_0^{(1)} = \frac{p-1}{2 \log p}(\log 2\pi - 1) - \frac{p+1}{4},$$

a 1-periodic continuous function of mean zero

$$\Phi^{(1)}(x) = \sum_{k \neq 0} c_k^{(1)} e^{2k\pi i x},$$

FIGURE 3. Periodic fluctuation for $p = 7$

and

$$c_k^{(1)} = -\frac{p-1}{\log p}\frac{1}{\chi_k(1+\chi_k)}\zeta(\chi_k), \quad \chi_k = \frac{2k\pi i}{\log p}$$

for $k \neq 0$.

This already leads to the simplification

$$v_p(T(N)) = \frac{1}{2}N\log_p N + N\Big(\log_p 2 + \frac{c_0^{(1)}}{p-1}\Big)$$

$$+ \frac{1}{p-1}2N\Phi^{(1)}(\log_p 2N) - \frac{1}{p-1}N\Phi^{(1)}(\log_p N) + \frac{1}{1-p}\sum_{j=0}^{N-1}S_p(3j+1). \quad (3)$$

In the rest of this paper we will analyse the sum

$$\frac{1}{1-p}\sum_{j=0}^{N-1}S_p(3j+1).$$

For $p = 3$, we have $S_3(3j+1) = 1 + S_3(j)$, so that another application of Lemma 2.1 directly yields the result described in Theorem 3.

In the following, we will therefore assume that $p \neq 3$. However, we will not follow Delange's original (elementary) approach, but rather use the Mellin-Perron approach as described in [2] and extended in [3].

3. Rewriting the Remaining Sum as an Integral

The aim of this section is to prove the following lemma.

Lemma 3.1. *We have*

$$\frac{1}{1-p}\sum_{n=0}^{N-1} S_p(3n+1) = -\frac{1}{1-p}\left(\frac{N}{2} - \frac{3N^2}{2}\right) + I \tag{4}$$

with

$$I = \frac{1}{2\pi i}\sum_{j=-1}^{1}\int_{2-i\infty}^{2+i\infty} \Lambda_j(s)\left(N + \frac{j}{3}\right)^{s+1}\frac{ds}{s(s+1)}$$

and

$$\Lambda_j(s) = \frac{\zeta\left(s, \frac{j}{3}\right) + p^s\zeta\left(s, \frac{uj}{3}\right)}{p^{2s} - 1}, \tag{5}$$

where $u \in \{\pm 1\}$ is chosen such that $p \equiv u \pmod 3$.

Proof. We intend to use the Mellin-Perron summation formula in the version

$$\sum_{\substack{n \\ -\alpha < n < N}} (N - n)a(n) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\left(\sum_{\substack{n \\ n > -\alpha}}\frac{a(n)}{(n+\alpha)^s}\right)(N+\alpha)^{s+1}\frac{ds}{s(s+1)}, \tag{6}$$

where $\alpha \in \mathbb{R}$ and C is in the half-plane of absolute convergence of the Dirichlet series $\sum_{n>-\alpha}\frac{a(n)}{(n+\alpha)^s}$, cf. [3, (4.3)]. Note that we do not impose the frequently used restriction $0 < \alpha \leq 1$, but we sum over $n > -\alpha$, which amounts to the same. This version, however, smoothes the following calculations.

For $j \in \{-1, 0, 1\}$, we set

$$a_j(n) = \frac{S_p(3n+j) - S_p(3n+j-1) - 1}{1-p}$$

and

$$\Lambda_j(s) = \sum_{n > -j/3}\frac{a_j(n)}{\left(n + \frac{j}{3}\right)^s}.$$

Note that $\Lambda_j(s)$ converges absolutely for $\operatorname{Re} s > 1$. The additional summand -1 in the numerator of $a_j(n)$ leads to simpler expressions for $\Lambda_j(s)$, the denominator also simplifies the expressions and is also present in (3).

To prove (4), we apply (6) with $a(n) = a_j(n)$ and $\alpha = j/3$ for $j \in \{-1, 0, 1\}$, sum up the result and use Abel summation. This yields

$$I = \sum_{n=1}^{N-1}(N - n)(a_1(n) + a_0(n) + a_{-1}(n)) + Na_1(0)$$

$$= \frac{1}{1-p}\sum_{n=1}^{N-1}(N-n)(S_p(3n+1) - S_p(3n-2) - 3) + \frac{1}{1-p}N(S_p(1) - S_p(0) - 1)$$

$$= \frac{1}{1-p}\left(S_p(3N-2) + \sum_{n=1}^{N-2}S_p(3n+1)(N-n-(N-(n+1)))\right)$$

$$- (N-1)S_p(1) + NS_p(1) + \frac{N}{2} - \frac{3N^2}{2} \Big)$$

$$= \frac{1}{1-p} \Big(S_p(3N-2) + \sum_{n=1}^{N-2} S_p(3n+1) + S_p(1) + \frac{N}{2} - \frac{3N^2}{2} \Big)$$

$$= \frac{1}{1-p} \sum_{n=0}^{N-1} S_p(3n+1) + \frac{1}{1-p} \Big(\frac{N}{2} - \frac{3N^2}{2} \Big),$$

as requested.

In order to compute I, we need explicit expressions for the Λ_j. It is well-known (and easy to see) that for a positive integer m, we have

$$S_p(m) - S_p(m-1) = 1 - (p-1)v_p(m),$$

which immediately results in

$$a_j(n) = v_p(3n+j).$$

Thus we get

$$\Lambda_j(s) = \sum_{n>-j/3} \frac{v_p(3n+j)}{\left(n+\frac{j}{3}\right)^s} = \sum_{k\geq 1} \sum_{\substack{n>-j/3 \\ v_p(3n+j)\geq k}} \frac{1}{\left(n+\frac{j}{3}\right)^s} = \sum_{k\geq 1} \sum_{\substack{n>-j/3 \\ p^k|3n+j}} \frac{1}{\left(n+\frac{j}{3}\right)^s}.$$

By definition of u, we have $\frac{3n+j}{p^k} \equiv ju^k \pmod 3$. Thus $3n+j$ is divisible by p^k if and only if there is an integer m such that $3n+j = p^k(3m+ju^k)$. This results in

$$\Lambda_j(s) = \sum_{k\geq 1} \sum_{m>-ju^k/3} \frac{1}{p^{ks}\left(m+\frac{ju^k}{3}\right)^s}.$$

We now split the sum over k according to k's parity and get

$$\Lambda_j(s) = \sum_{k\geq 1} \sum_{m>-j/3} \frac{1}{p^{2ks}\left(m+\frac{j}{3}\right)^s} + \sum_{k\geq 1} \sum_{m>-uj/3} \frac{1}{p^{(2k-1)s}\left(m+\frac{uj}{3}\right)^s}$$

$$= \frac{\zeta\left(s,\frac{j}{3}\right) + p^s\zeta\left(s,\frac{uj}{3}\right)}{p^{2s}-1}, \tag{7}$$

as claimed in (5). □

4. Computing the Asymptotic Main Terms

The further strategy is now to shift the line of integration to the left. The residues at the poles yield the main terms in the asymptotic expansion. We prove the following lemma.

Lemma 4.1. *We have*

$$I = -\frac{3N^2}{2(1-p)} - \frac{1}{2}N\log_p N + N\left(-\frac{1}{2}\log_p 6\pi + \frac{1}{2\log p} + \frac{1}{4}\right)$$
$$+ N\Phi^{(2)}(\log_p N) + \Psi(N) + f_1^{(2)}\log_p N + f_0^{(2)} \tag{8}$$
$$+ \frac{1}{2\pi i}\sum_{j=-1}^{1}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty}\Lambda_j(s)\left(N+\frac{j}{3}\right)^{s+1}\frac{ds}{s(s+1)},$$

where

- $\Psi(N)$ *has been defined in Theorems 1 and 2, respectively and has the properties given in these theorems;*
- *the quantities $f_1^{(2)}$ and $f_0^{(2)}$ depend on p modulo 3:*

$$f_1^{(2)} = 0, \qquad f_0^{(2)} = -\frac{1}{18} - \frac{1}{9(1-p)} \tag{9}$$

if $p \equiv -1 \pmod 3$ and

$$f_1^{(2)} = \frac{1}{9},$$
$$f_0^{(2)} = \frac{1}{3}\log_p\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right) + \frac{g_1(N) - g_{-1}(N)}{6} - \frac{1}{9\log p} - \frac{1}{18} - \frac{1}{9(1-p)} \tag{10}$$

if $p \equiv 1 \pmod 3$ with the convergent function $g_j(N)$ defined in (2);
- *the 1-periodic function*

$$\Phi^{(2)}(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} c_k^{(2)}e^{2k\pi ix} \tag{11}$$

with

$$c_k = \frac{3^{\chi_k}\zeta(\chi_k)}{\chi_k(1+\chi_k)\log p} \qquad \text{for } k \neq 0$$

is continuous.

Proof. We shift the line of integration to $\operatorname{Re} s = -1/4$,

$$I = \frac{1}{2\pi i}\sum_{j=-1}^{1}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty}\Lambda_j(s)\left(N+\frac{j}{3}\right)^{s+1}\frac{ds}{s(s+1)} + \sum_{s\text{ poles}} r_s,$$

where r_s denotes the residue of the integrand in s. Here, the usual growth estimates for the ζ function on vertical lines (cf. [5, § 13.51])

$$\zeta(s,\alpha) = O(|t|^{\tau(\sigma)}\log|t|) \qquad \text{with} \qquad \tau(\sigma) = \begin{cases} 0 & 1 \leq \sigma, \\ 1-\sigma & 1/2 \leq \sigma \leq 1, \\ 1/2 & 0 \leq \sigma \leq 1/2, \\ 1/2-\sigma & \sigma \leq 0 \end{cases} \tag{12}$$

have been used, where $\sigma = \operatorname{Re} s$ and $t = \operatorname{Im} s$, as usual.

The residue at $s = 1$ equals

$$r_1 = -\frac{3N^2}{2(1-p)} - \frac{1}{9(1-p)}.$$

The residue at $s = 0$ is calculated as

$$r_0 = -\frac{1}{2}N\log_p N + N\left(-\frac{1}{2}\log_p 6\pi + \frac{1}{2\log p} + \frac{1}{4}\right) + f_1^{(2)}\log_p N + f_0^{(2)} + \frac{1}{9(1-p)},$$

with the quantities $f_1^{(2)}$ and $f_0^{(2)}$ defined in (9) and (10).

Finally, there is a simple pole of the integrand at $s = \chi_{k/2} = k\pi i/\log p$ for $k \in \mathbb{Z}\setminus\{0\}$ with residue

$$r_{\chi_{k/2}} := \sum_{j=-1}^{1} \frac{\zeta(\chi_{k/2}, j/3) + (-1)^k\zeta(\chi_{k/2}, uj/3)}{2\chi_{k/2}(1+\chi_{k/2})\log p}\left(N + \frac{j}{3}\right)^{1+\chi_{k/2}}.$$

Approximating $(N + j/3)^{1+\chi_{k/2}}$ by $N^{1+\chi_{k/2}}$ and correcting the error yields

$$r_{\chi_{k/2}} = \frac{\sum_{j=-1}^{1}\left(\zeta(\chi_{k/2}, j/3) + (-1)^k\zeta(\chi_{k/2}, uj/3)\right)}{2\chi_{k/2}(1+\chi_{k/2})\log p}N^{1+\chi_{k/2}}$$

$$+ \sum_{j=-1}^{1} \frac{\zeta(\chi_{k/2}, j/3) + (-1)^k\zeta(\chi_{k/2}, uj/3)}{2\chi_{k/2}(1+\chi_{k/2})\log p}\left(\left(N + \frac{j}{3}\right)^{1+\chi_{k/2}} - N^{1+\chi_{k/2}}\right). \quad (13)$$

As $\zeta(s, -1/3) + \zeta(s, 0) + \zeta(s, 1/3) = 3^s\zeta(s)$, the first summand equals

$$\frac{1 + (-1)^k}{2}\frac{3^{\chi_{k/2}}\zeta(\chi_{k/2})}{\chi_{k/2}(1+\chi_{k/2})\log p}N^{1+\chi_{k/2}},$$

which leads to no contribution for odd k. Summing over all $k \neq 0$ yields

$$\sum_{k\neq 0} r_{\chi_{k/2}} = N\Phi^{(2)}(\log_p N) + \Psi(N),$$

where the 1-periodic function $\Phi^{(2)}$ is defined in (11)—it corresponds to the first summand in (13)—and $\Psi(N)$ is the function defined in Theorems 1 and 2.

As

$$\left(N + \frac{j}{3}\right)^{1+\chi_{k/2}} - N^{1+\chi_{k/2}} = O(\min\{N, |k|\})$$

and

$$\frac{\zeta(\chi_{k/2}, j/3) + (-1)^k\zeta(\chi_{k/2}, uj/3)}{2\chi_{k/2}(1+\chi_{k/2})\log p} = O(|k|^{-3/2}\log|k|)$$

by (12), we obtain

$$\Psi(N) \ll \sum_{1\leq|k|\leq N} |k|^{-1/2}\log|k| + \sum_{|k|>N} N|k|^{-3/2}\log|k|$$

$$\ll N^{1/2} \log N + N^{1/2} \log N \ll N^{1/2} \log N,$$

as claimed in Theorems 1 and 2.

As all involved Fourier series converge absolutely and uniformly, the periodic functions are continuous.

$\square$

5. REMAINDER INTEGRAL

Finally, we deal with the remainder integral in (8). It is clear that $R = O(N^{3/4})$, but even better, it can be computed explicitly.

Lemma 5.1. *We have*

$$R := \frac{1}{2\pi i} \sum_{j=-1}^{1} \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \Lambda_j(s) \left(N + \frac{j}{3}\right)^{s+1} \frac{ds}{s(s+1)} = \frac{2p}{9(p-1)}.$$

Proof. Expanding the denominator $p^{2s} - 1$ in a geometric series yields

$$R = -\frac{1}{2\pi i} \sum_{j=-1}^{1} \sum_{\ell \geq 0} \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \left(\zeta\left(s, \frac{j}{3}\right) + p^s \zeta\left(s, \frac{uj}{3}\right)\right) p^{2\ell s} \left(N + \frac{j}{3}\right)^{s+1} \frac{ds}{s(s+1)},$$

where exchanging summation and integration was legitimate due to absolute convergence.

Shifting the line of integration back to $\operatorname{Re} s = 2$ and taking the residues in the simple poles at $s = 0$ and $s = 1$ into account yields

$$R = \sum_{\ell \geq 0} \left(-N + \frac{u+1}{9} + (p+1)p^{2\ell} \left(\frac{3}{2}N^2 + \frac{1}{9}\right) \right.$$
$$\left. - \sum_{j=-1}^{1} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\zeta\left(s, \frac{j}{3}\right) + p^s \zeta\left(s, \frac{uj}{3}\right)\right) p^{2\ell s} \left(N + \frac{j}{3}\right)^{s+1} \frac{ds}{s(s+1)} \right).$$

We have

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta\left(s, \frac{vj}{3}\right) p^{ms} \left(N + \frac{j}{3}\right)^{s+1} \frac{ds}{s(s+1)}$$
$$= \frac{p^{-m}}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta\left(s, \frac{vj}{3}\right) \left(p^m N + \frac{(p^m - v)j}{3} + \frac{vj}{3}\right)^{s+1} \frac{ds}{s(s+1)}$$
$$= p^{-m} \sum_{-vj/3 < n < p^m N + (p^m - v)j/3} \left(p^m N + \frac{(p^m - v)j}{3} - n\right)$$

by the Mellin-Perron summation formula again. In the cases of interest, we have either $v = u$ and m odd or $v = 1$ and m even, which implies that $p^m - v$ is divisible by 3 in our cases. Thus we obtain

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta\left(s, \frac{vj}{3}\right) p^{ms} \left(N + \frac{j}{3}\right)^{s+1} \frac{ds}{s(s+1)}$$

$$= p^{-m} \frac{(p^m N + (p^m - v)j/3 + \lceil vj/3 \rceil - 1)(p^m N + (p^m - v)j/3 + \lceil vj/3 \rceil)}{2}.$$

Summing over j yields

$$R = \sum_{\ell \geq 0} \frac{2}{9} p^{-2\ell-1}(p+1) = \frac{2p}{9(p-1)}.$$

$\square$

Combining (3), Lemma 3.1, Lemma 4.1 and 5.1 yields the results given in Theorems 1 and 2.

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