# Selected partial inverse combinatorial optimization problems with forbidden elements

Elisabeth Gassner

*Project Area(s):*
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Institut für Optimierung und Diskrete Mathematik (Math B)

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**Abstract**

We study the computational complexity of some special partial inverse combinatorial optimization problems where the goal is to change a parameter at minimum cost such that there exists an optimal solution for the underlying combinatorial optimization problem with respect to the modified parameter that does not contain a prespecified set of forbidden elements. We show that the partial inverse problems that arise for the shortest path, minimum cut and assignment problem where the modification cost is measured by the weighted $L_1$- or weighted $L_\infty$-norm are strongly $\mathsf{NP}$-hard.

## 1 Introduction

Let $\Gamma = (w, \mathcal{S}, f)$ be an optimization problem where $w \in \mathbb{R}^n$ is a parameter, $\mathcal{S}$ denotes the set of feasible solutions and $f : \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}$ is an objective function. The goal of a classical optimization problem is to find a feasible solution with minimum objective value, i.e.,

$$\min_{x \in \mathcal{S}} f(w, x).$$

In the corresponding inverse version, a feasible solution $x^0 \in \mathcal{S}$ is already given. The task is to "minimally" modify $w$ to $\tilde{w}$ such that $x^0$ is an optimal solution of $\tilde{\Gamma} = (\tilde{w}, \mathcal{S}, f)$. The modification cost is usually measured by the (weighted) $L_1$ or (weighted) $L_\infty$ norm, i.e., there are given cost coefficients $c_i \in \mathbb{R}_+$ for $i = 1, \ldots, n$ and the modification cost is given by

$$\|w - \tilde{w}\|_{1,c} = \sum_{i=1}^{n} c_i |w_i - \tilde{w}_i|, \text{ or}$$

$$\|w - \tilde{w}\|_{\infty,c} = \max_{i=1,\ldots,n} c_i |w_i - \tilde{w}_i|$$

There is a lot of literature on inverse optimization problems and especially inverse network problems. The interested reader is referred to the comprehensive survey on inverse optimization by Heuberger [4].

Partial inverse problems are a generalization of inverse problems where a subset $I \subseteq \{1, \ldots, n\}$ and values $x_j^0$ for $j \in I$ are given. The task is to "minimally" modify $w$ to $\tilde{w}$ such that there exists an optimal solution $x^*$ of $\tilde{\Gamma} = (\tilde{w}, \mathcal{S}, f)$ with $x_j^* = x_j^0$ for all $j \in I$.

We are especially interested in partial inverse combinatorial optimization problems. A combinatorial optimization problem is usually given in the following form: There is given a ground set $E$ then $\mathcal{S} \subseteq \mathcal{P}(E)$ where $\mathcal{P}(E)$ is the potential set of $E$ and there are weights $w_e$ for $e \in E$. The objective function is of the form

$$f(w, F) = \sum_{e \in F} w_e$$

for $F \in \mathcal{S}$. Observe that this definition implies a binary decision, i.e., one has to decide if $e \in F$ or $e \notin F$.

Hence, for partial inverse combinatorial optimization problems there are given two sets $J^+, J^- \in \{1, \ldots, |E|\}$ with $J^+ \cap J^- = \emptyset$ and the task is to change the weights such that there exists an optimal solution $F^*$ of the underlying combinatorial optimization problem with respect to the modified weight vector such that $F^* \cap J^- = \emptyset$ and $J^+ \subseteq F^*$.

There are several known results for the special case where $J^- = \emptyset$, i.e., there is given a subset of elements $J^+$ and the task is to minimally change the weights such that there exists an optimal solution with respect to the modified weights that contains all elements in $J^+$. Especially, the papers by Orlin [6] and Lai and Orlin [5] consider partial inverse combinatorial optimization problems of this kind. In [6] several special partial inverse combinatorial optimization problems are considered. Most of them turn out to be strongly NP-hard. However, all hard instances of the considered partial inverse combinatorial optimization problems satisfy the property $J^+ \neq \emptyset$. Hence, the question arises if the problems become substantially easier if $J^+ = \emptyset$. In [5] the authors are interested in the special partial inverse problem where $|J^+| = 1$, $J^- = \emptyset$ with weighted $L_\infty$-norm which they call preprocessing problem.

We are interested in the computational complexity of special partial inverse network problems with $J^+ = \emptyset$, i.e., the task is to change weights such that there exists an optimal solution that contains no element of $J^-$. We introduce the notation partial anti-inverse network problem, PAIP for short, for the special case of a partial inverse network problem where $J^+ = \emptyset$.

# 2 Specific partial anti-inverse network problems

In this section, we consider partial anti-inverse versions of the shortest path problem, the minimum cut problem and the assignment problem.

## 2.1 Partial anti-inverse Path Problems

It is known that the partial inverse shortest path problem for the $L_1$- and $L_\infty$-norm is strongly $\mathsf{NP}$-hard even if $J^- = \emptyset$ (see Lai and Orlin [5] and Orlin [6]). We are, however, interested in the computational complexity of the partial anti-inverse path problem. Therefore, we take a closer look to the hard instances of the partial inverse problems constructed in [5] and [6]: Both instances (for the $L_1$- and $L_\infty$-norm) fulfill the property that every $(s-t)$-path either uses the arcs of $J^+$ or it uses at least one arc of a set $J \subset E$ (which they call shortcut arcs and mini-shortcut arcs, respectively). Therefore, the constructed hard instances of the partial inverse shortest path problem imply hard instances of the partial anti-inverse shortest path problem if the set of required arcs $J^+$ is replaced by the set $J$ of forbidden arcs. These observations imply the following theorem:

**Theorem 2.1.** *The partial anti-inverse shortest path problem is strongly* $\mathsf{NP}$*-hard for the* $L_1$*- and* $L_\infty$*-norm.*

## 2.2 Partial anti-inverse Cut Problems

Let us start with the partial anti-inverse cut problem for the $L_\infty$-norm:

**Theorem 2.2.** *The partial anti-inverse* $(s-t)$*-cut problem with* $L_\infty$*-norm is strongly* $\mathsf{NP}$*-hard.*

*Proof.* Lai and Orlin [5] showed that the partial inverse $(s-t)$-cut problem with $L_\infty$-norm is strongly $\mathsf{NP}$-hard even if $|J^+| = 1$ and $J^- = \emptyset$. The proof for the partial anti-inverse $(s-t)$-cut problem relays on the same ideas: Let $G = (V, E)$ be a digraph with source $s$ and sink $t$ and an arc $(p, q) \in E$. The restricted path problem is to find a path from $s$ to $t$ that traverses $(p, q)$. Based on the hardness results by Fortune, Hopcroft and Wyllie [2] it is easy to see that the restricted path problem is strongly $\mathsf{NP}$-hard.

We construct an instance $(G', w, J^-)$ of the partial anti-inverse network problem with $L_\infty$-norm in the following way: $G' = (V, E')$ where $E' = E \cup \{(s, p), (q, t)\}$, $J^- = \{(s, p), (q, t)\}$, $w_{(p,q)} = 3$, $w_{(s,p)} = w_{(q,t)} = 0$ and $w_e = 1$ otherwise. We show that there exists a path in $G$ from $s$ to $t$ that traverses $(p, q)$ if and only if there exists a weight modification $\delta$ such that there exists a minimum $(s-t)$-cut in $G'$

with respect to the weights $w+\delta$ that does not contain any $e \in J^-$ and $|\delta_e| \leq 1$ holds for all $e \in E'$.

Assume that there exists a path $P$ from $s$ to $t$ in $G$ that traverses $(p,q)$ then let $\delta_{(p,q)} = -1$, $\delta_{(s,p)} = \delta_{(q,t)} = 1$, $\delta_e = -1$ for all $e \in E \setminus P$ and $\delta_e = 0$ otherwise. This implies that there are two paths $P$ and $(s,p,q,t)$ that may carry flow of amount 1 while all arcs that are not on these two paths have weight 0. Obviously, there exists an $(s-t)$-cut that is minimum with respect to the modified weights and contains $(p,q)$ and hence does not contain $(s,p)$ or $(q,t)$.

Now assume that there exists a weight modification $\delta$ such that there exists a minimum cut $(X,\bar{X})$ that does not contain $(s,p)$ or $(q,t)$ and $|\delta_e| \leq 1$ for every $e \in E'$. We may assume without loss of generality that $\delta_{(s,p)} = \delta_{(q,t)} = 1$ and $\delta_{(p,q)} = -1$ because the minimum cut $(X,\bar{X})$ contains $(p,q)$ but does not contain $(s,p)$ and $(q,t)$. Since $(X,\bar{X})$ is a minimum cut and $(p,q) \in (X,\bar{X})$, $f_{(p,q)} = 2$ holds for every maximum $(s-t)$-flow $f$. On the other hand, a maximum flow can be found by using the augmenting path algorithm by Ford and Fulkerson [1]: If we first send 1 unit along path $(s,p,q,t)$ and then continue with the augmenting path algorithm then no further augmenting path traverses $(s,p)$ or $(q,t)$ in forward- or backward-direction because both arcs are already saturated and can not be part of a simple augmenting path. Hence, the augmenting path algorithm produces a maximum flow $f^*$ with $f^*_{(s,p)} = f^*_{(q,t)} = 1$ and $f^*_{(p,q)} = 2$. Consider a path decomposition of $f^*$ that contains path $(s,p,q,t)$. Since $f^*_{(p,q)} = 2 > 1 = f^*_{(s,p)} = f^*_{(q,t)}$ and every maximum flow sends 2 units of flow along $(p,q)$ there exists another path that traverses $(p,q)$ but uses neither $(s,p)$ nor $(q,t)$. Hence, there is a path from $s$ to $t$ in $G$ that traverses $(p,q)$. □

Let us now consider the partial anti-inverse $(s-t)$-cut problem with $L_1$-norm.

**Theorem 2.3.** *The partial anti-inverse $(s-t)$-cut problem with $L_1$-norm is strongly* NP*-hard.*

*Proof.* Gassner [3] proved that the partial inverse $(s-t)$-cut problem with $L_1$-norm is strongly NP-hard even if $J^- = \emptyset$ and all cost coefficients are equal to 1. The hard instance of the partial inverse $(s-t)$-cut problem of [3] has the following property: $J^- = J_1^- \cup J_2^-$, all arcs of $J_1^-$ are incident to $s$, no arc of $J_2^-$ is incident to $s$ or $t$ and $s$ is the only vertex that is an endpoint of more than one arc in $J^-$.

Let $(G, w, J^+, J^-)$ be an instance of the inverse $(s,t)$-cut problem with $J^- = \emptyset$ and the property of $J^-$ stated above. We construct a hard instance $(\tilde{G}, \tilde{w}, \tilde{J}^-)$ of the partial anti-inverse $(s-t)$-cut problem as follows: $\tilde{G} = (V, \tilde{E})$ with $\tilde{E} = E \cup E'$ and $E' = \tilde{J}^- = \tilde{J}_1^- \cup \tilde{J}_2^-$ with $\tilde{J}_1^- = \{(i,t) \mid (s,i) \in J_1^+\}$ and $\tilde{J}_2^- = \{(s,i),(j,t) \mid (i,j) \in J_2^+\}$. Moreover, $\tilde{w}_e = 0$ if $e \in E'$ and $\tilde{w}_e = w_e$ otherwise. Observe that every arc in $J_1^+$ uniquely corresponds to an arc in $\tilde{J}_1^-$ and every arc in $J_2^+$ uniquely corresponds to a set of two arcs in $\tilde{J}_2^-$.

First we show that this partial anti-inverse $(s-t)$-cut instance admits an optimal solution $\delta$ with $\delta_e = 0$ for all $e \in E'$. Assume that $\delta$ is an optimal solution that does not satisfy this property. Consider the new solution $\tilde{\delta}$ with

$$\tilde{\delta}_e = \begin{cases} \delta_{(s,i)} - \delta_{(i,t)} & \text{if } e = (s,i) \in J_1^+ \\ \delta_{(i,j)} - (\delta_{(s,i)} + \delta_{(j,t)}) & \text{if } e = (i,j) \in J_2^+ \\ 0 & \text{if } e \in \tilde{J}^- \\ \delta_e & \text{otherwise} \end{cases}$$

Since $\delta$ is an optimal solution, $\delta_e \geq 0$ for all $e \in \tilde{J}^-$ and $\delta_e \leq 0$ for all $e \in \tilde{J}^+$. Therefore, $\tilde{\delta}$ and $\delta$ the have the same modification cost. Observe that

$$\bigotimes_{e\in F}^{f} (w_e + \tilde{\delta}_e) \geq \bigotimes_{e\in F}^{f} (w_e + \tilde{\delta}_e) - \sum_{e\in \tilde{J}^-} \delta_e$$

holds for every $(s-t)$-cut $F$ because every cut contains either $(s,i) \in J_1^+$ or $(i,t) \in \tilde{J}_1^-$ and every cut contains either $(i,j) \in J_2^+$ or one or both arcs of $\{(s,i),(j,t)\}$. A cut $F$ that was optimal with respect to $w + \delta$ and satisfies $F \cap \tilde{J}^- = \emptyset$ is then also optimal with respect to $w + \tilde{\delta}$ because the above inequality is satisfied with equality for such cuts.

Hence, there exists an optimal solution $\delta$ of the partial anti-inverse $(s-t)$-cut with $\delta_e = 0$ for all $e \in E'$. Now it is obvious that there exists a feasible solution of the partial inverse $(s-t)$-cut instance with cost at most $k$ if and only if there exists a feasible solution of the partial anti-inverse $(s-t)$-cut problem with cost at most $k$ because $w_e + \delta_e = 0$ holds for all $e \in \tilde{J}^-$ and a cut in $G'$ contains all arcs of $J^+$ if and only if it contains no arc of $\tilde{J}^-$. ☐

## 2.3 Partial Anti-Inverse Assignment Problem

Lai and Orlin [5] proved strong NP-hardness of the partial inverse assignment problem with $L_\infty$-norm. In their NP-hardness proof they consider a node-splitting transformation in order to transform a hard partial inverse shortest path instance on an acyclic graph to a hard instance of the partial inverse assignment problem. Their node-splitting transformation guarantees a 1-1 correspondence between directed $(s-t)$-paths and assignments. Their hard instance $(G, c, w, J^+, J^-)$ of the partial inverse assignment problem with $L_\infty$-norm has the following property: $J^+ = \{(i'', j')\}$ and $J^- = \emptyset$. Moreover, vertex $i''$ has degree 2 (because vertex $i$ had only one outgoing arc and after the node-splitting transformation vertex $i''$ has then degree 2). Hence, $(G, c, w, \tilde{J}^-)$ is a hard instance of the partial anti-inverse assignment problem with $L_\infty$-norm for $\tilde{J}^- = \{(i', i'')\}$ (and $i' \neq j'$) because every assignment either contains $(i'', j')$ or $(i', i'')$.

The same ideas can also be applied to show that the partial anti-inverse assignment problem with weighted $L_1$-norm is NP-hard: Start with the hard instance of the partial shortest $(s-t)$-path problem with weighted $L_1$-norm and $J^- = \emptyset$ as suggested by Orlin [6] and apply the node-splitting transformation. The node-splitting arcs get weight 0 and cost 2 (using our terminology of weights and cost coefficients). It is straightforward to show that the resulting instance of the partial inverse assignment problem with weighted $L_1$-norm has a solution with cost at most $k$ if and only if the underlying instance of the Satisfiability Problem (which was used to construct the hard instance of the partial inverse $(s-t)$-path problem) is satisfiable. The constructed hard instance $(G, c, w, J^+, J^-)$ of the partial inverse assignment problem has the following properties: $J^- = \emptyset$ and every edge $e \in J^+$ is of the form $e = (i'', j')$ where $i''$ has degree 2. Hence, analoguous observations as above yield strong NP-hardness of the partial anti-inverse assignment problem with weighted $L_1$-norm.

Finally, Orlin [6] showed that the partial inverse assignment problem with $L_1$-norm is solvable in polynomial time. Since the partial anti-inverse version is a special case there also exists a polynomial time algorithm for the partial anti-inverse assignment problem.

**Theorem 2.4.** *The partial anti-inverse assignment problem with weighted $L_1$- or unweighted $L_\infty$-norm is strongly* NP*-hard but if the unweighted $L_1$-norm is used then a polynomial time algorithm exists.*

# References

[1] L. R. Ford, Jr. and D. R. Fulkerson. Maximal flow through a network. *Canad. J. Math.*, 8:399–404, 1956.

[2] S. Fortune, J. Hopcroft, and J. Wyllie. The directed subgraph homeomorphism problem. *Theoret. Comput. Sci.*, 10(2):111–121, 1980.

[3] E. Gassner. The partial inverse minimum cut problem with $l_1$-norm is strongly np-hard. Technical Report 2009-10, Graz University of Technology, 2009. `http://www.math.tugraz.at/fosp/pdfs/tugraz_0144.pdf`.

[4] C. Heuberger. Inverse combinatorial optimization: a survey on problems, methods, and results. *J. Comb. Optim.*, 8(3):329–361, 2004.

[5] T. Lai and J. Orlin. The complexity of preprocessing. Research Report of Sloan School of Management, MIT, 2003.

[6] J. Orlin. Partial inverse optimization problems. Working paper, Sloan School of Management, MIT.