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The computational complexity of continuous-discrete bilevel network problems

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Abstract

We study a bilevel approach for combinatorial optimization problems on graphs: In our model the follower has to solve a network problem. The leader is allowed to modify parameters of the follower's objective function and thereby influences the follower's decision and indirectly his own outcome. The main focus of this paper is to analyse the computational complexity of such bilevel network problems. We give several conditions on the underlying network problem that imply that the associated bilevel network problem is solvable in polynomial time or NP-hard. The computational complexity of bilevel spanning tree problems is fully characterized provided that the underlying objective functions are of sum- or bottleneck-type.

1 Introduction

Multilevel and especially bilevel optimization problems have reached increasing interest in the last years. The characteristic of bilevel optimization is the existence of two decision makers (the leader and the follower), each with his own objective function, who act in a hierarchy. The decision process starts with an action of the leader. Given the leader's action, the follower answers with a reaction that is an optimal solution of his own follower's optimization problem. However, the leader's action influences the follower's optimization problem in the sense that he can modify parameters of the follower's objective function or constraints. The goal is to find an action that optimizes the leader's outcome that depends on his own action and the follower's reaction. Formally, a bilevel optimization problem is given in the following form:

$$\begin{aligned} \min_{x \in S_1} \ & H_1(x, y(x)) \\ \text{s.t. } & y(x) \text{ is an optimal solution of} \\ & \min_{y \in S_2} \ H_2(x, y) \\ & \text{s.t. } \ (x, y) \in S \end{aligned} \tag{1}$$

where $S_i \subseteq \mathbb{R}^{n_i}$ $(i = 1, 2)$, $S \subseteq \mathbb{R}^{n_1+n_2}$ and $H_i : S \to \mathbb{R}$ $(i = 1, 2)$. Observe that we apply the so-called optimistic rule, i.e., in case of several follower-optimal reactions the follower chooses a most leader-friendly one. In contrast there also exists the pessimistic rule where in case of several follower-optimal reactions the follower chooses one that is worst for the leader's outcome (e.g., see Dempe [4] for a discussion of several possibilities about how to deal with non-unique follower-optimal solutions).

Most attention was paid to the bilevel linear programming problem, i.e., $H_i(x, y)$ are linear functions for $i = 1, 2$ and S_i $(i = 1, 2)$ and S are given by systems of linear (in)equalities. Hansen, Jaumard and Savard [11] proved strong NP-hardness of the bilevel linear programming problem. On the other hand, there are several solution approaches based on branch and bound, extreme point, penalty approaches and many other techniques. The interested reader is referred to the bibliography by Vicente and Calamai [15] or the books by Bard [1] and Dempe [4].

Less attention was paid to discrete bilevel programming problems where at least one decision vector x or y is restricted to lie in a discrete set. Questions concerning the existence of an optimal solution and efficient algorithms were addressed for different variants of discrete bilevel problems (e.g., see Bard [1] or Dempe [4]). Most investigations were done for discrete bilevel optimization problems where both objective functions H_i $(i = 1, 2)$ are linear, and the sets S_i $(i = 1, 2)$ and S are described by systems of linear (in)equalities with the additional constraint that either x and y or only x are discrete decision vectors. One reason for this is that the compactness of $(S_1 \times S_2) \cap S$ guarantees the existence of an optimal solution for these variants of discrete bilevel optimization problems. This is, however, not the case if x is continuous and y is discrete (e.g., see Dempe [3]).

Our purpose is to investigate bilevel network problems, i.e., the follower's optimization problem is a network problem like the shortest path problem or the minimum spanning tree problem. While (discrete) bilevel linear programming problems have reached a lot of attention, there has been done much less work in the area of bilevel combinatorial optimization problems. Gassner [8] studied a discrete bilevel approach for the minimum spanning tree problem where a partition of the set of edges into leader- and follower-edges is given. The leader's action is to choose a subset of his edges while the follower's reaction is to build up a spanning tree that includes the edges chosen by the leader. Hence, the leader's and follower's decision vectors are discrete. She obtained polynomial time algorithms for the cases where the leader has a bottleneck objective function while the follower has a sum or bottleneck objective function. An analoguous bilevel approach was considered for the assignment problem. Gassner and Klinz [10] showed several NP-hardness results of such bilevel assignment problems with sum- and bottleneck objective functions for the leader and follower.

Another direction of research are continuous-discrete bilevel combinatorial optimization problems, i.e., the leader has a continuous and the follower has a discrete decision vector. Dempe and Richter [5] and later Brotcorne, Hanafi and Mansi [2] considered a continuous-discrete bilevel knapsack problem where the leader chooses the capacity of the knapsack while the follower has to solve the knapsack instance with respect to the capacity chosen by the leader. This model allows the leader to modify a parameter of a constraint of the follower's optimization problem. The leader's objective function consists of the cost of parameter modification and an evaluation of the knapsack solution chosen by the follower.

We consider a similar model but the leader is allowed to change parameters of the follower's objective function instead of parameters of constraints. The follower's optimization problem is a network problem of the following form: Given a graph property π, the task of the follower is to choose a subset of edges F such that the graph induced by F satisfies π and the follower's objective value is minimum. The goal of this paper is to clarify the computational complexity of continuous-discrete bilevel network problems. In Section 2, we introduce continuous-discrete bilevel network problems and make some assumptions on the solvability of the underlying network problems. Section 3 deals with the computational complexity of continuous-discrete bilevel network problems. We state relationships to inverse and blocking network problems and thereby conclude some conditions that guarantee NP-hardness or polynomial solvability of the associated bilevel network problems. Special attention is paid to the cases where the evaluations of network solutions are of sum- and bottleneck-type. If at least one of the decision makers has a bottleneck objective function then we show relationships to partial inverse network and blocking network problems, respectively. Finally, in Section 4 the results of the previous section are applied to the spanning tree problem. We state the computational complexity for the cases that the modification cost, leader's evaluation and follower's evaluation of a feasible network solution are of sum- or bottleneck-type.

2 Problem Formulation

This section gives a formal definition of our model of continuous-discrete bilevel network problems, CDBP for short. CDBP is a special case of the bilevel optimization problem (1) where S_2 is the set of feasible solutions of a network problem. Each decision maker, the leader and the follower, have their own evaluation of a feasible network solution. In order to force the follower to choose a certain solution and hereby to optimize the leader's objective value, the leader is allowed to modify input parameters that influence the follower's evaluation of a feasible solution. However, the leader has to pay for such modifications. Hence, the task is to find an optimal modification strategy such that the sum of total cost of modifications and the leader's evaluation of a follower-optimal solution is minimum. In the context of general bilevel optimization problems, S_1 is the set of possible modification strategies, S_2 is the set of feasible solutions of the network problem and $S = S_1 \times S_2$.

We apply this type of continuous-discrete bilevel optimization problems to network problems that are given in the following way: There is a (di)graph $G = (V, E)$, element weights $w_e \in \mathbb{R}$ (for every $e \in E$), a graph property π and a composition $\otimes$. The task is to find a subset $F \subseteq E$ of edges such that the graph G_F induced by F satisfies π and $\bigotimes_{e \in F} w_e$ is minimum. Hence, the problem is defined by the tuple $(\pi, \otimes)$ while an associated instance is specified by (G, w). The set of feasible solutions is denoted by

$$\mathcal{F}^\pi(G) = \{F \subseteq E \mid G_F \text{ satisfies } \pi\}.$$

Observe that several well-known combinatorial optimization problems can be described in this way, e.g., the shortest path problem, the minimum linear assignment problem, the minimum cut problem or the minimum spanning tree problem.

A continuous-discrete bilevel network problem CDBP $(\pi, \oplus, \otimes^\ell, \otimes^f)$ is specified by a graph property π, a cost-composition $\oplus$ and the compositions of the leader's and follower's weights $\otimes^\ell$ and $\otimes^f$, respectively. An instance (G, c, w^ℓ, w^f) of the CDBP is given by a (di)graph G, a cost coefficient vector c, a weight-vector for the leader w^ℓ and a weight-vector for the follower w^f. The task is to find a modification δ of the follower's weights and an optimal solution $F \in \mathcal{F}^\pi(G)$ of the follower's reaction problem that minimizes the leader's objective value, i.e., the modification cost plus the leader's evaluation of the network solution F chosen by the follower. Formally, CDBP is to solve

$$\begin{aligned}
\min_{\delta} \ & \bigoplus_{e \in E} c_e |\delta_e| + \bigotimes_{e \in F}^{\ell} w_e^\ell \\
\text{s.t. } \ & F \text{ is an optimal solution of} \\
& \quad \min_{F \subseteq E} \ \bigotimes_{e \in F}^{f} (w_e^f + \delta_e) \\
& \quad \text{s.t. } \quad G_F \text{ satisfies } \pi
\end{aligned}$$

We complete this section by some notation and assumptions:

Let (G, c, w^ℓ, w^f) be an instance of CDBP $(\pi, \oplus, \otimes^\ell, \otimes^f)$: A solution (δ, F) of CDBP is feasible if $F \in \mathcal{F}^\pi(G)$ is an optimal solution of the instance $(G, w^f + \delta)$ of the follower's network problem $(\pi, \otimes^f)$, i.e.,

$$\bigotimes_{e\in F}^{f} (w_e^f + \delta_e) \leq \bigotimes_{e\in \tilde{F}}^{f} (w_e^f + \delta_e)$$

holds for all $\tilde{F} \in \mathcal{F}^\pi(G)$. A feasible solution (δ^*, F^*) is an optimal solution if

$$\bigoplus_{e\in E} c_e|\delta_e^*| + \bigotimes_{e\in F^*}^{\ell} w_e^\ell \leq \bigoplus_{e\in E} c_e|\delta_e| + \bigotimes_{e\in F}^{\ell} w_e^\ell$$

holds for every feasible solution (δ, F).

The aim of this paper is to characterize easy and hard cases of continuous-discrete bilevel network problems. Obviously, if the network problems $(\pi, \otimes^\ell)$ or $(\pi, \otimes^f)$ are NP-hard then CDBP $(\pi, \oplus, \otimes^\ell, \otimes^f)$ is also NP-hard or does not even lie in NP. Hence, we will only consider network problems $(\pi, \otimes^\ell)$ and $(\pi, \otimes^f)$ that are solvable in polynomial time.

Throughout the paper, we assume that all compositions $\oplus$, $\otimes^\ell$ and $\otimes^f$ are compatible with the order $\leq$, e.g., for $\oplus$ this means if $a \leq b$ then $a \oplus c \leq b \oplus c$ holds. Moreover, we assume that the cost coefficients c_e $(e \in E)$ and the cost composition satisfy $\bigoplus_{e\in E} c_e|\delta_e| \geq 0$ for all $\delta \in \mathbb{R}^{|E|}$ and 0 is the neutral element of this composition, i.e., $a \oplus 0 = a$. Finally, the leader's weights w_e^ℓ $(e \in E)$ and his evaluation function $\otimes^\ell$ satisfy $w_e^\ell \leq \bigotimes_{e\in F}^{\ell} w_e^\ell$ for all $e \in F$ and $F \in \mathcal{F}^\pi(G)$. This property is called leader's monotonicity.

We will frequently use the sum- and the maximum-composition. Observe that both compositions are compatible with the order $\leq$. Moreover, the nonnegativity assumption for the cost-composition as well as the leader's monotonicity (provided that the cost coefficients and leader's weights are nonnegative) hold for sum- and maximum-operators.

An optimization problem where the objective function is a composition with the maximum-operator is also called bottleneck optimization problem. Throughout the paper, the number of vertices of a graph $G = (V, E)$ is denoted by n while $m = |E|$.

Finally, we mention an obvious property of an optimal solution (δ^*, F^*) of CDBP: Since (δ^*, F^*) is feasible, F^* is an optimal solution of the follower's network problem with respect to the weight vector $w^f + \delta^*$. The compatibility of $\otimes^f$ with $\leq$ implies that F^* is also an optimal solution with respect to the weight vector $w^f + \tilde{\delta}$ where $\tilde{\delta}_e \leq \delta_e^*$ holds for all $e \in F^*$ and $\tilde{\delta}_e \geq \delta_e^*$ for all $e \in E \setminus F^*$. However, the goal is to minimize the sum of modification cost and the leader's evaluation of the follower's network solution. Therefore, we get the following lemma:

Lemma 2.1. *CDBP $(\pi, \oplus, \otimes^\ell, \otimes^f)$ admits an optimal solution (δ^*, F^*) such that $\delta_e^* \leq 0$ for all $e \in F^*$ and $\delta^* \geq 0$ for all $e \in E \setminus F^*$.*

3 The computational complexity for general bilevel network problems

In this section, we try to draw a borderline between easy and hard cases of CDBP. Intuitively, the task of CDBP is to find a follower's weight modification such that the leader forces the follower either to choose a leader-friendly network solution or at least to reject all these edges that have a bad leader's weight. Hence, there is a strong relationship to so-called (partial) inverse optimization problems.

A partial inverse network problem is specified by a network problem $(\pi, \otimes)$ together with a cost composition $\oplus$, i.e., there is given a triple $(\pi, \oplus, \otimes)$. An instance of such a partial inverse network problem is then given by (G, c, w, J^+, J^-) where (G, w) is an instance of the network problem, c is a cost vector and $J^+ \subseteq E$ and $J^- \subseteq E$ with $J^+ \cap J^- = \emptyset$ are subsets of edges. The task of a partial inverse network problem is to find a weight modification δ such that there exists an optimal solution F^* of the network problem instance $(G, w + \delta)$ such that $J^+ \subseteq F^*$ and $J^- \cap F^* = \emptyset$ hold and the total modification cost $\bigoplus_{e \in E} c_e |\delta_e|$ is minimum. Hence, the task is to change the weights such that there exists an optimal network solution that contains all elements of J^+ and excludes all elements of J^-.

If $J^+ \cup J^- = E$ then the problem is called inverse problem because the solution that should become optimal is fully given. There is a lot of literature on inverse optimization problems and especially inverse network problems. The interested reader is referred to the comprehensive survey on inverse optimization by Heuberger [12].

Since partial inverse network problems with $J^+ = \emptyset$ are considered frequently in this paper, we introduce the notation partial anti-inverse network problem, PAIP for short, for the special case of a partial inverse network problem where $J^+ = \emptyset$, i.e., the goal is to modify the weights such that there exists an optimal network solution that excludes a given set of edges.

3.1 Sufficient NP-hardness conditions

This subsection provides sufficient conditions for CDBP to be NP-hard. The main idea is to construct an instance that contains a set of edges that are unacceptable for the leader. Hence, his goal is to change the follower's weight such that the follower rejects these edges:

Theorem 3.1. *If the Partial Anti-Inverse Problem* $(\pi, \oplus, \otimes^f)$ *is* NP*-hard then CDBP* $(\pi, \oplus, \otimes^\ell, \otimes^f)$ *is also* NP*-hard.*

Proof. Consider an instance (G, c, w, J^-) of PAIP. We will construct an instance I of CDBP with the following property: There exists a feasible solution of PAIP with objective value at most k if and only if there exists a feasible solution of CDBP with leader's objective value at most k. The instance $I = (G, c, w^\ell, w^f)$ is defined in the following way: $w^\ell_e = k + 1$ for $e \in J^-$ and $w^\ell_e = 0$ otherwise and $w^f_e = w_e$ for $e \in E$. Observe that there is a feasible solution (δ, F) of the constructed CDBP-instance with leader's objective value at most k if and only if F does not contain any element of J^- and the modification cost is at most k (because of the leader's monotonicity). Hence, there exists a feasible solution (δ, F) of the CDBP-instance with objective value at most k if and only if there exists a follower's weight modification with cost at most k such that there exists a follower-optimal solution F that does not contain any element of J^-. This observation immediately implies the correctness of the theorem. □

Observe that Theorem 3.1 provides a sufficient NP-hardness condition that is based on the partial anti-inverse network problem. In the following, we will consider the partial inverse network problem. Moreover, we assume that the leader's composition is a sum-operation and π satisfies the property that all feasible network solutions have the same number of edges:

Theorem 3.2. *If the Partial Inverse Problem $(\pi, \oplus, \otimes^f)$ is NP-hard and all subgraphs of G that satisfy π have the same number of edges, then CDBP $(\pi, \oplus, \sum, \otimes^f)$ is also NP-hard.*

Proof. This proof is very similar to the proof of Theorem 3.1. Therefore, we only mention the main ideas of the construction of a hard instance (G, c, w^ℓ, w^f) of CDBP given an instance (G, c, w, J^+, J^-) of the partial inverse problem $(\pi, \oplus, \otimes^f)$: Assume that every feasible network solution contains exactly p edges. Then define $w^\ell_e = 2k(p + 2)$ for $e \in J^-$, $w^\ell_e = 0$ for $e \in J^+$ and $w^\ell_e = 2k$ otherwise. Moreover, $w^f_e = w_e$ for every $e \in E$. We show that there exists a feasible solution of the partial inverse network instance with objective value at most k if and only if there exists a feasible solution (δ, F) of CDBP with leader's objective value at most $k(2p - 2|J^+| + 1)$. Recall that we assume the leader's monotonicity. This implies that as soon as there is one edge of J^- or there are more than $p - |J^+|$ elements of $E \setminus (J^+ \cup J^-)$ in F then the leader's objective value is at least $2k(p+2) > k(2p-2|J^+|+1)$. Therefore, the leader's objective value is at most $k(2p - 2|J^+| + 1)$ if and only if there exists a follower-optimal solution F with $F \cap J^- = \emptyset$ and $J^+ \subseteq F$ and the modification cost are at most k. □

The following two subsections deal with the special cases where either the leader's or the follower's composition is equal to the maximum-operator, i.e., at least one of them has a bottleneck evaluation function.

3.2 Characterization if the leader's weight composition is of bottleneck-type

In this subsection, we assume that the leader's composition is equal to the maximum-operator, i.e., $\otimes^\ell = \max$ and hence the leader's objective function is equal to

$$\bigoplus_{e\in E} c_e|\delta_e| + \max_{e\in F} w_e^\ell.$$

Consider CDBP $(\pi, \oplus, \max, \otimes^f)$ and an instance (G, c, w^ℓ, w^f) thereof. In order to solve this problem, we may apply a threshold algorithm that is based on the calculation of an optimal follower's weight modification provided that the leader's evaluation of the follower-optimal network solution is fixed, i.e.,

- For every $z \in W := \{w_e^\ell \mid e \in E\}$ solve the instance $(G, c, w^f, J^-_{>z})$ of PAIP $(\pi, \oplus, \otimes^f)$ where
$$J^-_{>z} = \{e \in E \mid w_e^\ell > z\}.$$
This means, we are interested in an optimal weight modification such that the follower chooses only edges $e \in E$ with $w_e^\ell \leq z$. Let $\delta(z)$ be an optimal solution of this PAIP-instance with objective value $\text{cost}(z)$.

- Let $\min_{z\in W}(\text{cost}(z) + z) = \text{cost}(z^*) + z^*$ and let F^* be any follower-optimal solution with respect to $w^f + \delta(z^*)$ that contains no edge of $J^-_{>z^*}$. Then $(\delta(z^*), F^*)$ is an optimal solution of CDBP.

Observe that this algorithm runs in $\mathcal{O}(mT(\text{PAIP}))$ where $T(\text{PAIP})$ denotes the time to solve PAIP. Together with Theorem 3.1 we get the following result:

Corollary 3.3. *If PAIP $(\pi, \oplus, \otimes^f)$ is solvable in polynomial time then CDBP $(\pi, \oplus, \max, \otimes^f)$ is also solvable in polynomial time. However, if PAIP $(\pi, \oplus, \otimes^f)$ is* NP*-hard then CDBP $(\pi, \oplus, \max, \otimes^f)$ is also* NP*-hard.*

3.3 Characterization if the follower's weight composition is of bottleneck-type

In this subsection, we consider the special case of CDBP where the follower has a bottleneck objective function, i.e., the follower's objective function is of the form

$$\bigotimes_{e\in F}^{f}(w_e^f + \delta_e) = \max_{e\in F}(w_e^f + \delta_e).$$

First we show that there exists an optimal solution (δ^*, F^*) such that whenever the follower's weight of an edge is changed then its new weight is equal to the optimal follower's objective value:

Lemma 3.4. *There exists an optimal solution* (δ^*, F^*) *of CDBP* $(\pi, \oplus, \otimes^\ell, \max)$ *such that* $w_e^f + \delta_e^* = z^* \leq \max_{e \in E} w_e^f$ *holds for all* $e \in E$ *with* $\delta_e^* \neq 0$.

Proof. Let $z^* = \max_{e \in F^*}(w_e^f + \delta_e^*)$ be the optimal follower's objective value. According to Lemma 2.1, we have $z^* \leq \max_{e \in F^*} w_e^f \leq \max_{e \in E} w_e^f$. Since $\oplus$ is compatible with $\leq$, it can easily be seen that there exists an optimal solution with the following properties: Whenever $w_e^f > z^*$ then the follower-weight of e is either decreased to z^* or not changed at all. An analogue result holds for an edge e with $w_e^f \leq z^*$. $\square$

Observe that Lemma 3.4 implies that whenever the follower's optimal objective value is known to be equal to z^* then $\delta_e^* \in \{0, z^* - w_e^f\}$. Let $z \in \mathbb{R}$ then CDBP(z) is based on CDBP with the additional requirement that the follower's optimal objective value is equal to z. If opt(z) is the optimal objective value of CDBP(z) and $\min_{z \in \mathbb{R}} \text{opt}(z) = \text{opt}(z^*)$ then an optimal solution of CDBP(z^*) is an optimal solution of CDBP.

In order to solve CDBP(z), let us define

$$E_{\leq z} = \{e \in E \mid w_e^f \leq z\};$$
$$\mathcal{F}^\pi_{\leq z}(G) = \{F \in \mathcal{F}^\pi(G) \mid F \subseteq E_{\leq z}\}.$$

Then $(\tilde{\delta}, \tilde{F})$ is a feasible solution of CDBP(z) if and only if for every $F \in \mathcal{F}^\pi(G)$ there exists at least one element $e' \in F$ with $(w_{e'}^f + \tilde{\delta}_e) \geq z$ (this requirement makes sure that the optimal follower's objective value is at least z) and moreover $(w_e^f + \tilde{\delta}_e) \leq z$ holds for all $e \in \tilde{F}$ (this requirement guarantees that the optimal follower's objective value is at most z and F is an optimal follower reaction). If we use the fact that the cost-composition $\oplus$ is compatible with the order $\leq$ then we get the following property: There exists an optimal solution (δ^*, F^*) of CDBP(z) such that

1. for every $F \in \mathcal{F}^\pi_{\leq z}$ there exists at least one element $e' \in F$ with $w_{e'}^f + \delta_{e'}^* = z$ and
2. $w_e^f + \delta_e^* = z$ holds for all $e \in F^*$ with $w_e^f > z$.

Observe that the first condition is independent of F^*. It requires that there exists a set $X \subseteq E_{\leq z}$ such that $X \cap F \neq \emptyset$ holds for all $F \in \mathcal{F}^\pi_{\leq z}$ and $w_e^f + \delta_e^* = z^*$ for all $e \in X$, i.e., the task is to find a subset of elements such that the removal of all elements in the subset destroys all feasible network solutions in $\mathcal{F}^\pi_{\leq z}$. These observations lead to the so-called blocking network problem $(\pi, \oplus)$: An instance is given by (G', c') and the task is to find a subset $X \subseteq E'$ with $X \cap F \neq \emptyset$ for all $F \in \mathcal{F}^\pi(G')$ and $\bigoplus_{e \in X} c'_e$ is minimum. We consider the instance (G', c') with $G' = (V, E_{\leq z})$ and $c'_e = c_e(z - w_e^f)$ for $e \in E_{\leq z}$. The set of feasible solutions is then equal to

$$\mathcal{B}^\pi_{\le z} = \{X \subseteq E_{\le z} \mid F \cap X \neq \emptyset \;\forall F \in \mathcal{F}^\pi_{\le z}(G)\}.$$

Hence, CDBP(z) can be written in the following form:

$$\min_{\substack{X \in \mathcal{B}^\pi_{\le z} \\ F \in \mathcal{F}^\pi(G)}} \left(\bigoplus_{e \in X} c_e(z - w^f_e) \oplus \bigoplus_{e \in F} c_e \max\{w^f_e - z, 0\} \right) + \bigotimes_{e \in F}^{\ell} w^\ell_e.$$

Observe that the cost and leader's weight compositions as well as the sum-operation are compatible with the order $\le$ and hence the above formulation is equivalent to

$$\min_{F \in \mathcal{F}^\pi(G)} \left(\left(\min_{X \in \mathcal{B}^\pi_{\le z}} \bigoplus_{e \in X} c_e(z - w^f_e) \right) \oplus \bigoplus_{e \in F} c_e \max\{w^f_e - z, 0\} \right) + \bigotimes_{e \in F}^{\ell} w^\ell_e.$$

This means that we split CDBP(z) into two subproblems. The first subproblem is to solve

$$\text{block}(z) = \min_{X \in \mathcal{B}^\pi_{\le z}} \bigoplus_{e \in X} c_e(z - w^f_e)$$

which is a blocking network problem $(\pi, \oplus)$. Taking the optimal objective value of the corresponding blocking network instance into account, CDBP(z) is equal to

$$\min_{F \in \mathcal{F}^\pi(G)} \left(\text{block(z)} \oplus \bigoplus_{e \in F} c_e \max\{w^f_e - z, 0\} \right) + \bigotimes_{e \in F}^{\ell} w^\ell_e \tag{2}$$

The composition given in (2) is denoted by $\oplus + \otimes^\ell$. Hence, the task of the second subproblem is to solve the network problem $(\pi, \oplus + \otimes^\ell)$.

Assume that we are given a subset of κ candidates for an optimal z-value then the above discussion implies that CDBP can be solved by solving κ blocking problems $(\pi, \oplus)$ and network problems $(\pi, \oplus + \otimes^\ell)$.

The next task is to find a small set of candidates for an optimal z-value. Assume that the follower's weights are sorted $w^f_{e_1} \le w^f_{e_2} \le \cdots \le w^f_{e_m}$ and let $a = w^f_{e_i} \le z \le w^f_{e_{i+1}} = b$. Consider a restricted version of CDBP where the optimal follower's objective value is required to lie in $[a, b]$:

$$\min_{a \le z \le b} \min_{\substack{X \in \mathcal{B}^\pi_{\le z} \\ F \in \mathcal{F}^\pi(G)}} \left(\bigoplus_{e \in X} c_e(z - w^f_e) \oplus \bigoplus_{e \in F} c_e \max\{w^f_e - z, 0\} \right) + \bigotimes_{e \in F}^{\ell} w^\ell_e.$$

Observe that $E_{\leq z} = E_{\leq a}$ holds for all $a \leq z < b$ which implies $\mathcal{B}^{\pi}_{\leq z} = \mathcal{B}^{\pi}_{\leq a}$ for all $a \leq z < b$. Moreover,

$$\min_{X \in \mathcal{B}^{\pi}_{\leq b}} \bigoplus_{e \in X} c_e(z - w^f_e) = \min_{X' \in \mathcal{B}^{\pi}_{\leq a}} \bigoplus_{e \in X'} c_e(z - w^f_e) = \text{block}(z)$$

because if $X \in \mathcal{B}^{\pi}_{\leq b}$ then $X \cap E_{\leq a} \in \mathcal{B}^{\pi}_{\leq a}$ and if $X' \in \mathcal{B}^{\pi}_{\leq a}$ then $X \cup \{e \in E \mid w^f_e = b\} \in \mathcal{B}^{\pi}_{\leq b}$.

Assume that the cost composition is equal to the sum-operator. Then block(z) and $c_e \max\{w^f_e - z, 0\}$ are linear functions for $a \leq z \leq b$. Hence, there exists an optimal solution with $z \in \{a, b\} = \{w^f_{e_i}, w^f_{e_{i+1}}\}$ for every fixed $F \in \mathcal{F}^{\pi}(G)$. Since this property holds for every $F \in \mathcal{F}^{\pi}(G)$ there exists an optimal solution of CDBP that coincides with an optimal solution of CDBP(z) for $z \in \{w^f_e \mid e \in E\}$.

If the cost composition is equal to the maximum-operator then block(z) is a piecewise linear and convex function. Moreover, $c_e \max\{w^f_e - z, 0\}$ for $e \in F$ are also piecewise linear and convex. Therefore, the objective function is piecewise linear and convex for every fixed $F \in \mathcal{F}^{\pi}(G)$. It is well-known that there exists a minimum of a piecewise linear and convex function that is is either attained at the border of the interval $[w^f_{e_i}, w^f_{e_{i+1}}]$ or at the intersection point of two functions of the form $c_{e_k}(z - w^f_{e_k})$ and $c_{e_j} \max\{w^f_{e_j} - z, 0\}$ which implies that either $z = w^f_{e_k}$ or $z = \frac{c_{e_k} w^f_{e_k} + c_{e_j} w^f_{e_j}}{c_{e_k} + c_{e_j}}$. Hence, in this case there exists an optimal z-value in

$$\{w^f_e \mid e \in E\} \cup \left\{ \frac{c_{e_i} w^f_{e_i} + c_{e_j} w^f_{e_j}}{c_{e_i} + c_{e_j}} \middle| e_i, e_j \in E \right\}.$$

We conclude that for the sum- and maximum-operator there exist candidate-sets for z of polynomial size. Therefore, we get the following theorem:

Theorem 3.5. *Consider CDBP* $(\pi, \oplus, \otimes^{\ell}, \max)$ *with* $\oplus \in \{\sum, \max\}$. *Then CDBP can be solved in*

$$\mathcal{O}(\kappa(T(\mathit{Block}) + T(\mathit{Network})))$$

where $T(\mathit{Block})$ *is the running time of an algorithm that solves the blocking problem* $(\pi, \oplus)$, $T(\mathit{Network})$ *is the running time of an algorithm that solves the network problem* $(\pi, \oplus + \otimes^{\ell})$ *and* $\kappa = m$ *if the cost composition is the sum-operator and* $\kappa = m^2$ *if the cost composition is the maximum-operator.*

Theorem 3.5 provides a sufficient condition for a polynomial time algorithm for a large class of CDBPs. Assume that $\oplus, \otimes^{\ell} \in \{\max, \sum\}$ holds: If both compositions are equal to the sum-operator then the network problem $(\pi, \oplus + \otimes^{\ell})$ is equal to the network problem $(\pi, \sum)$ that is assumed to be solvable in polynomial time (cf. Section 2). If at least one of the two compositions is equal to the maximum-operator, say $\oplus = \max$, then a simple threshold algorithm would solve $(\pi, \oplus + \otimes^\ell)$ in polynomial time provided that the network problem $(\pi, \otimes^\ell)$ is solvable in polynomial time. These observations imply the following corollary:

Corollary 3.6. *If* $\oplus, \otimes^\ell \in \{\max, \sum\}$*, the network problems* $(\pi, \max)$ *and* $(\pi, \sum)$ *and the blocking network problem* $(\pi, \oplus)$ *are solvable in polynomial time then CDBP* $(\pi, \oplus, \otimes^\ell, \max)$ *is also solvable in polynomial time.*

Since we assume that our network problems with sum- or maximum-operator are solvable in polynomial time, Corollary 3.6 provides a sufficient condition for a polynomial time algorithm of CDBP that requires a polynomial time algorithm for the corresponding blocking network problem. Consider an instance (G, c) of the blocking network problem $(\pi, \max)$. Then $c_{e'}$ is the optimal objective value if and only if $\{e \in E \mid c_e \leq c_{e'}\}$ is an optimal solution of this blocking-instance, i.e.,

$$F \cap \{e \in E \mid c_e \leq c_{e'}\} = \{e \in F \mid c_e \leq c_{e'}\} \neq \emptyset$$

holds for every $F \in \mathcal{F}^\pi(G)$ but there exists a feasible solution $F' \in \mathcal{F}^\pi(G)$ such that $c_e \geq c_{e'}$ holds for all $e \in F'$. Hence, the blocking network problem $(\pi, \max)$ is equivalent to the so-called max-min network problem that is to solve $\max_{F \in \mathcal{F}^\pi(G)} \min_{e \in F} c_e$. A max-min network problem $(\pi, \max)$ can be solved by a threshold algorithm which runs in polynomial time whenever it is possible to decide whether a graph contains a subgraph that satisfies property π. On the other hand, there exist more efficient algorithms for many max-min network problems. We have shown that the blocking network problem $(\pi, \max)$ is solvable in polynomial time. Together with Corollary 3.6, we get

Corollary 3.7. *CDBP* $(\pi, \max, \otimes^\ell, \max)$ *for* $\otimes^\ell \in \{\max, \sum\}$ *can be solved in polynomial time if the network problem* $(\pi, \otimes^\ell)$ *is solvable in polynomial time.*

However, there are network problems whose blocking problem is not solvable in polynomial time. Since we are now interested in hard blocking network problems, we turn our attention to a sum-type cost-composition. The following NP-hardness result is restricted to certain graph properties. A graph property π is called *extendable* if every graph $G = (V, E)$ admits an extension $G' = (V', E')$ with $V \subseteq V'$, $E' = E \cup E_1 \cup E_2$ and

- $F \cup E_1 \in \mathcal{F}^\pi(G')$ for every $F \in \mathcal{F}^\pi(G)$ (old feasible solutions retain);
- Every $F \in \mathcal{F}^\pi(G')$ with $F \notin \{\hat{F} \cup E_1 \mid \hat{F} \in \mathcal{F}^\pi(G)\}$ satisfy $F \cap E_2 \neq \emptyset$ (new feasible solutions need edges from E_2);
- There exists a set $F \in \mathcal{F}^\pi(G')$ with $F \subseteq E_1 \cup E_2$ (a new solution without edges of E appears).

A lot of well-known graph properties turn out to be extendable. Consider the spanning tree property and let $G = (V, E)$ be a graph with $v_1 \in V$. Then $G' = (V', E')$ with $V' = V \cup \{v_0\}$ and $E' = E \cup E_1 \cup E_2$ with $E_1 = \{(v_1, v_0)\}$ and $E_2 = \{(v, v_0) \mid v \in V, v \neq v_1\}$ is an extension.

Theorem 3.8. *Let π be an extendable graph property. If the blocking network problem $(\pi, \sum)$ is* NP*-hard then the CDBP $(\pi, \sum, \otimes^\ell, \max)$ is also* NP*-hard.*

Proof. Consider an instance (G, c) of the blocking network problem $(\pi, \sum)$. We construct an instance (G', c, w^ℓ, w^f) of CDBP with the following property: There exists a feasible solution X of (G, c) with objective value at most k if and only if there exists a feasible solution (δ, F) of CDBP with leader's objective value at most k. The CDBP-instance is defined as follows: $G' = (V', E \cup E_1 \cup E_2)$ is an extension of G with respect to π. We define

$$c_e = \begin{cases} c_e & \text{if } e \in E, \\ k+1 & \text{otherwise;} \end{cases} \qquad w^\ell_e = \begin{cases} k+1 & \text{if } e \in E, \\ 0 & \text{otherwise;} \end{cases} \qquad w^f_e = \begin{cases} 0 & \text{if } e \in E \cup E_1, \\ 1 & \text{otherwise;} \end{cases}$$

Consider a solution X of the blocking network problem with objective value at most k and define $\delta_e = 1$ if $e \in X$ and $\delta_e = 0$ otherwise. Since G' is an extension of G, there are two types of feasible solutions in G': Either $F' \in \mathcal{F}^\pi(G')$ satisfies $F' = F \cup E_1$ for some $F \in \mathcal{F}^\pi(G)$ or $E_2 \subseteq F'$. A solution of type $F' = F \cup E_1$ has follower's objective value 1 because there exists at least one $e \in F$ with $w^f_e + \delta_e = 0 + 1 = 1$. On the other hand the follower's objective value of a solution F' with $E_2 \subseteq F'$ is also equal to 1 because $w^f_e = 1$ for $e \in E_2$. Hence, every feasible solution $F' \in \mathcal{F}^\pi(G')$ is follower-optimal. Choose any $F \in \mathcal{F}^\pi(G')$ with $F \subseteq E_1 \cup E_2$, then (δ, F) is a feasible solution with leader's objective value equal to $\sum_{e \in X} c_e \leq k$.

Now assume that there exists a feasible solution (δ, F) of the constructed instance of CDBP with leader's objective value at most k. Recall that $w^\ell_e = k + 1$ for $e \in E$. The leader's monotonicity and the fact that the leader's objective value is at most k imply $F^* \subseteq E_1 \cup E_2$.

Since the cost-composition is equal to the sum-operator there exists an optimal solution of CDBP with follower's objective value out of $\{w^f_e \mid e \in E\}$. In our case there exists an optimal solution with follower's objective value equal to 0 or 1. Let (δ^*, F^*) be an optimal solution with $\max_{e \in F^*}(w^f_e + \delta^*_e) = 0$. Since $F^* \subseteq E_1 \cup E_2$ and G' is an extension we have $F^* \cap E_2 \neq \emptyset$. Hence, there exists at least one element $e' \in E_2$ with $\delta^*_{e'} = 1$ and therefore the leader's objective value is then at least $c_{e'} \delta^*_{e'} = k + 1 > k$ which leads to a contradiction. It follows that $\max_{e \in F^*}(w^f_e + \delta^*_e) = 1$ and $F^* \subseteq E_1 \cup E_2$ hold. However, F^* is follower-optimal with objective value 1 if and only if there exists an element $e \in F'$ with $\delta^*_e = 1$ for every $F' = F \cup E_1$ and $F \in \mathcal{F}^\pi(G)$. If $\delta^*_e = 1$ for $e \in E_1$ then the cost would be at least $k + 1$ which contradicts the bound k on the leader's objective value. Hence, $X = \{e \in E \mid \delta^*_e = 1\}$ is a feasible solution of the blocking network problem with total cost at most k. ☐

4 Case study: The bilevel spanning tree problem

In this section, we discuss continuous-discrete bilevel approaches for a special combinatorial optimization problem, the minimum spanning tree problem, which is the network problem $(\pi_{ST}, \otimes)$ where $\mathcal{F}^{\pi_{ST}}(G)$ is equal to the set of spanning trees of G.

It is well-known that the corresponding network problems $(\pi_{ST}, \sum)$ and $(\pi_{ST}, \max)$, i.e., the classical minimum spanning tree problem and the bottleneck spanning tree problem, can be solved in polynomial time, e.g., in $\mathcal{O}(m + n \log n)$ time using Dijkstra's algorithm with Fibonacci-heaps (Fredman and Tarjan [6]).

In the following, we are interested in CDBP $(\pi_{ST}, \oplus, \otimes^\ell, \otimes^f)$ for $\oplus, \otimes^\ell, \otimes^f \in \{\sum, \max\}$.

We will start with the case that $\otimes^\ell = \max$ and make use of the results of Subsection 3.2. In order to apply Corollary 3.3, we have to investigate the partial anti-inverse spanning tree problem.

Orlin [14] showed that the partial inverse spanning tree problem $(\pi_{ST}, \max, \sum)$ is solvable in polynomial time if $J^+ = \emptyset$ or $J^- = \emptyset$. Thus, there exists a polynomial time algorithm for the partial anti-inverse spanning tree problem $(\pi_{ST}, \max, \sum)$. Hence, Corollary 3.3 immediately implies

Corollary 4.1. *The bilevel spanning tree problem* $(\pi_{ST}, \max, \max, \sum)$ *is solvable in polynomial time.*

In a next step, we are interested in the computational complexity of the partial anti-inverse spanning tree problem $(\pi_{ST}, \sum, \sum)$. Unfortunately, it turns out that this partial anti-inverse problem is strongly NP-hard:

Theorem 4.2. *The partial anti-inverse spanning tree problem* $(\pi_{ST}, \sum, \sum)$ *is strongly* NP*-hard.*

Proof. Consider the Steiner Tree Problem which is defined as follows: $G = (V, E)$ is a graph with cost coefficients $c_e \in \mathbb{R}_+$ (for $e \in E$) and $S \subseteq V$ is a subset of vertices (the socalled terminals). The task is to find a subset of edges $F \subseteq E$ such that all terminals lie in the same connected component in the graph G_F induced by F and $\sum_{e \in F} c_e$ is minimum. The Steiner Tree Problem is known to be strongly NP-hard (Garey and Johnson [7]). Observe that the Steiner Tree Problem remains strongly NP-hard even if there is no edge $(i, j) \in E$ with $i, j \in S$.

Let (G, c, S) be an instance of the Steiner Tree Problem such that there is no edge $(i, j) \in E$ with $i, j \in S$. We construct an instance (G', c', w', J^-) of the partial anti-inverse spanning tree problem $(\pi_{ST}, \sum, \sum)$ as follows: $G' = (V, E \cup E')$ with $E' = \{(i, j) \mid i, j \in S\}$, $c'_{(i,j)} = \sum_{e \in E} c_e$ for $(i, j) \in E'$ and $c'_e = c_e$ otherwise, $w'_e = 0$ if $e \in E'$ and $w'_e = 1$ otherwise and $J^- = E'$. We show that there exists a Steiner Tree with objective value at most k if and only if the partial anti-inverse spanning tree instance admits a feasible solution with cost at most k.

Assume that $F \subseteq E$ is a Steiner Tree with cost at most k. Then define $\delta_e = -1$ if $e \in F$ and $\delta_e = 0$ otherwise. Since F is a Steiner Tree there exists a path $P(i,j)$ in G_F between every pair of terminals $i, j \in S$, i.e., for every $(i,j) \in J^-$ there exists a path $P(i,j)$ in G' such that $w_e + \delta_e = 1 - 1 \leq 0 + 0 = w_{(i,j)} + \delta_{(i,j)}$ holds for all $e \in P(i,j)$. Hence, there exists a minimum spanning tree T with respect to $w+\delta$ with $T \cap J^- = \emptyset$. Moreover, the cost of solution δ is equal to $\sum_{e \in E} c_e |\delta_e| = \sum_{e \in F} c_e \leq k$.

Now assume that the partial anti-inverse spanning tree problem admits a feasible solution δ with cost at most k and let T be a minimum spanning tree with respect to $w + \delta$ with $T \cap J^- = \emptyset$. We may assume without loss of generality that $\delta_e \geq 0$ holds for all $e \in J^-$ and $\delta_e \leq 0$ for all $e \in T$. Assume that $\delta_e > 0$ holds for an edge $e \in J^-$. Then we can define a new solution $\tilde{\delta}$ with

$$\tilde{\delta}_e = \begin{cases} 0 & \text{if } e \in J^- \\ \delta_e - \sum_{(i,j) \in J^-} \delta_{(i,j)} & \text{if } e \in T \\ \delta_e & \text{otherwise} \end{cases}$$

Observe that T is a minimum spanning tree with respect to $w + \tilde{\delta}$ and

$$\begin{aligned} \sum_{e \in E} c'_e |\tilde{\delta}_e| &= \sum_{e \in E} c'_e |\delta_e| - \sum_{e \in J^-} c'_e \delta_e + \sum_{e \in T} c'_e \sum_{e \in J^-} \delta_e \\ &= \sum_{e \in E} c'_e |\delta_e| - \sum_{e \in J^-} \left(\sum_{e \in E} c_e \right) \delta_e + \sum_{e \in T} c_e \sum_{e \in J^-} \delta_e \leq \sum_{e \in E} c'_e |\delta_e| \leq k. \end{aligned}$$

Hence, there exists a feasible solution $(\tilde{\delta}, T)$ of the bilevel problem with cost at most k and $\tilde{\delta}_e = 0$ for all $e \in J^-$. Since T is a minimum spanning tree with respect to $w + \tilde{\delta}$ there exists a unique path $P(i,j)$ in T between every pair $i, j \in S$ with $1 + \tilde{\delta}_e \leq 0 = w_{(i,j)}$ for all $e \in P(i,j)$, i.e., all terminals lie in the same connected component in the graph G_F induced by $F = \{e \in E \mid \tilde{\delta}_e \leq -1\}$. Hence, F is a Steiner Tree with cost

$$\sum_{e \in F} c_e \leq \sum_{e \in F} c_e |\tilde{\delta}_e| \leq \sum_{e \in E'} c_e |\tilde{\delta}_e| \leq k.$$

□

Theorem 4.2 and Corollary 3.3 imply the following hardness-result:

Corollary 4.3. *The bilevel spanning tree problem* $(\pi_{ST}, \sum, \max, \sum)$ *is strongly* NP-*hard.*

Now we consider the bilevel spanning tree problem where the follower's composition is the maximum-operator and apply the results of Subsection 3.3.

Recall that CDBP is solvable in polynomial time if the cost composition $\oplus$ and the follower's composition $\otimes^f$ are maximum-operators, the leader's composition $\otimes^\ell$ is equal to the sum- or maximum-operator and the network problem $(\pi, \otimes^\ell)$ is solvable in polynomial time (cf. Corollary 3.7). Therefore, we immediately get the following result:

Corollary 4.4. *The bilevel spanning tree problem* $(\pi_{ST}, \max, \otimes^\ell, \max)$ *with* $\otimes^\ell \in \{\sum, \max\}$ *is solvable in polynomial time.*

In order to apply Corollary 3.6 we have to investigate the blocking spanning tree problem $(\pi_{ST}, \sum)$: The set of minimal (with respect to inclusion) feasible solutions of the blocking spanning tree problem $(\pi_{ST}, \sum)$ is equal to the set of cuts. Hence, the blocking spanning tree problem $(\pi_{ST}, \sum)$ is equal to the minimum weighted cut problem that can be solved in $\mathcal{O}(mn + n^2 \log n)$ (cf. Nagamochi and Ibaraki [13]). Corollary 3.6 implies

Corollary 4.5. *The bilevel spanning tree problem* $(\pi_{ST}, \sum, \otimes^\ell, \max)$ *with* $\otimes^\ell \in \{\max, \sum\}$ *is solvable in polynomial time.*

Finally, we are interested in the bilevel spanning tree problem where the leader's and follower's composition is equal to the sum-operator. Observe that every spanning tree has the same cardinality. Hence, we apply Theorem 3.2:

Corollary 4.6. *The bilevel spanning tree problem* $(\pi_{ST}, \oplus, \sum, \sum)$ *with* $\oplus \in \{\max, \sum\}$ *is in general strongly* NP*-hard.*

Proof. Orlin [14] proved strong NP-hardness of the partial inverse minimum spanning tree problem if the modification cost is measured by the sum-operation (L_1-norm) or by the maximum-operation (L_∞-norm). Hence, the partial inverse spanning tree problem $(\pi_{ST}, \oplus, \sum)$ with $\oplus \in \{\max, \sum\}$ is strongly NP-hard. Observe that every spanning tree has $(n-1)$ edges. Therefore, Theorem 3.2 implies strong NP-hardness of all considered bilevel spanning tree problems. □

Table 1 gives an overview about the complexity status of bilevel spanning tree problems.

The results for the bilevel spanning tree problem can also be generalized to bilevel minimum weight matroid basis problems, i.e., where the underlying combinatorial optimization problem is equal to the problem of finding a basis of a matroid of minimum weight.

Moreover, similar results concerning the computational complexity for other special network problems can be obtained by analyzing the associated partial anti-inverse and blocking problems. Table 2 and 3 contain several results for bilevel shortest path (graph property π_P), minimum cut (graph property π_C) and minimum assignment problems (graph property π_A) for $\otimes^\ell \in \{\max, \sum\}$:

leader's composition: $\sum$		follower's composition $\sum$	max
cost	$\sum$	NP-hard (Corollary 4.6)	P (Corollary 4.5)
composition	max	NP-hard (Corollary 4.6)	P (Corollary 4.4)
leader's composition: max		**follower's composition $\sum$**	**max**
cost	$\sum$	NP-hard (Corollary 4.3)	P (Corollary 4.5)
composition	max	P (Corollary 4.1)	P (Corollary 4.4)

Table 1: Computational complexity of bilevel spanning tree problems

		follower's composition $\sum$	max
cost	$\sum$	NP-hard	P
composition	max	NP-hard	P

Table 2: Computational complexity of bilevel shortest path and bilevel cut problems

		follower's composition $\sum$	max
cost	$\sum$	NP-hard	NP
composition	max	NP-hard	P

Table 3: Computational complexity of bilevel assignment problems

To obtain the results of Table 2 and 3 one makes use of the fact that the partial anti-inverse network problems $(\pi, \oplus, \sum)$ are strongly NP-hard for $\oplus \in \{\sum, \max\}$ and $\pi \in \{\pi_P, \pi_C, \pi_A\}$ (see [9] for the NP-hardness proofs). Moreover, the blocking $(s-t)$-path problem $(\pi, \sum)$ for nonnegative weights is equivalent to the minimum $(s-t)$-cut problem which can be solved in polynomial time. The blocking $(s-t)$-cut problem $(\pi, \sum)$ is also solvable in polynomial time because it is equivalent to the shortest $(s-t)$-path problem. The blocking assignment problem $(\pi_A, \sum)$ is the only blocking network problem in this paper that turns out to be strongly NP-hard: Zenklusen et al. [16] show that it is strongly NP-hard to find a smallest set of edges whose removal destroys all perfect matchings in a bipartite graph. Moreover, the graph property π_A is extendable: Let $G = (U, V, E)$ with $U = \{u_i \mid i = 1, \ldots, n\}$ and $V = \{v_i \mid i = 1, \ldots, n\}$. Then $G' = (U', V', E \cup E_1 \cup E_2)$ with $U' = U \cup \{u'_i \mid i = 1, \ldots, n\}$, $V' = V \cup \{v'_i \mid i = 1, \ldots, n\}$ and $E_1 = \{(u'_i, v'_i) \mid i = 1, \ldots, n\}$ and $E_2 = \{(u_i, v'_i), (v_i, u'_i) \mid i = 1, \ldots, n\}$ is an extension with respect to π_A.

5 Conclusion

This paper deals with continuous-discrete bilevel network problems (CDBP for short). We present several necessary and/or sufficient conditions for CDBP to be solvable in polynomial time or to be NP-hard. These general results can be applied to bilevel network problems that occur for arbitrary underlying network problems. We present several results for specific continuous-discrete bilevel network problems.

There are still a lot of challenging open questions. The purpose of this paper was to clarify the computational complexity of bilevel network problems. However, efficient algorithms for those cases that are shown to be solvable in polynomial time are still to be developed. Moreover, most NP-hardness results only hold for arbitrary cost coefficients $c_e \in \mathbb{R}_+$ (for $e \in E$). It would be interesting to investigate the computational complexity of bilevel, partial (anti-)inverse and blocking network problems for unit cost coefficients.

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