ALL SOLUTIONS TO THOMAS' FAMILY OF THUE EQUATIONS OVER IMAGINARY QUADRATIC NUMBER FIELDS

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Abstract. We completely solve the family of relative Thue equations

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \mu,$$

where the parameter t, the root of unity μ and the solutions x and y are integers in the same imaginary quadratic number field. This is achieved using the hypergeometric method for $|t| \geq 53$ and Baker's method combined with a computer search using continued fractions for the remaining values of t.

1. Introduction

Let F be an irreducible form of degree at least 3 with integral coefficients and m be a nonzero integer. Then the Diophantine equation

$$F(x, y) = m$$

is called a *Thue* equation in honour of Thue [20] who proved that it has only finitely many solutions over the integers. Algorithms for solving single Thue equations over $\mathbb{Z}$ have been developed, see Bilu and Hanrot [1].

Starting with Thomas [19] in 1990, several families of parametrized Thue equations (of positive discriminant) have been solved, cf. the surveys [8, 7].

In the last years, a few parametrized families of relative Thue equations where the parameter and the solutions are elements of an imaginary quadratic number field have been studied by Heuberger, Pethő, and Tichy [11], by Ziegler [23, 24], and by Jadrijević and Ziegler [13].

In this paper, we return to the parametrized family of Thue equations

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \mu, \qquad x, y \in \mathbb{Z}_{\mathbb{Q}(t)},\ t \text{ imaginary quadratic integer}, \quad \mu \text{ a root of unity in } \mathbb{Z}_{\mathbb{Q}(t)} \tag{1}$$

studied in [11]. This is the family that Thomas [19] and Mignotte [16] solved completely in the rational integer case. In [11], all solutions for $|t| > 3.023 \cdot 10^9$ have been found using Baker's method. Furthermore, all solutions for $\operatorname{Re} t = -1/2$ were claimed to be listed. However, as announced in [10], the proof of [11, Theorem 3] is incorrect (more precisely, the argument for excluding the possibility $\Lambda = 0$ in [11, Section 7] is invalid) and some solutions are missing in [11, Table 2].

Instead of performing a large computer search in the missing case in order to correct the result for $\operatorname{Re} t = -1/2$, we use this opportunity to solve (1) *completely* for *all values of* t. As far as we know, this is the first instance of a family of relative Thue equations to be solved completely.

This is achieved by combining the hypergeometric method due to Thue and Siegel (for values $|t| \geq 53$) and lower bounds for linear forms in logarithms ("Baker's method") together with a computer search (using continued fraction expansions) for $|t| < 53$.

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We now give an overview on the main results and the structure of the present paper.

The first step will be an effective measure of irrationality for the smallest root of $X^3-(t-1)X^2-(t+2)X-1$.

Theorem 1. *Let t be an imaginary quadratic integer of absolute value at least 48 and α be the unique root of*

$$f_t(X)=X^3-(t-1)X^2-(t+2)X-1$$

with absolute value at most $1/4$.

Then we have

$$\left|\alpha-\frac{p}{q}\right|>\frac{1}{746|t||q|^{\kappa+1}}\quad \textit{with}\quad \kappa=\frac{\log|t|+0.83+\log\left(1+\frac{2.66}{|t|}\right)}{\log|t|-1.3+\log\left(1-\frac{7.86}{|t|}\right)}$$

for all algebraic integers p and q in $\mathbb{Q}(t)$ with $|q|\geq 0.0773|t|$.

For $|t|\geq 48$, we have $\kappa<2$ and

$$\kappa\leq 1+\frac{2.13}{\log|t|}+\frac{6.8}{\log^2|t|}.$$

For rational integer parameters t, such a measure of irrationality has been provided by Lettl, Pethő, and Voutier [15].

The proof of this theorem is spread over Sections 2 and 3: Section 2 collects the auxiliary results known as the Hypergeometric Method in the form of [15] or adapts them to the case of an imaginary quadratic parameter. In Section 3, they are applied to our specific equation.

The remainder of the paper is devoted to the proof of our main result:

Theorem 2. *Let t be an integer in an imaginary quadratic number field, $t\notin\{(-1\pm 3\sqrt{-3})/2\}$, $\mathbb{Z}_{\mathbb{Q}(t)}$ be the ring of integers of $\mathbb{Q}(t)$,*

$$F_t(X,Y)=X^3-(t-1)X^2Y-(t+2)XY^2-Y^3\in\mathbb{Z}_{\mathbb{Q}(t)}[X,Y],$$

and μ be a root of unity in $\mathbb{Q}(t)$.

Then all solutions $(x,y)\in\mathbb{Z}^2_{\mathbb{Q}(t)}$ to

$$F_t(x,y)=\mu \tag{2}$$

are listed in Table 1 (solutions independent of t) and in the online Table [9] (solutions for specific values of t). A short summary of these 732 "sporadic" solutions is given in Table 2.

x	y	μ	x	y	μ	x	y	μ
0	1	-1	i	0	$-i$	$-1+\omega_3$	$1-\omega_3$	-1
-1	0	-1	0	i	i	ω_3	0	-1
1	-1	-1	$-i$	0	i	0	$1-\omega_3$	1
0	-1	1	i	$-i$	i	0	ω_3	1
-1	1	1	0	$-\omega_3$	-1	$-\omega_3$	0	1
1	0	1	0	$-1+\omega_3$	-1	$1-\omega_3$	$-1+\omega_3$	1
0	$-i$	$-i$	$-\omega_3$	ω_3	-1	$-1+\omega_3$	0	1
$-i$	i	$-i$	$1-\omega_3$	0	-1	ω_3	$-\omega_3$	1

Table 1. Solutions (if contained in $\mathbb{Q}(t)$) to (2) for all t, where $\omega_3=(1+\sqrt{-3})/2$.

Remark. If $t\in\{(-1\pm 3\sqrt{-3})/2\}$ then $F_t(X,Y)$ is the cube of a linear polynomial. Thus (2) has infinitely many solutions (x,y) for all roots of unity $\mu\in\mathbb{Q}(\sqrt{-3})$ in this case.

The solutions for $\operatorname{Re} t=-1/2$ are also listed in [10].

t	Number of solutions	$\max\{\lvert x\rvert^2, \lvert y\rvert^2\}$
-4	6	81
-2	6	9
-1	12	81
0	12	81
1	6	9
3	6	81
$-1 \pm 2i$	24	5
$-1 \pm 3i$	24	5
$\pm 2i$	24	5
$\pm 3i$	24	5
$-1 \pm \sqrt{-2}$	6	9
$-1 \pm 2\sqrt{-2}$	6	3
$\pm\sqrt{-2}$	6	9
$\pm 2\sqrt{-2}$	6	3
$-2 \pm 2\sqrt{-3}$	12	688
$(-3 \pm 3\sqrt{-3})/2$	24	7
$-1 \pm \sqrt{-3}$	24	3
$-1 \pm 2\sqrt{-3}$	6	1
$(-1 \pm \sqrt{-3})/2$	18	27
$\pm\sqrt{-3}$	24	3
$\pm 2\sqrt{-3}$	6	1
$(1 \pm 3\sqrt{-3})/2$	24	7
$1 \pm 2\sqrt{-3}$	12	688
$-2 \pm \sqrt{-5}$	6	86
$1 \pm \sqrt{-5}$	6	86
$-1 \pm \sqrt{-7}$	12	4
$(-1 \pm \sqrt{-7})/2$	6	7
$\pm\sqrt{-7}$	12	4
$(-3 \pm \sqrt{-11})/2$	6	20
$(1 \pm \sqrt{-11})/2$	6	20
$(-1 \pm \sqrt{-19})/2$	6	19
$(-1 \pm \sqrt{-31})/2$	6	98
$(-1 \pm \sqrt{-35})/2$	6	611

TABLE 2. Overview on sporadic solutions to (2) for specific t.

Section 4 is devoted to elementary results on the Thue equation (2). For instance, transformation properties are proved which reduce the number of cases drastically. A computer search for "small" solutions is also performed in this section. These results are used both for the hypergeometric (large $|t|$) and for Baker's method (small $|t|$).

Combining the irrationality measure with elementary upper bounds for $|\alpha - \frac{x}{y}|$ for solutions (x, y) to (1) yields upper bounds for $|y|$. The contradicting lower bounds for $|y|$ are derived in Section 5 using a gap principle (due to Ziegler [23]) based on concrete Padé approximations to a root of f_t. This replaces the continued fraction expansion arguments which are usually used in the case of rational integer parameters. This leads to the proof of Theorem 2 for $|t| \geq 53$.

Section 6 proves Theorem 2 for $|t| < 53$. This uses lower bounds for linear forms in two logarithms for deriving a (large) upper bound for $|y|$. Solutions up to this bound are searched using the continued fraction expansion of the quotient of the two logarithms of the linear form considered. For some values of t a brute force search for medium sized solutions (up to $|y| \leq 99$) is required. Note that we do not need Wildanger's [22] ellipsoid method which can be used for enumerating small solutions of general single relative Thue equations, cf. Gaál and Pohst [6].

We remark that the results contained in this paper open two possible directions for generalizations. It is possible to consider inequalities instead of equations. Other "simple" families of degrees 4 and 6 can be studied, as this has been done by Lettl, Pethő, and Voutier [15] in the rational integer case. All this is planned for future work.

Several parts of this paper need heavy symbolic manipulations. One instance is the gap principle in Lemma 5.2. Here, precise asymptotic expressions were handled in Mathematica® in a similar way as in [12]. The determination of the small solutions in Lemma 4.2 (which gives 720 of the sporadic solutions) has also been performed in Mathematica. Since the numerical verifications of Section 6 were not expected to require extreme code efficiency and thus a specialized program, they were also performed in Mathematica. Here it was very convenient that Mathematica keeps track of the precision of floating point numbers.

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2. The Hypergeometric Method

In this section, we collect several auxiliary results which the hypergeometric method relies on. Although we only use the case $n = 3$ in this paper, we state the results in general form where possible in order to facilitate future reference.

For positive integers n and r, we set

$$\mathcal{X}_{n,r}(X) = {}_2\mathcal{F}_1\left(-r, -r - \frac{1}{n}; 1 - \frac{1}{n}; X\right) \in \mathbb{Q}[X] \quad \text{and} \quad \mathcal{X}^*_{n,r}(X,Y) = Y^r \mathcal{X}_{n,r}\left(\frac{X}{Y}\right) \in \mathbb{Q}[X,Y].$$

Here, ${}_2\mathcal{F}_1$ denotes the classical hypergeometric function. Therefore, $\mathcal{X}_{n,r}$ is a polynomial of degree r and $\mathcal{X}^*_{n,r}$ is its homogenization.

The basis of the method is Thue's fundamental lemma.

Lemma 2.1 (Thue)**.** *Let K be a field of characteristic 0, $\mathbf{P} \in K[X]$ be a squarefree polynomial of degree $n \geq 2$ and assume that there is a squarefree quadratic polynomial $\mathbf{U} \in K[X]$ such that*

$$\mathbf{U}\mathbf{P}'' - (n-1)\mathbf{U}'\mathbf{P}' + \frac{n(n-1)}{2}\mathbf{U}''\mathbf{P} = 0 \tag{3}$$

holds, where the prime denotes differentiation with respect to the indeterminate X. We set $\lambda = \frac{1}{4}\operatorname{disc}(\mathbf{U})$, where $\operatorname{disc}(\mathbf{U}) = \mathbf{U}'^2 - 2\mathbf{U}\mathbf{U}'' \in K$ is the discriminant of $\mathbf{U}$. We define the polynomials contained in $K(\sqrt{\lambda})[X]$ by

$$\begin{aligned}
&\mathbf{Y} = 2\mathbf{U}\mathbf{P}' - n\mathbf{U}'\mathbf{P}, \\
&\mathbf{a} = \frac{n^2-1}{6}(\sqrt{\lambda}\mathbf{U}' + 2\lambda), && \mathbf{c} = \frac{n^2-1}{6}(\sqrt{\lambda}(\mathbf{U}'X - 2\mathbf{U}) + 2\lambda X), \\
&\mathbf{b} = \frac{n^2-1}{6}(\sqrt{\lambda}\mathbf{U}' - 2\lambda), && \mathbf{d} = \frac{n^2-1}{6}(\sqrt{\lambda}(\mathbf{U}'X - 2\mathbf{U}) - 2\lambda X), \\
&\mathbf{u} = \frac{1}{2}\left(\frac{\mathbf{Y}}{2n\sqrt{\lambda}} - \mathbf{P}\right), && \mathbf{z} = \frac{1}{2}\left(\frac{\mathbf{Y}}{2n\sqrt{\lambda}} + \mathbf{P}\right).
\end{aligned}$$

Finally, we set $\mathbf{w} = \mathbf{z}/\mathbf{u} \in K(\sqrt{\lambda})(X)$.

Let r be a positive integer. Then the polynomials $\mathbf{A}_r$, $\mathbf{B}_r$ given by

$$\begin{aligned}
(\sqrt{\lambda})^r\mathbf{A}_r &= \mathbf{a}\mathcal{X}^*_{n,r}(\mathbf{z},\mathbf{u}) - \mathbf{b}\mathcal{X}^*_{n,r}(\mathbf{u},\mathbf{z}), \\
(\sqrt{\lambda})^r\mathbf{B}_r &= \mathbf{c}\mathcal{X}^*_{n,r}(\mathbf{z},\mathbf{u}) - \mathbf{d}\mathcal{X}^*_{n,r}(\mathbf{u},\mathbf{z})
\end{aligned} \tag{4}$$

are elements of the polynomial ring $K[X]$ over K. For every root α of $\mathbf{P}$, the polynomial

$$\mathbf{C}_r = \alpha\mathbf{A}_r - \mathbf{B}_r$$

is divisible by $(X-\alpha)^{2r+1}$.

This lemma has been proved (in slightly different notation) by Thue [21]. Cf. also Chudnovsky [4] (see Lemma 7.1 and the remarks that follow, pages 364–366) and Chen and Voutier [3, Lemma 2.1]. Following Lettl, Pethő, and Voutier [15, Proposition 1], we only quoted those parts which are necessary in our application and emphasized all polynomials by using boldface letters. Note that Thue's assumption that $\mathbf{P}$ is squarefree does not appear in all those references; on the other hand, the assumptions $n \geq 2$ and $\mathbf{U}$ squarefree (and thus $\lambda \neq 0$) are necessary for our formulation. Thue's proof holds over all ground fields K of characteristic 0.

We will use Lemma 2.1 to construct a sequence of good rational approximations $\mathbf{B}_r/\mathbf{A}_r$ to α. We have to be sure that we do not generate the same approximation for two consecutive values of r:

Lemma 2.2 ([3, Lemma 2.7])**.** *Let $\mathbf{A}_r$, $\mathbf{B}_r$, $\mathbf{P}$ and $\mathbf{U}$ be defined as in Lemma 2.1. If $\mathbf{U}(\xi)\mathbf{P}(\xi) \neq 0$ for a given $\xi \in \mathbb{C}$, then for all positive integers r, we have*

$$\mathbf{A}_{r+1}(\xi)\mathbf{B}_r(\xi) \neq \mathbf{A}_r(\xi)\mathbf{B}_{r+1}(\xi).$$

This is the special case $a = d = 1$ and $b = c = 0$ of Lemma 2.7 in [3].

Estimating the quality of the rational approximations requires estimates for $\mathbf{C}_r$, for instance. One of the tools is the following analytic representation of $\mathbf{C}_r$.

Lemma 2.3 ([3, Lemma 2.3])**.** *We use the notation of Lemma 2.1. For any nonzero complex ξ such that $\mathbf{w}(\xi)$ is not a negative real number or zero, we have*

$$(\sqrt{\lambda})^r \mathbf{C}_r(\xi) = (\alpha(\mathbf{a}(\xi)\mathbf{w}(\xi)^{1/n} - \mathbf{b}(\xi)) - (\mathbf{c}(\xi)\mathbf{w}(\xi)^{1/n} - \mathbf{d}(\xi)))\mathcal{X}^*_{n,r}(\mathbf{u}(\xi), \mathbf{z}(\xi)) - (\alpha\mathbf{a}(\xi) - \mathbf{c}(\xi))\mathbf{u}(\xi)^r R_{n,r}(\mathbf{w}(\xi)), \tag{5}$$

with

$$R_{n,r}(w) := \frac{\Gamma(r+1+1/n)}{r!\Gamma(1/n)} \int_1^w (1-\zeta)^r(\zeta - w)^r \zeta^{-(r+1-1/n)}\, d\zeta, \tag{6}$$

where the path of integration is the straight line from 1 to w.

The proof of [3] does not depend on any algebraic information and is therefore valid in our situation.

The estimates for $R_{n,r}$ and for $\mathcal{X}_{n,r}$ contained in [3, Lemma 2.5 and Lemma 2.6] are not applicable in our case since $\mathbf{w}(\xi)$ cannot be guaranteed to be of absolute value 1. However, w will be a number close to 1. Therefore, we will have to produce our own estimates for such w.

Lemma 2.4. *Let w be a complex number with $|1 - w| < 1$, n, r be positive integers and $R_{n,r}$ be defined by* (6)*. Then*

$$|R_{n,r}(w)| \leq \frac{\Gamma(r+1+1/n)}{r!4^r\Gamma(1/n)} \cdot \frac{|w-1|^{2r+1}}{(1-|w-1|)^{r+1-1/n}}.$$

Proof. Setting $\zeta = (1-\lambda) + \lambda w$ yields

$$R_{n,r}(w) = \frac{\Gamma(r+1+1/n)}{r!\Gamma(1/n)}(w-1)^{2r+1} \int_0^1 \frac{\lambda^r(1-\lambda)^r}{(1+\lambda(w-1))^{r+1-1/n}}\, d\lambda.$$

Using the estimates $\lambda(1-\lambda) \leq 1/4$ and $|1 + \lambda(w-1)| \geq 1 - |w-1|$ yields the result. □

We now estimate $\mathcal{X}_{n,r}(w)$ in a neighbourhood of 1:

Lemma 2.5. *Let w be a complex number with $|1 - w| < 1$. Then we have*

$$|\mathcal{X}_{n,r}(w)| \leq \frac{\Gamma(1-1/n)r!}{\Gamma(r+1-1/n)} \left(4^r \left(\frac{1+|w|}{2}\right)^{r+1/n} + \frac{4e^{2/n}}{\pi} \cdot \frac{1}{(2\sqrt{3})^{r+1}} |1-w|^{2r+1}|w|^{1/n} \right).$$

Proof. By residue calculus, we obtain (cf. [3, Lemma 2.6])

$$\mathcal{X}_{n,r}(w) = (-1)^r \frac{\Gamma(1-1/n)r!}{2\pi i\Gamma(r+1-1/n)} \oint_{\mathcal{C}} \zeta^{-r-1}(1-\zeta)^{r-1/n}(1-w\zeta)^{r+1/n}\, d\zeta,$$

where $\mathcal{C}$ is a circle around the origin with radius smaller than $\max\{1, 1/|w|\}$ and the branches $(1-\zeta)^{r-1/n}$ and $(1-w\zeta)^{r+1/n}$ are chosen such that for $\zeta = 0$, both functions equal 1, and they are analytically continued in the appropriate way.

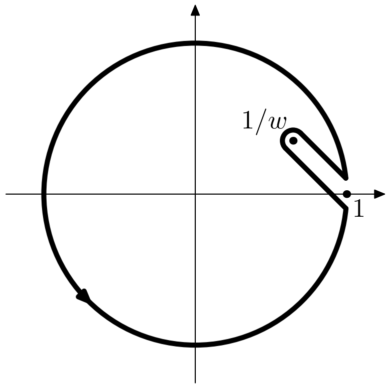

FIGURE 1. Integration path $\mathcal{C}'$ for Lemma 2.5.

It is more convenient to estimate the integral on a circle of radius 1. Thus we extend the circle $\mathcal{C}$ to a cycle $\mathcal{C}'$, which is a circle of radius 1 except that if $|w| > 1$, it excludes the point $1/w$, cf. Figure 1.

On the circle with radius 1, the integrand can be estimated by $2^{r-1/n}(1+|w|)^{r+1/n}$. Each of the two line segments has length $|1-1/w|$. The angle between the line connecting 1 and $1/w$ and the line connecting 0 and $1/w$ is at least $\pi/3$ since $|1/w| < 1$ and $|1-w| < 1$. Thus $|\zeta| \geq \sqrt{3}/2 \cdot |1/w|$ on the line segments. To estimate the remaining factors of the integrand, we note that the inequality between the weighted geometric and arithmetic means implies

$$|1-\zeta|^{r-1/n}|\zeta - 1/w|^{r+1/n} \leq \left(\frac{r-1/n}{2r}|1-\zeta| + \frac{r+1/n}{2r}|\zeta - 1/w|\right)^{2r} \leq \frac{|1-1/w|^{2r}}{2^{2r}}\left(1+\frac{1}{nr}\right)^{2r} \leq \frac{|1-1/w|^{2r}}{2^{2r}}e^{2/n}.$$

Collecting all these estimates yields the assertion of the lemma. □

Inserting a rational integer ξ close to α in the polynomials $\mathbf{A}_r$, $\mathbf{B}_r \in K[X]$ constructed in Lemma 2.1 gives us $\mathbf{A}_r(\xi)$, $\mathbf{B}_r(\xi) \in K$ such that $|\alpha\mathbf{A}_r(\xi) - \mathbf{B}_r(\xi)|$ is small. In the sequel, we need integers instead; therefore, we will clear the denominators of $\mathbf{A}_r$ and $\mathbf{B}_r$ and estimate the effect of this operation by the following lemma.

Lemma 2.6 ([15, Proposition 2])**.** *Let r be a positive integer, $\Delta_{3,r}$ be the least common multiple of the denominators of the coefficients of $\mathcal{X}_{3,r}$, and $N_{3,r}$ be the greatest common divisor of the numerators of the coefficients of $\mathcal{X}_{3,r}(1-3\sqrt{3}X)$. Then $\Delta_{3,r}/N_{3,r}\mathcal{X}_{3,r}(1-3\sqrt{3}X)$ is a polynomial with (algebraic) integer coefficients and we have*

$$\frac{4^r\Delta_{3,r}}{N_{3,r}} \cdot \frac{\Gamma(2/3)r!}{\Gamma(r+2/3)} < 1.9e^{0.83r} \qquad \text{and} \qquad \left(\frac{27}{4}\right)^r \cdot \frac{\Delta_{3,r}}{N_{3,r}} \cdot \frac{\Gamma(r+4/3)}{\Gamma(1/3)r!} < 0.87e^{1.3r}.$$

Here, we exactly use the result of [15].

Finally, the second key lemma is that if we have a sequence of good approximations of α, we obtain an effective irrationality measure:

Lemma 2.7 ([3, Lemma 2.8])**.** *Let t be an integer in an imaginary quadratic number field and let $\alpha \in \mathbb{C}$. Suppose that there exist real numbers k_0, $\ell_0 > 0$ and E, $Q > 1$ such that for all positive integers r there are integers p_r, $q_r \in \mathbb{Z}_{\mathbb{Q}(t)}$ with $|q_r| < k_0 Q^r$ and $|q_r\alpha - p_r| \leq \ell_0 E^{-r}$ satisfying $p_r q_{r+1} \neq p_{r+1} q_r$ for all r. Then for any integers p and q in $\mathbb{Z}_{\mathbb{Q}(t)}$ with $|q| \geq 1/(2\ell_0)$, we have*

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{c|q|^{\kappa+1}}, \text{ where } c = 2k_0 Q(2\ell_0 E)^\kappa \text{ and } \kappa = \frac{\log Q}{\log E}.$$

In [3], this lemma has been proved for rational integers t. However, the only point where this assumption is used is in the conclusion that any nonzero integer p has absolute value at least 1. This is also true in imaginary quadratic number fields, hence their proof can be copied letter by letter.

3. An effective measure of irrationality (Proof of Theorem 1)

We now concentrate on the polynomial $f_t(X) = F_t(X,1) = X^3 - (t-1)X^2 - (t+2)X - 1$ for an imaginary quadratic integer t and prove the effective measure of irrationality stated in Theorem 1.

For $|t| \geq 9$, Rouché's theorem with the comparison function $(t+2)X + 1$ shows that $f_t(X)$ indeed has exactly one root α with absolute value at most $1/4$.

We set $n = 3$, choose $\mathbf{P} = f_t$ and $\mathbf{U}(X) = X^2 + X + 1$ and check that (3) holds. As an approximation ξ to a root of $\mathbf{P}(X)$ we choose $\xi = 0$. We calculate the parameters of Lemma 2.1:

$$\begin{aligned} \lambda &= -\frac{3}{4}, & \mathbf{Y}(0) &= -1-2t, \\ \mathbf{a}(0) &= \frac{2}{3}(\sqrt{-3}-3), & \mathbf{c}(0) &= -\frac{4}{3}\sqrt{-3}, \\ \mathbf{b}(0) &= \frac{2}{3}(\sqrt{-3}+3), & \mathbf{d}(0) &= -\frac{4}{3}\sqrt{-3}, \\ u := \mathbf{u}(0) &= \frac{(2t+1)\sqrt{-3}+9}{18}, & z := \mathbf{z}(0) &= \frac{(2t+1)\sqrt{-3}-9}{18}. \end{aligned}$$

For positive integers r, we set

$$M_r = \left(\frac{9}{2}\right)^r \cdot \frac{3}{4} \cdot \frac{\Delta_{3,r}}{N_{3,r}}, \qquad p_r = M_r \mathbf{B}_r(0), \qquad q_r = M_r \mathbf{A}_r(0),$$

where $\Delta_{3,r}$ and $N_{3,r}$ have been defined in Lemma 2.6 and $\mathbf{A}_r$ and $\mathbf{B}_r$ have been defined in Lemma 2.1.

Lemma 3.1. *p_r and q_r are elements of $\mathbb{Z}_{\mathbb{Q}(t)}$.*

Proof. Since $M_r \in \mathbb{Q}$ and $\mathbf{A}_r(0)$ and $\mathbf{B}_r(0)$ are elements of $\mathbb{Q}(t)$ by Lemma 2.1, we immediately see that p_r and q_r are also elements of the field $\mathbb{Q}(t)$.

We have

$$\begin{aligned} q_r = \sqrt{-3}(-1)^r \Bigg(& \zeta_6 \left(3\sqrt{-3}u\right)^r \frac{\Delta_{3,r}}{N_{3,r}} \mathcal{X}_{3,r} \left(1 - 3\sqrt{3}i \frac{1}{3\sqrt{-3}u}\right) \\ & - \overline{\zeta}_6 \left(3\sqrt{-3}z\right)^r \frac{\Delta_{3,r}}{N_{3,r}} \mathcal{X}_{3,r} \left(1 - 3\sqrt{3}(-i) \frac{1}{3\sqrt{-3}z}\right)\Bigg), \end{aligned}$$

where $\zeta_6 = (1+\sqrt{-3})/2$, $\overline{\zeta}_6 = (1-\sqrt{-3})/2$ is its complex conjugate and where the relations $z/u = 1 - (u-z)/u = 1 + \mathbf{P}(0)/u = 1 - 1/u$ and $u/z = 1 + 1/z$ have been used.

We note that the numbers

$$3\sqrt{-3}u = 3\zeta_6 - t - 2, \qquad 3\sqrt{-3}z = -3\zeta_6 - t + 1,$$

are algebraic integers.

From Lemma 2.6, we conclude that q_r is an algebraic integer, hence it belongs to $\mathbb{Z}_{\mathbb{Q}(t)}$. The computation for p_r runs along the same lines. □

We estimate q_r using

$$|u|, |z| \leq \frac{|t|}{3\sqrt{3}} + 0.51, \qquad |w-1| = \frac{1}{|u|} \leq \frac{3\sqrt{3}}{|t|} + \frac{14.6}{|t|^2}, \qquad \left|\frac{1}{w} - 1\right| = \frac{1}{|z|} \leq \frac{3\sqrt{3}}{|t|} + \frac{14.6}{|t|^2}. \tag{7}$$

Equation (4), Lemma 2.5 and Lemma 2.6 yield

$$|q_r| \leq k_0 Q^r \text{ for } Q = e^{0.83}|t| + 6.08 \text{ and } k_0 = 6.59 + \frac{6.19}{|t|}. \tag{8}$$

Before estimating $q_r\alpha - p_r$, we verify that

$$\rho := \frac{\mathbf{c}(0)\mathbf{w}(0)^{1/3} - \mathbf{d}(0)}{\mathbf{a}(0)\mathbf{w}(0)^{1/3} - \mathbf{b}(0)}$$

is a root of $\mathbf{P}(X)$ satisfying

$$|\rho| \leq \frac{1}{|t|} + \frac{1.03}{|t|^2}. \tag{9}$$

Thus we have $\rho = \alpha$ which means that the coefficient of $\mathcal{X}^*_{n,r}(\mathbf{u}(0), \mathbf{z}(0))$ in (5) vanishes.

Using (7), (9), Lemma 2.4 and Lemma 2.6 yields

$$|q_r\alpha - p_r| \leq \ell_0 E^{-r} \text{ for } E = e^{-1.3t}|t| - 2.15, \text{ and } \ell_0 = \frac{7.83}{|t|} + \frac{65.16}{|t|^2}. \tag{10}$$

Combining (8), (10), Lemma 3.1 and Lemma 2.2 shows that we can apply Lemma 2.7 and obtain the assertions of Theorem 1.

4. Transformation Properties of the Thue Equation

In this section, we study elementary properties of solutions to (2) which will be used in both the hypergeometric and the linear form method. Among them, we will prove transformation properties of $F_t(X, Y)$ which help to reduce the number of cases that must be considered.

We recall that $f_t(X) := F_t(X, 1)$ and define the maps

$$\begin{aligned} \Phi &: (X, Y) \mapsto (-(X+Y), X) \\ \phi &: \qquad X \mapsto -1 - 1/X. \end{aligned}$$

Then we note the following identities.

Lemma 4.1. *(1) $F_t \circ \Phi = F_t$,*
(2) $f_t(\phi(X)) = X^{-3} f_t(X)$,
(3) $\Phi \circ \Phi \circ \Phi = \mathrm{id}$,
(4) $\phi \circ \phi \circ \phi = \mathrm{id}$,
(5) Φ acts on the solutions to $F_t(x, y) = \mu$,
(6) ϕ acts on the roots of f_t,
(7) Tables 1 and the online Table [9] are invariant under Φ,
(8) $F_{-1-t}(-Y, -X) = F_t(X, Y)$,
(9) Online Table [9] is invariant under the map $(t, x, y) \mapsto (-1-t, -y, -x)$.

Proof. The assertions can be verified by straightforward computations. □

We first deal with very small solutions to the Thue equation.

Lemma 4.2. *All solutions $(x, y) \in \mathbb{Z}_{\mathbb{Q}(t)}$ to (2) with $\min\{|y|, |x+y|, |x|\} < 3$ are listed in Table 1 or in online Table [9].*

Proof. In view of Lemma 4.1, we only have to prove this assertion for $|y| < 3$. This has been done for $|t| > 4$ and $|y| < \sqrt{5}$ in [11, Lemma 5].

To extend this to $\sqrt{5} \leq |y| < 3$ or $|t| \leq 4$, we follow the lines of that proof, perform the necessary computations (less than 1 minute CPU time on a Pentium IV with 2 GHz using Mathematica®) and collect the arising special cases in online Table [9]. □

In the case $\operatorname{Re} t = -1/2$, we can use the identities of Lemma 4.1 to derive a certain amount of information on the roots of f_t.

Lemma 4.3. *Let* $\operatorname{Re} t = -1/2$. *Then the roots* $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$ *of* f_t *can be numbered in such a way that* $\alpha^{(k+1)} = \phi(\alpha^{(k)})$, $k \in \{1,2\}$, *and*

$$\begin{aligned} \operatorname{Re}\alpha^{(1)} &= -1/2,\\ |\alpha^{(2)}| &= 1,\\ \alpha^{(1)}\overline{\alpha^{(3)}} &= 1,\\ \operatorname{Im}\alpha^{(2)} &= \operatorname{Im}\alpha^{(3)},\\ \operatorname{Re}\alpha^{(2)} + \operatorname{Re}\alpha^{(3)} &= -1. \end{aligned}$$

Proof. We have $(-1-t) = \bar{t}$. Thus every root α of f_t satisfies

$$0 = F_t(\alpha, 1) = F_{-1-t}(-1, -\alpha) = -\alpha^3 F_{\bar{t}}\left(\frac{1}{\alpha}, 1\right) = -\alpha^3 \overline{F_t\left(\frac{1}{\overline{\alpha}}, 1\right)},$$

which implies that $1/\overline{\alpha}$ is also a root of f_t. It follows that there must be at least one root of f_t, say $\alpha^{(2)}$, satisfying $\alpha^{(2)}\overline{\alpha^{(2)}} = 1$. We set $\alpha^{(3)} = \phi(\alpha^{(2)})$ and $\alpha^{(1)} = \phi(\alpha^{(3)})$. From $\alpha^{(2)} = \phi(\alpha^{(1)})$ we conclude that

$$1 = \alpha^{(2)}\overline{\alpha^{(2)}} = \left(-1 - \frac{1}{\alpha^{(1)}}\right)\left(-1 - \frac{1}{\overline{\alpha^{(1)}}}\right),$$

which yields $\operatorname{Re}\alpha^{(1)} = -1/2$. Furthermore, we have

$$\alpha^{(3)} = -1 - \frac{1}{\alpha^{(2)}} = -1 - \overline{\alpha^{(2)}}.$$

In particular, this means $\operatorname{Im}\alpha^{(2)} = \operatorname{Im}\alpha^{(3)}$ and $\operatorname{Re}\alpha^{(2)} + \operatorname{Re}\alpha^{(3)} = -1$. This implies that $\alpha^{(2)}$ and $\alpha^{(3)}$ lie symmetrically with respect to the line $\operatorname{Re} z = -1/2$. Finally, we have $\overline{\alpha^{(3)}} = -1 - \alpha^{(2)}$ and $\alpha^{(1)} = -1/(1+\alpha^{(2)})$, which implies that $\alpha^{(1)}\overline{\alpha^{(3)}} = 1$. $\square$

We also need asymptotic estimates for the roots.

Lemma 4.4. *Let* $|t| \geq 6$. *Then the roots of* f_t *can be numbered such that* $\alpha^{(k+1)} = \phi(\alpha^{(k)})$, $k \in \{1,2\}$, *and*

$$\begin{aligned} \left|\alpha^{(1)} - \left(t + \frac{2}{t} - \frac{1}{t^2} - \frac{3}{t^3} + \frac{5}{t^4}\right)\right| &\leq \frac{159}{20|t|^4},\\ \left|\alpha^{(2)} - \left(-1 - \frac{1}{t} + \frac{2}{t^3} - \frac{1}{t^4}\right)\right| &\leq \frac{541}{100|t|^4},\\ \left|\alpha^{(3)} - \left(-\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} - \frac{4}{t^4}\right)\right| &\leq \frac{193}{100|t|^4}. \end{aligned}$$

If $\operatorname{Re} t = -1/2$, *this numbering coincides with that of Lemma 4.3.*

Proof. Set $B(1/t) = t + 2t^{-1} - t^{-2} - 3t^{-3} + 5t^{-4}$, $h(z) := f_t(B(1/t) + z)$, and $h_1(z) = h(z) - h(0)$. We check that for $|z| = r$ with $r = (159/20)|t|^{-4}$, we have $|h(0)| = |h_1(z) - h(z)| < |h_1(z)|$. Thus, by Rouché's theorem, the number of zeros of $h(z)$ for $|z| < r$ equals the number of zeros of $h_1(z)$ in the same region. This number is at least 1, because we obviously have $h_1(0) = 0$. Thus we fix such a root of f_t and call it $\alpha^{(1)}$.

Applying ϕ twice, we get the other two estimates. $\square$

Similar estimates have been proved in [11, Lemma 4].

From now on, we assume that the roots $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$ of $f_t(X)$ are numbered such that $\alpha^{(k+1)} = \phi(\alpha^{(k)})$ for $k \in \{1,2\}$, i.e.,

$$\alpha^{(2)} = -1 - \frac{1}{\alpha^{(1)}}, \qquad \alpha^{(3)} = -\frac{1}{\alpha^{(1)} + 1}, \tag{11}$$

and additionally according to Lemma 4.3 or Lemma 4.4, if those are applicable, i.e., if $\operatorname{Re} t = -1/2$ or $|t| \geq 6$.

Obviously, we have

$$F_t(X,Y)=\prod_{k=1}^{3}(X-\alpha^{(k)}Y). \tag{12}$$

Let $(x,y)\in\mathbb{Z}^2_{\mathbb{Q}(t)}$ be a solution to (2). For $k\in\{1,2,3\}$, we set $\beta^{(k)}:=(x-\alpha^{(k)}y)$. We say that (x,y) is a solution of type $j\in\{1,2,3\}$, if

$$|\beta^{(j)}|=\min_{1\le k\le 3}|\beta^{(k)}|.$$

For $k\neq j$ we have

$$|y|\cdot|\alpha^{(k)}-\alpha^{(j)}|\le|\beta^{(k)}|+|\beta^{(j)}|\le 2|\beta^{(k)}|,$$

which implies that

$$|\beta^{(j)}|=\frac{1}{\prod_{k\neq j}|\beta^{(k)}|}\le\frac{4}{|y|^2\prod_{k\neq j}|\alpha^{(j)}-\alpha^{(k)}|}=\frac{4}{|y|^2|f_t'(\alpha^{(j)})|}. \tag{13}$$

Lemma 4.5. *Let $(x,y)\in\mathbb{Z}^2_{\mathbb{Q}(t)}$ be a solution of type j to (2) with $|y|\ge 3$. Then $\Phi(x,y)$ is a solution of type j' to (2) with $j'\equiv j+1 \pmod 3$.*

Proof. Let $r_j=\frac{4}{3^3|f_t'(\alpha^{(j)})|}$. Then (13) implies that

$$\left|\frac{x}{y}-\alpha^{(j)}\right|\le r_j.$$

We check that $|r_j|<|\alpha^{(j)}|$ (using Lemma 4.4 for $|t|\ge 6$ or a direct calculation for $|t|<6$). Then the image of the circle with center $\alpha^{(j)}$ and radius r_j under the Möbius transform ϕ is again a circle C. For $|t|<6$, we directly check that this circle and the union of the circles with center $\alpha^{(k)}$ and radius r_k for $k\equiv j,j-1 \pmod 3$ are disjoint.

For $|t|\ge 6$, we do the same using the expressions from Lemma 4.4 and the circle C' with center $\alpha^{(j')}$ and radius $r_j/(|\alpha^{(j)}|^2-|\alpha^{(j)}|r_j)$ instead of C. This is sufficient since $C\subseteq C'$. □

Note that for $|t|\ge 20$, the assertion has already been proved in [11, Lemma 10].

We may summarize the findings of this section as follows:

- It is sufficient to find solutions with $\min\{|x|,|y|,|x+y|\}\ge 3$. (Lemma 4.2).
- It is sufficient to consider t with $\operatorname{Re} t\ge -1/2$. (Lemma 4.1).
- It is sufficient to consider t with $\operatorname{Im} t>0$ (Complex conjugation and the fact that the real case has already been solved by Thomas [19] and Mignotte [16]).
- For every t, it is sufficient to look for solutions of one type j.

5. Proof of Theorem 2 for large t

In this section, we consider solutions of type 3 for $|t|\ge 53$ with $xy\neq 0$. We first prove lower bounds for y. Then we find a contradiction to the irrationality measure in Theorem 1.

For this purpose, we need a very precise estimate of $\alpha^{(3)}$:

Lemma 5.1. *Let $|t|\ge 53$. Define $B(1/t)$ as shown in Table 3. Then*

$$|\alpha^{(3)}-B(1/t)|\le 51387359556548586062194688143796511703.04\cdot\frac{1}{|t|^{100}}.$$

Proof. Analogous to Lemma 4.4. □

Inserting this (or a weaker) estimate for $\alpha^{(3)}$ in (13) yields the bound

$$|x-\alpha^{(3)}y|\le\frac{4.01}{|t|\cdot|y|^2}. \tag{14}$$

We now use a gap principle motivated by ideas of Ziegler [23]:

$$\begin{aligned}
B(1/t) = &-t^{-1} + t^{-2} + t^{-3} - 4t^{-4} + 17t^{-6} - 16t^{-7} - 68t^{-8} + 146t^{-9} + 221t^{-10} - 1003t^{-11} - 273t^{-12} + 5939t^{-13} - 4106t^{-14} \\
&- 30707t^{-15} + 53687t^{-16} + 131578t^{-17} - 449767t^{-18} - 365365t^{-19} + 3102531t^{-20} - 747043t^{-21} - 18490366t^{-22} + 22545272t^{-23} \\
&+ 93871236t^{-24} - 232750387t^{-25} - 366783648t^{-26} + 1831430044t^{-27} + 553107227t^{-28} - 12227384755t^{-29} + 8360097237t^{-30} \\
&+ 70376016509t^{-31} - 122421114260t^{-32} - 334950561638t^{-33} + 1119069945497t^{-34} + 1080046128147t^{-35} - 8329158252177t^{-36} \\
&+ 1105044611909t^{-37} + 53275465086270t^{-38} - 59337912806662t^{-39} - 290843808206224t^{-40} + 674000307789011t^{-41} \\
&+ 1250796152304132t^{-42} - 5660515728916384t^{-43} - 2693819426015571t^{-44} + 39991821417302559t^{-45} - 21428344452354721t^{-46} \\
&- 243529333325012000t^{-47} + 380947435597026427t^{-48} + 1240741800080374625t^{-49} - 3747242449142200944t^{-50} \\
&- 4563352468999555078t^{-51} + 29409538432556248817t^{-52} + 1475509292965010539t^{-53} - 197445650597602094289t^{-54} \\
&+ 184335486824133183968t^{-55} + 1136845252802719037358t^{-56} - 2350255865887809649771t^{-57} - 5289222240740725862365t^{-58} \\
&+ 20935387440978698590499t^{-59} + 14811215682522276167814t^{-60} - 154901679465719078131456t^{-61} \\
&+ 53240613187491143890133t^{-62} + 987189533409762322299664t^{-63} - 1330349465515614374931604t^{-64} \\
&- 5323107819731629993156598t^{-65} + 14193414957566188396175161t^{-66} + 21823085777661564707596931t^{-67} \\
&- 117078371813275261131394113t^{-68} - 31534314874155685846975099t^{-69} + 820227434658729444572777582t^{-70} \\
&- 596599700203890683086638862t^{-71} - 4945546779999781803498158756t^{-72} + 8908381074139735647794765915t^{-73} \\
&+ 24608747011035812888024367920t^{-74} - 84490452633231465005329664604t^{-75} - 82940280269070852659131077343t^{-76} \\
&+ 653351565330431854943165524363t^{-77} - 84074889959597667599424329177t^{-78} - 4339726070414328571607429024602t^{-79} \\
&+ 4856670004168898880028778593555t^{-80} + 24602992608326918458537314000031t^{-81} - 57229058383679980895278648633348t^{-82} \\
&- 110257792051864925259267151235014t^{-83} + 497161472272266025422918934264323t^{-84} \\
&+ 256690198726876186452861054516233t^{-85} - 3628472501163804383745830422086839t^{-86} \\
&+ 1844672227398687956652697315541176t^{-87} + 22822968487971727859190213653191982t^{-88} \\
&- 35105954268010446646386193041307353t^{-89} - 120410748397105340371323345245135991t^{-90} \\
&+ 357810231901105490450393113172907881t^{-91} + 465009372845331408407375001252961914t^{-92} \\
&- 2894517278313339440318990407506909496t^{-93} - 300667075265930836764696467457051625t^{-94} \\
&+ 20002866122431575104808322798458168713t^{-95} - 17748919264893387400845600433650214749t^{-96} \\
&- 118686846824009920970473048714039532863t^{-97} + 238418246427009426528297626827357533696t^{-98} \\
&+ 572887677557407929811670502226635548670t^{-99} - 2192555813110355101344909628196350021615t^{-100}.
\end{aligned}$$

TABLE 3. Approximation $B(1/t)$ to the root $\alpha^{(3)}$.

Lemma 5.2. *Let $(x, y) \in \mathbb{Z}_{\mathbb{Q}(t)}$ be a solution to $F_t(x, y) = \mu$ of type $j = 3$ with $xy \neq 0$ and $|t| \geq 53$. Then we have*

$$|y| \geq \frac{1}{67 \cdot 10^{18}} |t|^{42}.$$

Proof. We consider sequences k_m and C_m ($m = 1, \ldots, 25$) defined as follows:

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
k_m	0	1	1	2	2	3	3	4	4	5	6	6	7	8	10
C_m	1	$\frac{25}{126}$	$\frac{100}{103}$	$\frac{20}{43}$	$\frac{100}{207}$	$\frac{100}{731}$	$\frac{100}{729}$	$\frac{25}{158}$	$\frac{100}{631}$	$\frac{1}{152}$	$\frac{10}{557}$	$\frac{5}{278}$	$\frac{1}{696}$	$\frac{1}{702}$	$\frac{1}{5340}$

m	16	17	18	19	20	21	22	23	24	25
k_m	12	14	16	18	20	23	27	30	36	42
C_m	$\frac{1}{60700}$	$\frac{1}{153 \cdot 10^3}$	$\frac{1}{168 \cdot 10^4}$	$\frac{1}{663 \cdot 10^6}$	$\frac{1}{743 \cdot 10^5}$	$\frac{1}{209 \cdot 10^8}$	$\frac{1}{531 \cdot 10^{11}}$	$\frac{1}{221 \cdot 10^{10}}$	$\frac{1}{18 \cdot 10^{14}}$	$\frac{1}{67 \cdot 10^{18}}$

Note that $k_{m+1} \leq 3k_m/2 + 1$ for all $m \in \{1, \ldots, 24\}$.

We will inductively prove that $|y| \geq C_m |t|^{k_m}$ for $m = 1, \ldots, 25$. For $m = 1$, there is nothing to prove. So we assume that the assertion holds for some $m \in \{1, \ldots, 24\}$.

To simplify the presentation, we omit the subscript m in k_m, C_m where no ambiguity can arise.

From (14) we conclude that

$$|x - \alpha^{(3)} y| \leq \frac{4.01}{C^3} \cdot \frac{1}{|t|^{3k+1}} \cdot |y|. \tag{15}$$

Combining this with Lemma 5.1, we obtain

$$|x - B(1/t)y| \le \frac{c_1}{|t|^{\min\{3k+1,100\}}} \cdot |y|, \tag{16}$$

where c_1, c_2, ... will denote positive constants depending on k_m, C_m and k_{m+1}, but not on t, x, or y.

We set $\ell = k_{m+1} - 1$ and calculate polynomials $U_\ell(1/T)$, $V_\ell(1/T) \in \mathbb{Z}[1/T]$ of degree at most ℓ such that

$$|U_\ell(1/t) - B(1/t)V_\ell(1/t)| \le \frac{c_2}{|t|^{2\ell+1}}. \tag{17}$$

Note that $U_\ell(1/T)/V_\ell(1/T)$ is the Padé approximant of order (ℓ, ℓ) of $B(1/T)$, in particular, the polynomials U_ℓ and V_ℓ always exist.

Multiplying (16) by $|t^\ell V_\ell(1/t)| \le c_3 |t|^\ell$ yields

$$|V_\ell(1/t)t^\ell x - B(1/t)V_\ell(1/t)t^\ell y| \le \frac{c_1 c_3}{|t|^{\min\{3k+1-\ell,100-\ell\}}} \cdot |y|.$$

By replacing $B(1/t)V_\ell(1/t)$ with $U_\ell(1/t)$—i.e., by applying (17)—, we obtain

$$|V_\ell(1/t)t^\ell x - U_\ell(1/t)t^\ell y| \le \frac{c_4}{|t|^{\min\{3k+1-\ell,100-\ell,\ell+1\}}} \cdot |y| = \frac{c_4}{|t|^{\ell+1}} \cdot |y|. \tag{18}$$

We first consider the case that the left hand side vanishes, which is equivalent to

$$x = \frac{U_\ell(1/t)}{V_\ell(1/t)} y.$$

Inserting this in F_t and using our assumption $|y| \ge C|t|^k$, it turned out that

$$\left| F_t \left(\frac{U_\ell(1/t)}{V_\ell(1/t)} y, y \right) \right| > 1. \tag{19}$$

This is a contradiction to $|F_t(x, y)| = 1$.

By construction, the left hand side of (18) is the absolute value of an algebraic integer in $\mathbb{Z}_{\mathbb{Q}(t)}$. Since it does not vanish, it is at least 1. We obtain

$$|y| \ge \frac{1}{c_4}|t|^{\ell+1} \ge C_{m+1}|t|^{k_{m+1}} \text{ for } C_{m+1} \le \frac{1}{c_4}.$$

$\square$

Remark. Asymptotically, one should choose $\ell = k_{m+1} = \lfloor 3k_m/2 \rfloor$ in each step. For the rather small bound $|t| \ge 53$, however, it turned out that the rather small increases yield better constants C_m. Note that several of the first steps even have $k_{m+1} = k_m$ and only improve the constant C_m slightly. In fact, we built up a directed graph of such improvement steps and just displayed a path leading to an optimal (in the sense of the proposition below) bound. This took less than 4 minutes of CPU time.

The fact that (19) holds in every step is not completely surprising: It is quite probable that $|U_\ell(1/t) - B(1/t)V_\ell(1/t)|$ is not only bounded by $c_2/|t|^{2\ell+1}$ from above, but also by $c_5/|t|^{2\ell+1}$ from below. If we assume this and use (15), we get

$$\frac{4.01}{C^3|t|^{3k+1}} \ge \frac{4.01}{|y|^3|t|} \ge \left| \frac{x}{y} - \alpha^{(3)} \right| \ge \frac{c_6}{|t|^{2\ell+1}},$$

which implies

$$|t|^{3k-2\ell} \le \frac{4.01}{C^3 c_6}. \tag{20}$$

As $2\ell < 3k$ and $3k - 2\ell$ is not too small in our cases, it is only a question of some luck that (20) yields a contradiction for $|t| \ge 53$. Actually, the condition (19) is equivalent to the conclusion of this consideration. Here, another reason why ℓ should not be too large (compared to k) becomes evident.

Proposition 5.3. *Let $|t| \geq 53$ and $(x, y) \in \mathbb{Z}_{\mathbb{Q}(t)}$ be a solution to $F_t(x, y) = \mu$ of type $j = 3$. Then we have $xy = 0$.*

Proof. We assume that $y \neq 0$, hence $|y| \geq 1$. By Lemma 5.2, this implies $|y| \geq |t|$. Thus we can apply Theorem 1. Together with (14), we obtain

$$\frac{1}{746|t| \cdot |y|^{\kappa}} \leq |x - \alpha^{(3)} y| \leq \frac{4.01}{|t| \cdot |y|^2}.$$

This is equivalent to

$$|y|^{2-\kappa} \leq 2991.46.$$

It is easily seen that κ decreases monotonically in $|t|$ for $|t| \geq 53$. Thus inserting the lower bound for $|y|$ from Lemma 5.2, we get $|t| \leq 49$, a contradiction. ☐

Combining Lemma 4.2, Lemma 4.5 and Proposition 5.3 proves Theorem 2 for $|t| \geq 53$.

6. Proof of Theorem 2 for small t

This section is devoted to the proof of the remaining case ($|t| < 53$) of Theorem 2.

We will now consider the values of t with $t \in \mathcal{T}$ where

$$\mathcal{T} := \left\{ t \text{ imaginary quadratic integer} : \operatorname{Re} t \geq -1/2, \operatorname{Im} t > 0, |t| < 53, t \neq \frac{-1 + 3\sqrt{-3}}{2} \right\}$$

and we assume that (x, y) is a solution to (2) of type 1 with $|y| \geq 3$ for some fixed value $t \in \mathcal{T}$. We will use the "$\ll$" notation, where the implied constants may depend on t. Similarly, we will denote constants possibly depending on t by $c_1, \ldots$. These constants are not required to be positive.

We extend the result of Lemma 4.2 to $\min\{|y|, |x+y|, |x|\} < 9$, which requires less than 4 hours of CPU time. Therefore, we may assume that $|y| \geq 9$.

Lemma 6.1. *Let $t \in \mathcal{T}$ and α be a root of f_t. Then α and $\alpha + 1$ are independent units of $\mathbb{Z}[\alpha]$. The index I of $\langle \zeta, \alpha, \alpha + 1 \rangle$ in the unit group of $\mathbb{Z}[\alpha]$ is bounded by*

$$I \leq 19.5R \text{ with } R = \left| \det \begin{pmatrix} \log|\alpha^{(1)}| & \log|\alpha^{(1)} + 1| \\ \log|\alpha^{(2)}| & \log|\alpha^{(2)} + 1| \end{pmatrix} \right|,$$

where ζ is a primitive 14^{th} root of unity, if $t = \sqrt{-7}$, and a root of unity in $\mathbb{Q}(t)$, otherwise.

Proof. We have

$$1 = \alpha(\alpha^2 - (t-1)\alpha - (t+2)) = (\alpha + 1)(2 + t\alpha - \alpha^2),$$

which implies that α and $(\alpha + 1)$ are units in $\mathbb{Z}[\alpha]$.

For $|t| \geq 6$, Lemma 4.4 yields $R \geq 0.786 \log^2 |t| > 0$. For all $|t| < 6$, we obtain $R > 0.1$. Thus α and $\alpha + 1$ are independent units.

Let ζ be a root of unity in $\mathbb{Q}(\alpha)$. Thus $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a cyclic Galois extension. From [11, Theorem 7] we conclude that the only cyclic Galois extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\alpha)$ is $\mathbb{Q}(t)$ unless $t = \sqrt{-7}$, where $\mathbb{Q}(\alpha)/\mathbb{Q}$ itself is cyclic. Since $\mathbb{Q}(\sqrt{-7})$ is not a subset of the field generated by the 9^{th} or 18^{th} roots of unity, ζ must be a 7^{th} or 14^{th} root of unity. The second possibility is more general.

By Friedman [5, Theorem B], the regulator of $\mathbb{Q}(\alpha)$ can be bounded by $\operatorname{Reg} \mathbb{Z}_{\mathbb{Q}(\alpha)} \geq 0.2052$. From Pohst and Zassenhaus [18, p. 361], we conclude that

$$I = [\mathbb{Z}[\alpha]^{\times} : \langle \zeta, \alpha, \alpha + 1 \rangle] \leq [\mathbb{Z}_{\mathbb{Q}(\alpha)}^{\times} : \langle \zeta, \alpha, \alpha + 1 \rangle] = \frac{\operatorname{Reg}\langle \zeta, \alpha, \alpha + 1 \rangle}{\operatorname{Reg} \mathbb{Z}_{\mathbb{Q}(\alpha)}} \leq \frac{4R}{0.2052} \leq 19.5R.$$

☐

Unfortunately, the factor 4 in the numerator has been omitted in the corresponding result [11, Lemma 9].

Since $\beta^{(1)}$ is a unit by (12), there are rational integers u_0, u_1 and u_2 such that

$$(\beta^{(k)})^I = (\zeta^{(k)})^{u_0} (\alpha^{(k)})^{u_1} (\alpha^{(k)} + 1)^{u_2} \tag{21}$$

for $k \in \{1, 2, 3\}$.

For $k \neq 1$, we have

$$\log|\beta^{(k)}| = \log|y| + \log|\alpha^{(1)} - \alpha^{(k)}| + \delta_k, \tag{22}$$

where

$$\delta_k = \log\left|1 + \frac{\beta^{(1)}}{y(\alpha^{(1)} - \alpha^{(k)})}\right| \ll \frac{1}{|y|^3} \tag{23}$$

by (13).

Siegel's identity states

$$\frac{\beta^{(2)}}{\beta^{(3)}} \cdot \frac{\alpha^{(1)} - \alpha^{(3)}}{\alpha^{(1)} - \alpha^{(2)}} = 1 - \frac{\beta^{(1)}}{\beta^{(3)}} \cdot \frac{\alpha^{(2)} - \alpha^{(3)}}{\alpha^{(2)} - \alpha^{(1)}}.$$

We set

$$\Lambda_1 := \log\left(\frac{\beta^{(2)}}{\beta^{(3)}} \cdot \frac{\alpha^{(1)} - \alpha^{(3)}}{\alpha^{(1)} - \alpha^{(2)}}\right),$$
$$\Lambda := \operatorname{Re}\Lambda_1 = \log\left|\frac{\beta^{(2)}}{\beta^{(3)}} \cdot \frac{\alpha^{(1)} - \alpha^{(3)}}{\alpha^{(1)} - \alpha^{(2)}}\right|,$$

where log denotes the principal branch of the complex logarithm, i.e., $-\pi < \operatorname{Im}\log z \leq \pi$.

From (13), (22), and (23) we conclude that

$$|\Lambda| \leq |\Lambda_1| \ll \frac{1}{|y|^3}. \tag{24}$$

Using (11) and (21), we rewrite the linear form Λ_1 as

$$\Lambda_1 = \frac{v_1}{I}\log(\alpha^{(1)}) + \frac{v_2}{I}\log(\alpha^{(2)}) + \frac{v_0}{wI}\log(-1), \tag{25}$$

where $v_1 = u_1 - u_2$, $v_2 = 2u_1 + u_2 - I$, v_0 is a rational integer, and

$$w = \begin{cases} 7, & \text{if } t = \sqrt{-7}, \\ 1, & \text{otherwise.} \end{cases}$$

Taking logarithms of the absolute values of (21) for $k \in \{2, 3\}$ and using (22) yields

$$\frac{v_2}{I} = c_1 \log|y| + c_2\delta_2 + c_3\delta_3 + c_4. \tag{26}$$

Unfortunately, the argument for excluding $\Lambda = 0$ in [11] is incorrect.

Let G denote the Galois group $\operatorname{Gal}(\mathbb{Q}(\alpha^{(1)}, \overline{\alpha^{(1)}})/\mathbb{Q})$. In [11], this Galois group has been computed depending on t. We first consider the case $|G| = 18$ or $G \simeq C_6$. We check that

$$\det\begin{pmatrix} \log\left(\alpha^{(1)}\overline{\alpha^{(1)}}\right) & \log\left(\alpha^{(2)}\overline{\alpha^{(2)}}\right) \\ \log\left(\alpha^{(2)}\overline{\alpha^{(2)}}\right) & \log\left(\alpha^{(3)}\overline{\alpha^{(3)}}\right) \end{pmatrix} \neq 0 \text{ or } \det\begin{pmatrix} \log\left(\alpha^{(1)}\overline{\alpha^{(1)}}\right) & \log\left(\alpha^{(2)}\overline{\alpha^{(2)}}\right) \\ \log\left(\alpha^{(3)}\overline{\alpha^{(3)}}\right) & \log\left(\alpha^{(1)}\overline{\alpha^{(1)}}\right) \end{pmatrix} \neq 0,$$

respectively. Together with [11, Theorem 7], this implies that $\log|\alpha^{(1)}|$ and $\log|\alpha^{(2)}|$ are linearly independent over $\mathbb{Q}$. In this case, $\Lambda = 0$ would imply that $v_1 = v_2 = 0$. It turns out that this is a contradiction to (26) for $|y| \geq 9$.

We proceed as in [11]: We use Mignotte's version [17, Remark 4] of the estimate for linear forms in two logarithms due to Laurent, Mignotte, and Nesterenko [14] and apply it to Λ. This gives an upper bound for $|y|$ and also an upper bound c_6 for $|v_2|$ by (26) and Lemma 6.1. From (24), (25), Lemma 6.1, and (26) we conclude that

$$\left|\frac{v_1}{v_2} + \frac{\log|\alpha^{(2)}|}{\log|\alpha^{(1)}|}\right| \leq \frac{c_5}{|y|^3|v_2|} \leq \frac{1}{2|v_2|^2} \tag{27}$$

for $|y| \geq c_7 > 0$. This implies that v_1/v_2 is a convergent to $-\frac{\log|\alpha^{(2)}|}{\log|\alpha^{(1)}|}$ by Lagrange's theorem.

We compute $-\frac{\log|\alpha^{(2)}|}{\log|\alpha^{(1)}|}$ numerically with high precision (100 decimal digits always suffice) and its convergents with denominator at most c_6. It turns out that for $|y| \geq c_8 \geq c_7 > 0$, these convergents do not satisfy (27). This verification took almost 6 hours on a Pentium 4 with 2.8 GHz. For all but 267 choices of t, the constant c_8 already equals 9, which concludes the proof in these cases. The highest encountered value of c_8 has been 99. In the remaining cases of t (usually with large absolute value of the discriminant) we considered all values of y with $9 \leq |y| < c_8$. For each of them, (13) gives a list of a few possible values of x. It turned out that there are no such solutions. This took around two minutes on the above mentioned computer.

Next, we deal with the case $G \simeq S_3$. In these cases, $\log|\alpha^{(1)}|$ and $\log|\alpha^{(2)}|$ are linearly dependent over $\mathbb{Q}$. This is immediately obvious for $\operatorname{Re} t = -1/2$ by Lemma 4.3. For the remaining 4 cases, this can also be verified. Choose integers a and b such that $a\log|\alpha^{(1)}| - b\log|\alpha^{(2)}| = 0$. We conclude that $bI\Lambda = (v_1 b + v_2 a)\log|\alpha^{(1)}|$. We check that (24) implies

$$v_1 b + v_2 a = 0. \tag{28}$$

Thus $\Lambda = 0$ and we switch to the complex linear form Λ_1. Since we now have the relation (28), Λ_1 can also be written as a linear form in two logarithms. We therefore use the same procedure as outlined above (except that we now use Mignotte's bound in [2, A.13] which exactly deals with this kind of linear form). The computation took almost 10 minutes, 25 cases of t needed the special search for solutions with $|y| \geq 9$. In one case this search had to be extended up to $|y| \leq 59$. During this procedure, one pair of solutions with $|y| = \sqrt{140}$ has been found and listed in online Table [9]. This completes the proof of Theorem 2.

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