HWANG'S QUASI-POWER-THEOREM IN DIMENSION TWO

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Abstract. In a frequently used theorem, H.-K. Hwang proved convergence rates for the central limit theorem of a class of random variables whose moment generating function has a "quasi-power" structure. We generalise this result to random vectors of two variables.

Gaussian laws in large random combinatorial structures are a frequently observed pattern. In his "quasi-power-theorem", Hwang [2] proved asymptotic normality for a certain class of random variables whose moment generating function satisfies an asymptotic expression which is almost (apart from an error term) of the form $e^{W_n(s)}$. He also includes the convergence rates, cf. Theorem 1 below. His result turned out to be particularly useful and frequently used.

The purpose of this note is to provide a version of Hwang's theorem in the case of random vectors of dimension 2, again including the convergence rate. For this, we use a two-dimensional Berry-Esseen-estimate proved by Sadikova [3], cf. Lemma 3.

Although there is a generalisation of Sadikova's result to higher dimensions by Gamkrelidze [1], it seems to be non-trivial to use it for a further generalisation of the quasi-power theorem to higher dimensions.

We will use boldface letters for vectors and $\|\cdot\|$ will denote the maximum norm $\|\mathbf{s}\| = \max\{|s_j|\}$.

Hwang's result is the following.

Theorem 1 (Hwang [2]). *Let $\{\Omega_n\}_{n\geq 1}$ be a sequence of integral random variables. Suppose that the moment generating function satisfies the asymptotic expression*

$$M_n(s) := \mathbb{E}(e^{\Omega_n s}) = \sum_{m\geq 0} \mathbb{P}(\Omega_n = m)e^{ms} = e^{W_n(s)}(1 + O(\kappa_n^{-1})),$$

the O-term being uniform for $|s| \leq \tau$, $s \in \mathbb{C}$, $\tau > 0$, where

1. *$W_n(s) = u(s)\phi(n) + v(s)$, with $u(s)$ and $v(s)$ analytic for $|s| \leq \tau$ and independent of n; and $u''(0) \neq 0$;*
2. *$\lim_{n\to\infty} \phi(n) = \infty$;*
3. *$\lim_{n\to\infty} \kappa_n = \infty$.*

Then the distribution of Ω_n is asymptotically normal, i.e.,

$$\mathbb{P}\left(\frac{\Omega_n - u'(0)\phi(n)}{\sqrt{u''(0)\phi(n)}} < x\right) = \Phi(x) + O\left(\frac{1}{\sqrt{\phi(n)}} + \frac{1}{\kappa_n}\right),$$

uniformly with respect to x, $x \in \mathbb{R}$, where Φ denotes the standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}y^2\right) dy.$$

We intend to prove the following 2-dimensional version of Theorem 1.

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Theorem 2. *Let $\{\mathbf{\Omega}_n\}_{n\geq 1}$ be a sequence of two dimensional integral random vectors. Suppose that the moment generating function satisfies the asymptotic expression*

$$M_n(\mathbf{s}) := \mathbb{E}(e^{\langle \mathbf{\Omega}_n, \mathbf{s}\rangle}) = \sum_{\mathbf{m}\geq 0} \mathbb{P}(\mathbf{\Omega}_n = \mathbf{m}) e^{\langle \mathbf{m}, \mathbf{s}\rangle} = e^{W_n(\mathbf{s})}(1 + O(\kappa_n^{-1})),$$

the O-term being uniform for $\|\mathbf{s}\|_\infty \leq \tau$, $\mathbf{s} \in \mathbb{C}^2$, $\tau > 0$, where

1. *$W_n(\mathbf{s}) = u(\mathbf{s})\phi(n) + v(\mathbf{s})$, with $u(\mathbf{s})$ and $v(\mathbf{s})$ analytic for $\|\mathbf{s}\| \leq \tau$ and independent of n; and the Hessian $H_u(\mathbf{0})$ of u at the origin is nonsingular;*
2. *$\lim_{n\to\infty} \phi(n) = \infty$;*
3. *$\lim_{n\to\infty} \kappa_n = \infty$.*

Then, the distribution of $\mathbf{\Omega}_n$ is asymptotically normal, i.e.,

$$\mathbb{P}\left(\frac{\mathbf{\Omega}_n - \operatorname{grad} u(\mathbf{0})\phi(n)}{\sqrt{\phi(n)}} \leq \mathbf{x}\right) = \Phi_{H_u(\mathbf{0})}(\mathbf{x}) + O\left(\frac{1}{\sqrt{\phi(n)}} + \frac{1}{\kappa_n}\right),$$

where Φ_Σ denotes the distribution function of the two dimensional normal distribution with mean $\mathbf{0}$ and variance-covariance matrix Σ, i.e.,

$$\Phi_\Sigma(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det \Sigma}} \iint_{\mathbf{y}\leq\mathbf{x}} \exp\left(-\frac{1}{2}\mathbf{y}^t\Sigma^{-1}\mathbf{y}\right) d\mathbf{y},$$

where $\mathbf{y} \leq \mathbf{x}$ means $y_1 \leq x_1$ and $y_2 \leq x_2$.

The proof of Theorem 2 relies on the following two-dimensional Berry-Esseen-inequality.

Lemma 3 (Sadikova [3])**.** *Let $\mathbf{X}$ and $\mathbf{Y}$ be two-dimensional random vectors with distribution functions F and G and characteristic functions f and g, respectively.*

Let

$$\hat{f}(s_1, s_2) = f(s_1, s_2) - f(s_1, 0)f(0, s_2), \qquad \hat{g}(s_1, s_2) = g(s_1, s_2) - g(s_1, 0)g(0, s_2), \tag{1}$$

and

$$A_1 = \sup_{x_1,x_2} \frac{\partial G(x_1, x_2)}{\partial x_1}, \qquad A_2 = \sup_{x_1,x_2} \frac{\partial G(x_1, x_2)}{\partial x_2}.$$

Then for any $T > 0$, we have

$$\begin{aligned} \sup_{x,y} |F(x, y) - G(x, y)| \leq{}& \frac{2}{(2\pi)^2} \iint_{\|\mathbf{s}\|\leq T} \left|\frac{\hat{f}(s_1, s_2) - \hat{g}(s_1, s_2)}{s_1 s_2}\right| d\mathbf{s} \\ &+ 2 \sup_x |F(x, \infty) - G(x, \infty)| + 2 \sup_y |F(\infty, y) - G(\infty, y)| + \frac{2(A_1 + A_2)}{T}\left(3\sqrt{2} + 4\sqrt{3}\right). \end{aligned} \tag{2}$$

Proof of Theorem 2. We define $E_n(\mathbf{s})$ by the relation $M_n(\mathbf{s}) = e^{W_n(\mathbf{s})}(1 + E_n(\mathbf{s}))$ and note that by assumption, $E_n(\mathbf{s}) = O(\kappa_n^{-1})$ uniformly for $\|\mathbf{s}\| \leq \tau$. We note that this implies $u(\mathbf{0}) = v(\mathbf{0}) = 0$ and therefore $E_n(\mathbf{0}) = 0$.

Let $\boldsymbol{\mu}_n = \phi(n) \operatorname{grad} u(\mathbf{0})$ and $\Sigma = H_u(\mathbf{0})$. We define the random vector $\mathbf{\Omega}_n^* = \phi(n)^{-1/2}(\mathbf{\Omega}_n - \boldsymbol{\mu}_n)$ with distribution function $F_n(\mathbf{x})$ and characteristic function

$$f_n(\mathbf{s}) = M_n\left(i\phi(n)^{-1/2}\mathbf{s}\right) \exp\left(-i\phi(n)^{-1/2}\langle\boldsymbol{\mu}_n, \mathbf{s}\rangle\right) = \exp\left(-\frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s} + W_n^*(\mathbf{s})\right)(1+E_n(i\phi(n)^{-1/2}\mathbf{s}))$$

with

$$W_n^*(\mathbf{s}) = u(i\phi(n)^{-1/2}\mathbf{s})\phi(n) + v(i\phi(n)^{-1/2}\mathbf{s}) - i\phi(n)^{-1/2}\langle\boldsymbol{\mu}_n, \mathbf{s}\rangle + \frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s}.$$

We consider the univariate analytic functions $u_j(s_j)$, $v_j(s_j)$, $E_{n,j}(s_j)$ for $j = 1, 2$ and the bivariate analytic functions $u_0(\mathbf{s})$, $v_0(\mathbf{s})$, $E_{n,0}(\mathbf{s})$ satisfying

$$\begin{aligned}
u(\mathbf{s}) &= \langle \operatorname{grad} u(\mathbf{0}), \mathbf{s}\rangle + \frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s} + s_1^2 u_1(s_1) + s_2^2 u_2(s_2) + s_1 s_2 u_0(\mathbf{s}),\\
v(\mathbf{s}) &= v_1(s_1) + v_2(s_2) + s_1 s_2 v_0(\mathbf{s}),\\
E_n(\mathbf{s}) &= E_{n,1}(s_1) + E_{n,2}(s_2) + s_1 s_2 E_{n,0}(\mathbf{s}),\\
0 &= u_j(0) = v_j(0) = E_{n,j}(0), \qquad j \in \{1, 2\},\\
0 &= u_0(\mathbf{0}).
\end{aligned}$$

Let c be a positive constant less than $\max\{\tau/2, 1\}$ which will be specified later and let $T_n = c\sqrt{\phi(n)}$. With these notations, we have

$$\begin{aligned}
W_n^*(\mathbf{s}) =& - s_1^2 u_1(i\phi(n)^{-1/2}s_1) - s_2^2 u_2(i\phi(n)^{-1/2}s_2) - s_1 s_2 u_0(i\phi(n)^{-1/2}\mathbf{s})\\
&+ v_1(i\phi(n)^{-1/2}s_1) + v_2(i\phi(n)^{-1/2}s_2) - \frac{s_1 s_2}{\phi(n)} v_0(i\phi(n)^{-1/2}\mathbf{s})\\
=&O(\rho_n(\mathbf{s})).
\end{aligned}$$

for $\|\mathbf{s}\| < T_n$, where

$$\rho_n(\mathbf{s}) := \frac{\|\mathbf{s}\|^3 + \|\mathbf{s}\|}{\sqrt{\phi(n)}}.$$

Since $E_n((s_1, 0)) = E_{n,1}(s_1) = O(\kappa_n^{-1})$ and $E_n((0, s_2)) = E_{n,2}(s_2) = O(\kappa_n^{-1})$, we also have $s_1 s_2 E_{n,0}(\mathbf{s}) = O(\kappa_n^{-1})$. By Cauchy's integral formula, we also get $E_{n,0}(\mathbf{s}) = O(\kappa_n^{-1})$ for $\|\mathbf{s}\| < \tau/2$. Similarly, we have $E_{n,j} = s_j O(\kappa_n^{-1})$ for $\|\mathbf{s}\| < \tau/2$ and $j = 1, 2$.

Note that

$$\lim_{n\to\infty} f_n(\mathbf{s}) = \exp\left(-\frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s}\right) =: g(\mathbf{s})$$

for $\mathbf{s} \in \mathbb{R}^2$, which implies that in distribution, $\mathbf{\Omega}_n^*$ converges to the normal distribution with mean zero and variance-covariance matrix Σ. Although we have to refine our estimates for applying Lemma 3, we conclude immediately that Σ is positive definite (since it is nonsingular).

We now estimate $\hat{f}(\mathbf{s})$ as defined in (1) for $\|\mathbf{s}\| < T_n$:

$$\begin{aligned}
\hat{f}(\mathbf{s}) =& \exp\left(-\frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s}\right)\exp(W_n^*(s))\\
&\times \Big(1 + E_{n,1}(i\phi(n)^{-1/2}s_1) + E_{n,2}(i\phi(n)^{-1/2}s_2) - s_1 s_2\phi(n)^{-1}E_{n,0}(i\phi(n)^{-1/2}\mathbf{s})\\
&\quad - \exp\big(s_1 s_2(\sigma_{12} + u_0(i\phi(n)^{-1/2}\mathbf{s}) + \phi(n)^{-1}v_0(i\phi(n)^{-1/2}\mathbf{s}))\big)\\
&\qquad \times \big(1 + E_{n,1}(i\phi(n)^{-1/2}s_1) + E_{n,2}(i\phi(n)^{-1/2}s_2) + E_{n,1}(i\phi(n)^{-1/2}s_1)E_{n,2}(i\phi(n)^{-1/2}s_2)\big)\Big)\\
=& \exp\Big(-\frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s}\Big)\exp(W_n^*(s))\Big(1 - \exp(s_1 s_2\sigma_{12})\big(1 + s_1 s_2 O(\|\mathbf{s}\|\phi(n)^{-1/2})\exp(O(\rho_n(\mathbf{s})))\big)\Big)\\
&\quad \times \big(1 + E_{n,1}(i\phi(n)^{-1/2}s_1) + E_{n,2}(i\phi(n)^{-1/2}s_2)\big)\\
&+ s_1 s_2 \exp\Big(-\frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s} + O(\rho_n(\mathbf{s}))\Big)O(\kappa_n^{-1}\phi(n)^{-1})\\
&+ s_1 s_2 \exp\Big(-\frac{1}{2}(\sigma_{11}s_1^2 + \sigma_{22}s_2^2) + O(\rho_n(\mathbf{s}))\Big)O(\kappa_n^{-2}\phi(n)^{-1})\\
=& \exp\Big(-\frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s}\Big)\big(1 - \exp(s_1 s_2\sigma_{12})\big)\\
&+ s_1 s_2 \exp\Big(-\frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s} + O(\rho_n(\mathbf{s}))\Big)O(\kappa_n^{-1}\phi(n)^{-1})\\
&+ s_1 s_2 \exp\Big(-\frac{1}{2}(\sigma_{11}s_1^2 + \sigma_{22}s_2^2) + O(\rho_n(\mathbf{s}))\Big)O(\kappa_n^{-1} + \rho_n(\mathbf{s}))
\end{aligned}$$

where the inequality $|\exp(w) - 1| \le |w|\exp(|w|)$ for all complex w has been used repeatedly.

In order to apply Lemma 3, we estimate $|\hat{f}(\mathbf{s}) - \hat{g}(\mathbf{s})|/|s_1 s_2|$ for $\|\mathbf{s}\| < T_n$:

$$\begin{aligned}\left|\frac{\hat{f}(\mathbf{s}) - \hat{g}(\mathbf{s})}{s_1 s_2}\right| &= \exp\Bigl(-\frac{1}{2}\mathbf{s}^t\Sigma\mathbf{s} + O(\rho_n(\mathbf{s}))\Bigr)O(\kappa_n^{-1}\phi(n)^{-1}) \\ &\qquad + \exp\Bigl(-\frac{1}{2}(\sigma_{11}s_1^2 + \sigma_{22}s_2^2) + O(\rho_n(\mathbf{s}))\Bigr)O(\kappa_n^{-1} + \rho_n(\mathbf{s})).\end{aligned}$$

We choose c sufficiently small such that for $\|\mathbf{s}\| < T_n$:

$$\left|\frac{\hat{f}(\mathbf{s}) - \hat{g}(\mathbf{s})}{s_1 s_2}\right| = \exp\Bigl(-\frac{1}{4}\mathbf{s}^t\Sigma\mathbf{s}\Bigr)O(\kappa_n^{-1}\phi(n)^{-1}) + \exp\Bigl(-\frac{1}{4}(\sigma_{11}s_1^2 + \sigma_{22}s_2^2)\Bigr)O(\kappa_n^{-1} + \rho_n(\mathbf{s})).$$

For a constant $k \geq 0$, we have

$$\int_0^\infty \exp(-x^2)x^k\,dx = \frac{1}{2}\Gamma\left(\frac{k+1}{2}\right)$$

and we conclude that

$$\iint\limits_{\|\mathbf{s}\| \leq T_n} \left|\frac{\hat{f}(\mathbf{s}) - \hat{g}(\mathbf{s})}{s_1 s_2}\right| d\mathbf{s} = O\left(\frac{1}{\sqrt{\phi(n)}} + \frac{1}{\kappa_n}\right).$$

For estimating the second and the third summand in (2) we simply use Hwang's result in dimension 1 (Theorem 1) to see that they are also bounded by $O(\phi(n)^{-1/2} + \kappa_n^{-1})$. The fourth summand is bounded by $O(\phi(n)^{-1/2})$. □

References

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