PARAMETERIZED NORM FORM EQUATIONS WITH ARITHMETIC PROGRESSIONS

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1. Introduction

Buchmann and Pethő [5] observed that following algebraic integer

$$10 + 9\alpha + 8\alpha^2 + 7\alpha^3 + 6\alpha^4 + 5\alpha^5 + 4\alpha^6,$$

with $\alpha^7 = 3$ is a unit. Since the coefficients form an arithmetic progressions they have found a solution to the Diophantine equation

$$\mathrm{N}_{K/\mathbb{Q}}(x_0 + \alpha x_1 + \cdots + x_6\alpha^6) = \pm 1, \tag{1}$$

such that $(x_0, \dots, x_6) \in \mathbb{Z}^7$ is an arithmetic progression.

Recently Bérczes and Pethő [3] considered the Diophantine equation

$$\mathrm{N}_{K/\mathbb{Q}}(x_0 + \alpha x_1 + \cdots + x_{n-1}\alpha^{n-1}) = m, \tag{2}$$

where α is an algebraic number of degree n, $K = \mathbb{Q}(\alpha)$, $m \in \mathbb{Z}$ and $(x_0, x_1, \dots, x_{n-1}) \in \mathbb{Z}^n$ is nearly an arithmetic progression. The sequence $(x_0, \dots, x_{n-1})$ is said to be nearly an arithmetic progression if there exists $d \in \mathbb{Z}$ and $0 < \delta \in \mathbb{R}$ such that

$$|(x_i - x_{i-1}) - d| \leq (\max\{|x_0|, \dots, |x_{n-1}|\})^{1-\delta}, \qquad (i = 1, \dots, n-1).$$

They proved that equation (2) has only finitely many solutions provided $\beta := \frac{n\alpha^n}{\alpha^n - 1} - \frac{\alpha}{\alpha - 1}$ is an algebraic number of degree at least 3. Moreover they showed that the solution found by Buchmann and Pethő is the only solution to (1).

Bérczes and Pethő also considered arithmetic progressions arising from the norm equation (2), where α is a root of $X^n - a$, with $n \geq 3$ and $2 \leq a \leq 100$ (see [2]).

Let $f_a \in \mathbb{Z}[X]$ be the Thomas polynomial

$$f_a := X^3 - (a-1)X^2 - (a+2)X - 1.$$

The aim of this paper is to prove the following theorem:

Theorem 1. *Let α be a root of the polynomial f_a, with $a \in \mathbb{Z}$. Then the only solutions to the norm form inequality*

$$\left|\mathrm{N}_{K/\mathbb{Q}}(x_0 + x_1\alpha + x_2\alpha^2)\right| \leq |2a+1| \tag{3}$$

such that $x_0 < x_1 < x_2$ is an arithmetic progression and (x_1, x_2, x_3) is primitive are either $(x_1, x_2, x_3) = (-2, -1, 0), (-1, 0, 1)$ and $(0, 1, 2)$, or they are sporadic solutions that are listed in table 1.

In table 1 we only list solutions, where the parameter is non-negative. Furthermore m denotes the value of the norm, i.e. $\mathrm{N}_{K/\mathbb{Q}}(x_0 + x_1\alpha + x_2\alpha^2) = m$. Lemma 1 will show that it suffices to study the norm inequality (3) only for $a \geq 0 \in \mathbb{Z}$. Moreover, Lemma 1 gives a correspondence between solutions for a and $-a - 1$.

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Table 1. Sporadic solutions to (3) with $a \geq 0$.

a	m	x_0	x_1	x_2	a	m	x_0	x_1	x_2
1	3	−7	−2	3	1	3	−3	−1	1
1	−3	−7	−3	1	2	5	−97	−35	27
2	5	−36	−13	10	2	5	−27	−10	7
2	5	−19	−7	5	2	−5	−97	−36	25
2	−5	−35	−13	9	2	−5	−25	−9	7
2	−5	−14	−5	4	2	−5	−5	−2	1
2	1	−11	−4	3	2	−1	−8	−3	2
2	−1	−3	−1	1	3	1	−5	−2	1
3	1	−3	−1	1	4	9	−7	−2	3
4	9	−3	−1	1	4	−9	−7	−3	1
5	−1	−4	−1	2	7	−15	−5	−1	3
16	−33	−28	−3	22					

2. Notations and Thue Equations

Let us prove first that we may assume $a \geq 0$.

Lemma 1. *Let $\alpha(a)$ denote a zero of $f_a(x)$ and put $K(a) = \mathbb{Q}(\alpha(a))$. Then*

$$\mathrm{N}_{K(a)/\mathbb{Q}}(x_0 + x_1\alpha(a) + x_2\alpha(a)^2) = m$$

holds if and only if

$$\mathrm{N}_{K(-a-1)/\mathbb{Q}}(-x_2 - x_1\alpha(-a-1) - x_0\alpha(-a-1)^2) = -m.$$

In particular each solution to (3) for a yields a solution for $-a-1$.

Proof. It is easy to see that $\alpha(a)$ is a root of $f_a(x)$ if and only if $\frac{1}{\alpha(a)}$ is a root of $f_{-a-1}(x)$. As $N_{K(a)/\mathbb{Q}}(-\alpha(a)) = -1$ the assertion follows immediately. □

Next, we want to transform the norm form inequality (3) into a Thue inequality. Since x_0, x_1, x_2 form an arithmetic progression we may write $x_0 = X - Y, x_1 = X$ and $x_2 = X + Y$. Using this notation in (3) we obtain

$$\left|\mathrm{N}_{K/\mathbb{Q}}(X(1+\alpha+\alpha^2) - Y(1-\alpha^2))\right| \leq |2a+1|.$$

Expanding the norm on the left side to a polynomial in X and Y we obtain the Thue inequality

$$\left|(a^2+a+7)X^3 - (a^2+a+7)XY^2 - (2a+1)Y^3\right| \leq |2a+1|. \tag{4}$$

Since we have the restrictions $x_0 < x_1 < x_2$ and (x_0, x_1, x_2) is primitive, we are only interested in solutions with $Y \geq 1$ and (X, Y) is primitive.

For the rest of this paper we will use the following notations: We denote by $f_a \in \mathbb{Z}[X]$ the Thomas polynomial, which is defined as follows:

$$f_a(X) := X^3 - (a-1)X^2 - (a+2)X - 1.$$

Let $\alpha := \alpha_1 > \alpha_3 > \alpha_2$ be the three distinct real roots of f_a. Furthermore we define $\gamma := 1+\alpha+\alpha^2$, $\delta := 1-\alpha^2$ and $\epsilon := \delta/\gamma$ and denote by $\gamma_1 := \gamma, \gamma_2, \gamma_3$, $\delta_1 := \delta, \delta_2, \delta_3$ and $\epsilon_1 := \epsilon, \epsilon_2, \epsilon_3$ their conjugates respectively. Moreover we define $G_a \in \mathbb{Z}[X, Y]$ and $g_a \in \mathbb{Z}[X]$ by

$$G_a(X,Y) := (a^2+a+7)X^3 - (a^2+a+7)XY^2 - (2a+1)Y^3, \tag{5}$$

$$g_a(X) := G_a(X,1) = (a^2+a+7)X^3 - (a^2+a+7)X - (2a+1). \tag{6}$$

Let us remark that ϵ_1, ϵ_2 and ϵ_3 are exactly the roots of g_a.

If (X, Y) is a solution to (4) then we define $\beta := X\gamma - Y\delta$ and we denote by $\beta_1 := \beta, \beta_2, \beta_3$ the conjugates of β. As one can easily see β_i is an element of the order $\mathbb{Z}[\alpha_i]$ for all $i = 1, \ldots, 3$. In fact the orders $\mathbb{Z}[\alpha_i]$ are all the same (see [14, 16, 17] or Section 4).

There are a lot of well known facts about the number fields $K := \mathbb{Q}(\alpha)$, which we will state in Section 4.

We will use the following variant of the usual O-notation: For two functions $g(t)$ and $h(t)$ and a positive number t_0 we will write $g(t) = L_{t_0}(h(t))$ if $|g(t)| \leq h(t)$ for all t with absolute value at least t_0. We will use this notation in the middle of an expression in the same way as it is usually done with the O-notation. Sometimes we omit the index t_0. This will happen only in theoretical results, and it means that there exists a (computable) t_0 with the desired property.

This L-notation will help us to state asymptotic results in a comfortable way.

3. Asymptotic expansions

From various papers ([16, 17, 11, 6]) we know that

$$\alpha_1 \sim a, \qquad \alpha_2 \sim -1, \qquad \alpha_3 \sim -1/a.$$

We apply Newton's method to the polynomial f_a with starting points $a, -1$ and 0. After 4 steps of Newton's method and an asymptotic expansion of the resulting expressions we get

$$\begin{aligned} \tilde{\alpha}_1 :=& a + \frac{2}{a} - \frac{1}{a^2} - \frac{3}{a^3} + \frac{5}{a^4} \simeq \alpha_1, \\ \tilde{\alpha}_2 :=& -1 - \frac{1}{a} + \frac{2}{a^3} - \frac{1}{a^4} \simeq \alpha_2, \\ \tilde{\alpha}_3 :=& -\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} - \frac{4}{a^4} \simeq \alpha_3. \end{aligned} \tag{7}$$

We consider the quantities $-f_a(\tilde{\alpha}_i + e_i/a^5) f_a(\tilde{\alpha}_i - e_i/a^5)$ with $e_1 = 10, e_2 = 8$ and $e_3 = 18$. These quantities are all positive provided that $a \geq 8, a \geq 7$ and $a \geq 10$ respectively, hence

$$\begin{aligned} \alpha_1 =& a + \frac{2}{a} - \frac{1}{a^2} - \frac{3}{a^3} + \frac{5}{a^4} + L_8\left(\frac{10}{a^5}\right), \\ \alpha_2 =& -1 - \frac{1}{a} + \frac{2}{a^3} - \frac{1}{a^4} + L_7\left(\frac{8}{a^5}\right), \\ \alpha_3 =& -\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} - \frac{4}{a^4} + L_{10}\left(\frac{18}{a^5}\right). \end{aligned} \tag{8}$$

Since $\alpha_1 + \alpha_2 + \alpha_3 = a - 1$ is an integer we also obtain

$$\alpha_3 = -\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} - \frac{4}{a^4} + L_8\left(\frac{18}{a^5}\right).$$

In order to keep the error terms low from now on we assume that $a \geq 1000$. Using these asymptotic expansions we obtain for the γ's

$$\begin{aligned} \gamma_1 =& a^2 + a + 5 - \frac{3}{a^2} - \frac{3}{a^3} + L_{1000}\left(\frac{36.037}{a^4}\right), \\ \gamma_2 =& 1 + \frac{1}{a} + \frac{1}{a^2} - \frac{2}{a^3} - \frac{3}{a^4} + L_{1000}\left(\frac{26.021}{a^5}\right), \\ \gamma_3 =& 1 - \frac{1}{a} + \frac{2}{a^2} - \frac{1}{a^3} - \frac{5}{a^4} + L_{1000}\left(\frac{28.044}{a^5}\right), \end{aligned} \tag{9}$$

and similarly for the δ's

$$\begin{aligned} \delta_1 =& -a^2 - 3 + \frac{2}{t} + \frac{2}{a^2} - \frac{6}{a^3} + L_{1000}\left(\frac{31.027}{a^4}\right), \\ \delta_2 =& -\frac{2}{a} - \frac{1}{a^2} + \frac{4}{a^3} + \frac{2}{a^4} + L_{1000}\left(\frac{18.021}{a^5}\right), \\ \delta_3 =& 1 - \frac{1}{a^2} + \frac{2}{a^3} + \frac{1}{a^4} - \frac{10}{a^5} + L_{1000}\left(\frac{43.045}{a^6}\right), \end{aligned} \tag{10}$$

and for the ϵ's

$$\begin{aligned}\epsilon_1 =& -1+\frac{1}{a}+\frac{1}{a^2}-\frac{4}{a^3}-\frac{2}{a^4}+\frac{22}{a^5}+L_{1000}\left(\frac{108.886}{a^6}\right),\\ \epsilon_2 =& -\frac{2}{a}+\frac{1}{a^2}+\frac{5}{a^3}-\frac{8}{a^4}+L_{1000}\left(\frac{67.81}{a^5}\right),\\ \epsilon_3 =& 1+\frac{1}{a}-\frac{2}{a^2}-\frac{1}{a^3}+L_{1000}\left(\frac{36.385}{a^4}\right).\end{aligned} \tag{11}$$

We will also use the asymptotic expansions of the logarithms of the α's. Therefore we recall a simple fact from analysis: If $|t|>|r|$ then

$$\log|t+r| = \log|t| - \sum_{i=1}^{N}\frac{(-r/t)^i}{i} + L\left(\left|\frac{r}{t}\right|^{N+1}\frac{1}{N+1}\cdot\left|\frac{t}{t-r}\right|\right).$$

We have omitted the index t_0 since this index depends on the L-Term of the quantity r. Let us write

$$\alpha = \overbrace{a}^{=:t} + \overbrace{\frac{2}{a}-\frac{1}{a^2}-\frac{3}{a^3}+\frac{5}{a^4}+L_{1000}\left(\frac{10}{a^5}\right)}^{=:r}.$$

We can write similar expressions for α_2 and α_3, too. Using the above formula we get

$$\begin{aligned}\log|\alpha_1| =& \log a - \frac{2}{a^2}+\frac{1}{a^3}+\frac{5}{a^4}-\frac{7}{a^5}+L_{1000}\left(\frac{18.184}{a^6}\right),\\ \log|\alpha_2| =& -\frac{1}{a}+\frac{1}{2a^2}+\frac{5}{3a^3}-\frac{11}{4a^4}+L_{1000}\left(\frac{11.035}{a^5}\right),\\ \log|\alpha_3| =& -\log a+\frac{1}{a}-\frac{3}{2a^2}+L_{1000}\left(\frac{3.514}{a^3}\right).\end{aligned} \tag{12}$$

4. Auxiliary results

Let us recall first some well known facts about the number field $K=\mathbb{Q}(\alpha)$, where α is a root of the Thomas polynomial f_a (these results can be found in [14, 16, 17, 9]).

Lemma 2. *Let α be a root of the polynomial f_a. Then we have the following facts:*

1. *The polynomials f_a are irreducible for all $a\in\mathbb{Z}$. Moreover all roots of f_a are real.*
2. *The number fields $K=\mathbb{Q}(\alpha)$ are cyclic Galois extensions of degree three of $\mathbb{Q}$ for all $a\in\mathbb{Z}$.*
3. *The roots of f_a are permuted by the map $\alpha\mapsto -1-\frac{1}{\alpha}$.*
4. *Any two of $\alpha_1,\alpha_2,\alpha_3$ form a fundamental system of units of the order $\mathbb{Z}[\alpha]$, where $\alpha_1,\alpha_2,\alpha_3$ denote the conjugates of α.*
5. *Let $a\geq 0$. If $|\mathrm{N}_{K/\mathbb{Q}}(\gamma)|\leq 2a+1$ then γ is either associated to a rational integer or associated to a conjugate of $\alpha-1$.*

Proof. Proofs of these statements can be found in [14, 16, 17, 9] except statement (5) in the case of $a=0$ and $a=1$. The case $a=0$ is trivial. So let us consider the case $a=1$.

If γ fulfills $|\mathrm{N}_{K/\mathbb{Q}}(\gamma)|\leq 3$ and if γ is not a unit of $\mathbb{Z}[\alpha]$ then $(\gamma)|(2)$ or $(\gamma)|(3)$. According to [7, Chapter I, Proposition 25] we have $(3)=\mathfrak{p}_1^3$ with $\mathfrak{p}_1=(\alpha_1-1)+(3)=(\alpha_1-1)$ and $(2)=\mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals. Therefore γ is a multiple of α_1-1 or 2. Computing the norms yields that γ is associated to α_1-1 or is 0. Therefore we have proved the statement for $a=1$. □

Part (5) of Lemma 2 shows that we only have to consider algebraic integers, that are associated to a rational integer or associated to a conjugate of $\alpha-1$. Let us exclude the case that $\gamma=n\epsilon$ with $n\neq\pm1\in\mathbb{Z}$ and $\epsilon\in\mathbb{Z}[\alpha]^*$ and γ yields a solution to (3). Since $\gamma=x_0+x_1\alpha+x_2\alpha^2$ with unique $x_0,x_1,x_2\in\mathbb{Z}$, also $\epsilon=\frac{x_0}{n}+\frac{x_1}{n}\alpha+\frac{x_2}{n}\alpha^2$ yields a solution to (3) . Therefore $n|x_0,x_1,x_2$. However, (x_0,x_1,x_2) is primitive, thus γ cannot be associated to a rational integer $\neq\pm1$.

We have to solve the Diophantine inequality (4), therefore we start to exclude all small values of Y.

Lemma 3. *Let (X, Y) be a solution to* (4) *such that $Y = 1$, then (X, Y) only yields solutions stated in Theorem 1.*

Proof. We insert $Y = 1$ into (4) and obtain

$$|(a^2+a+7)(X^2-1)X-(2a+1)| \leq 2a+1.$$

If we assume $X \geq 2$, respectively $X \leq -2$, then

$$6(a^2+a+7)-(2a+1) \leq |(a^2+a+7)(X^2-1)X-(2a+1)| \leq 2a+1$$

yields a contradiction. Therefore $|X| \leq 1$ and we only obtain solutions stated in Theorem 1. □

Now we investigate approximation properties of solutions (X, Y) to (4). We distinguish three types of solutions. We say that (X, Y) is of type j, if

$$\left|\frac{X}{Y}-\epsilon_j\right| = \min_{i=1,2,3}\left(\left|\frac{X}{Y}-\epsilon_i\right|\right).$$

A specific case j will be called by its roman number. Let us assume that (X, Y) is a solution of type j. Then we have (remember $\beta_i = X\gamma_i - Y\delta_i$)

$$2\left|\frac{\beta_i}{\gamma_i}\right| \geq \left|\frac{\beta_i}{\gamma_i}\right| + \left|\frac{\beta_j}{\gamma_j}\right| = |X-Y\epsilon_i| + |X-Y\epsilon_j| \geq |Y||\epsilon_i-\epsilon_j|.$$

Since $|\beta_1\beta_2\beta_3| \leq 2a+1$ by the above inequality we obtain

$$|\beta_j| \leq \frac{2a+1}{\prod_{i\neq j}|\beta_i|} \leq \frac{8a+4}{|Y|^2\prod_{i\neq j}|\gamma_i||\epsilon_j-\epsilon_i|}$$

or equivalently

$$\left|\frac{\beta_j}{\gamma_j}\right| \leq \frac{8a+4}{|Y|^2|\mathrm{N}_{K/\mathbb{Q}}\gamma|\prod_{i\neq j}|\epsilon_j-\epsilon_i|} =: \frac{c_1}{|Y|^2} \tag{13}$$

and we also get

$$\mathrm{sign}(y)\epsilon_j - \frac{c_1}{|Y|^3} \leq \frac{X}{|Y|} \leq \mathrm{sign}(y)\epsilon_j + \frac{c_1}{|Y|^3},$$

hence

$$\left|\frac{\beta_i}{\gamma_i}\right| = |Y||\epsilon_j-\epsilon_i| + L\left(\frac{c_1}{Y_0^2}\right) = |Y|\left(|\epsilon_j-\epsilon_i| + L\left(\frac{c_1}{Y_0^3}\right)\right), \tag{14}$$

where Y_0 is some lower bound for $|Y|$. Because of Lemma 3 we may assume $Y_0 \geq 2$. Using the asymptotic expansions (8), (9), (10) and (11) we find

- $c_1 = \frac{4}{a} + L_{1000}\left(\frac{10.011}{a^2}\right)$ if $j = 1$;
- $c_1 = \frac{8}{a} + L_{1000}\left(\frac{4.044}{a^2}\right)$ if $j = 2$;
- $c_1 = \frac{4}{a} + L_{1000}\left(\frac{14.035}{a^2}\right)$ if $j = 3$;

Now we can prove a new lower bound Y_0 for $|Y|$.

Lemma 4. *If $a \geq 1000$ and (X, Y) is a primitive solution to* (4) *such that $Y > 1$ then $Y \geq \frac{a}{3.01}$.*

Proof. We have to distinguish between three cases $j = 1, j = 2$ and $j = 3$. We find from (13) and (11):

$$\left|X - Y\left(-1 + L_{1000}\left(\frac{1.002}{a}\right)\right)\right| \leq \frac{4.011}{Y^2 a},$$

$$\left|X - YL_{1000}\left(\frac{2.002}{a}\right)\right| \leq \frac{8.005}{Y^2 a},$$

$$\left|X - Y\left(1 + L_{1000}\left(\frac{1.003}{a}\right)\right)\right| \leq \frac{4.015}{Y^2 a}.$$

Some straightforward calculations yield

$$\begin{aligned} |X+Y| &\leq \frac{4.011}{Y^2a} + \frac{Y1.002}{a} < \frac{1.51Y}{a}, \\ |X| &\leq \frac{8.005}{Y^2a} + \frac{Y2.002}{a} < \frac{3.01Y}{a}, \\ |X-Y| &\leq \frac{4.015}{Y^2a} + \frac{Y1.003}{a} < \frac{1.51Y}{a}. \end{aligned}$$

We conclude that $X+Y=0$, $X-Y=0$ or $X=0$ if $Y < \frac{a}{3.01}$. But if $X+Y=0$, $X-Y=0$ or $X=0$ we get a contradiction, hence $Y \geq \frac{a}{3.01}$. □

Let σ be the automorphism of $K=\mathbb{Q}(\alpha)$ that is induced by $\alpha \mapsto -1-\frac{1}{\alpha}$. Then we have $\alpha_i = \sigma^{i-1}\alpha$. From part (5) of Lemma 2 we know that β is either a unit, associated to a rational integer or associated to a conjugate of $\alpha_1 - 1$. By the discussion after Lemma 2 we know that β is not associated to a rational integer $\neq 1$. Furthermore α_1 and α_2 form a fundamental system of units of the relevant order $\mathbb{Z}[\alpha]$, hence the linear system

$$\log|\beta_i| = b_1 \log|\sigma^{i-1}\alpha_1| + b_2 \log|\sigma^{i-1}\alpha_2| + \log|\sigma^{i-1}\mu| \qquad i \neq j \tag{15}$$

with μ associated to one of $1, \alpha_1 - 1, \alpha_2 - 1$ or $\alpha_3 - 1$, has a unique integral solution (b_1, b_2). Solving (15) by Cramer's rule we find

$$\begin{aligned} B := \max\{|b_1|,|b_2|\} &\leq 2\frac{\max_{i\neq j}\left|\log|\beta_i| - \log|\sigma^{i-1}\mu|\right| \max_{i=1,2,3}|\log|\alpha_i||}{\mathrm{Reg}(\alpha_1,\alpha_2)} \\ &:= \max_{i\neq j}\left|\log|\beta_i| - \log|\sigma^{i-1}\mu|\right| c_2 \\ &\leq \log|Y| c_2 \left(1 + \left|\frac{\log\left|\max_{i\neq j}\frac{|\gamma_i|}{|\sigma^{i-1}\mu|}\left(|\epsilon_j - \epsilon_i| + \frac{c_1}{Y_0^3}\right)\right|}{\log Y_0}\right|\right) \\ &:= \log|Y| c_3 \end{aligned} \tag{16}$$

We will compute the quantity c_3 in Section 5, when we have a better lower bound $Y_0 \leq Y$.

Now we will investigate Siegel's identity. Therefore choose $i,k \in \{1,2,3\}$ such that i,j,k are all pairwise distinct. We consider the quantity

$$\frac{\beta_i}{\gamma_i}(\epsilon_j - \epsilon_k) + \frac{\beta_j}{\gamma_j}(\epsilon_k - \epsilon_i) + \frac{\beta_k}{\gamma_k}(\epsilon_i - \epsilon_j) = 0.$$

Taking into account (13) and (14) we find after some manipulations that

$$\begin{aligned} \left|\frac{\frac{\beta_j}{\gamma_j}}{\frac{\beta_i}{\gamma_i}} \cdot \frac{\epsilon_k - \epsilon_i}{\epsilon_k - \epsilon_j}\right| &= \left|1 - \frac{\frac{\beta_k}{\gamma_k}}{\frac{\beta_i}{\gamma_i}} \cdot \frac{\epsilon_j - \epsilon_i}{\epsilon_j - \epsilon_k}\right| \\ &\leq \frac{c_1}{|Y|^2} \cdot \left|\frac{\epsilon_k - \epsilon_i}{\epsilon_k - \epsilon_j}\right| \cdot \frac{1}{|Y|(|\epsilon_j - \epsilon_i| - \frac{c_1}{Y_0^3})} := \frac{c_4}{|Y|^3}. \end{aligned} \tag{17}$$

By the asymptotic expansions (8), (9), (10) and (11) together with the bounds for c_1 and Lemma 4, we see that for any choice of i,j,k except $(i,j,k) = (3,2,1)$ we have $c_4 \leq 4.035a$ provided that $a \geq 1000$. In the exceptional case we get $c_4 \leq 4.055a$. Note that this exceptional case, will not occur in this paper.

5. A first bound for the parameter

In this section we will derive a first upper bound for a such that (4) has no primitive solution (X,Y) with $Y>1$. First we consider

$$\begin{aligned}\Lambda_{i,j,k} :=& \log\left|\frac{\frac{\beta_k}{\gamma_k}}{\frac{\beta_i}{\gamma_i}}\cdot\frac{\epsilon_j-\epsilon_i}{\epsilon_j-\epsilon_k}\right| \\ =& \log\left|\frac{\gamma_i}{\gamma_k}\cdot\frac{\epsilon_j-\epsilon_i}{\epsilon_j-\epsilon_k}\right| + b_1\log\left|\frac{\sigma^{k-1}\alpha}{\sigma^{i-1}\alpha}\right| + b_2\log\left|\frac{\sigma^{k}\alpha}{\sigma^{i}\alpha}\right| + \log\left|\frac{\sigma^{k-1}\mu}{\sigma^{i-1}\mu}\right|.\end{aligned}$$

From Siegel's identity (17) and the fact that $\log|x| < 2|1-x|$ provided that $|1-x|<1/3$ we obtain

$$|\Lambda_{i,j,k}| < 2\left|1-\frac{\frac{\beta_k}{\gamma_k}}{\frac{\beta_i}{\gamma_i}}\cdot\frac{\epsilon_j-\epsilon_i}{\epsilon_j-\epsilon_k}\right| \leq \frac{2c_4}{|Y|^3}. \tag{18}$$

Let $\theta_{i,j,k} := \frac{\gamma_i}{\gamma_k}\cdot\frac{\epsilon_j-\epsilon_i}{\epsilon_j-\epsilon_k}$. We want to write $\Lambda_{i,j,k}$ as a linear combination of the logarithms of $\theta_{i,j,k}\frac{\sigma^{k-1}\mu}{\sigma^{i-1}\mu}, \alpha_1$ and α_2. Therefore we have to distinguish between several cases. In particular, we consider the three linear forms:

$$\Lambda_1 := B_1\log|\alpha_1| + B_2\log|\alpha_2| + \log\left|\theta_{3,1,2}\frac{\sigma\mu}{\sigma^2\mu}\right| \qquad (i,j,k)=(3,1,2), \tag{19}$$

$$\Lambda_2 := B_1\log|\alpha_1| + B_2\log|\alpha_2| + \log\left|\theta_{1,2,3}\frac{\sigma^2\mu}{\mu}\right| \qquad (i,j,k)=(1,2,3), \tag{20}$$

$$\Lambda_3 := B_1\log|\alpha_1| + B_2\log|\alpha_2| + \log\left|\theta_{1,3,2}\frac{\sigma\mu}{\mu}\right| \qquad (i,j,k)=(1,3,2), \tag{21}$$

where

$$\begin{aligned} &B_1 := b_1-2b_2 && B_2 := 2b_1-b_2 && \text{in case of } \Lambda_1,\\ &B_1 := -2b_1+b_2 && B_2 := -b_1-b_2 && \text{in case of } \Lambda_2,\\ &B_1 := b_1+b_2 && B_2 := -b_1+2b_2 && \text{in case of } \Lambda_3.\end{aligned}$$

Let us find relations between B_2 and B. These will be used in view of (16). Below we will distinguish between the case of $B_1=0$ and $B_1\neq 0$. Let us consider case I: Since $B=\max\{|b_1|,|b_2|\}$ we have trivially $|B_2|\leq 3B$. If we assume $B_1=0$ then we have $b_1=2b_2$ and therefore $B=|b_1|$. Inserting this relation in the equation for B_2 we get $B_2=\frac{3}{2}b_1$, hence $|B_2|=\frac{3}{2}B$. The two other cases are similar and the relations are given in table 2.

Table 2. Relations between B and $|B_2|$.

	Case I	Case II	Case III
$B_1\neq 0$	$\lvert B_2\rvert \leq 3B$	$\lvert B_2\rvert \leq 2B$	$\lvert B_2\rvert \leq 3B$
$B_1=0$	$\lvert B_2\rvert = \frac{3}{2}B$	$\lvert B_2\rvert = \frac{3}{2}B$	$\lvert B_2\rvert = B$

We have to distinguish between 12 cases (three linear forms and for each linear form four possible choices for μ). Since all 12 cases can be treated similarly, we only consider the case of Λ_1 and μ being associated to α_2-1. We choose this case because it is representative for most of the other cases. The computed quantities for the other cases are presented in tables. To say that μ is associated to some quantity α we use the notation $\mu\sim\alpha$.

From (18) we have

$$\begin{aligned}|\Lambda_1| =& B_1\left(\log a - \frac{2}{a^2} + \frac{1}{a^3} + \frac{5}{a^4} - \frac{7}{a^5} + L\left(\frac{18.18370123}{a^6}\right)\right)\\ &+ B_2\left(-\frac{1}{a} + \frac{1}{2a^2} + \frac{5}{3a^3} - \frac{11}{4a^4} + L\left(\frac{11.035}{a^5}\right)\right) + \log\left|\theta_{3,1,2}\frac{\sigma\mu}{\sigma^2\mu}\right|\\ \le& \frac{c_4}{Y^3} \le \frac{220.1}{a^2}.\end{aligned}$$

From this inequality we see that B_2 has to be large, except when a cancellation of the two quantities $B_1 \log|\alpha_1|$ and $\log|\theta_{3,1,2}\sigma\mu/\sigma^2\mu|$ arises. We want to choose μ such that cancellation may only occur if $B_1 = 0$. Since $\theta_{3,1,2} = \log 2 + \cdots$ we have to choose μ such that $\mu \sim \alpha_2 - 1$ and $\sigma\mu/\sigma^2\mu = O(1)$. With this constraints we choose $\mu = (\alpha_2 - 1)\alpha_1$. The other choices for μ are given in table 3.

Table 3. Choices for μ.

	$\mu \sim 1$	$\mu \sim \alpha_1 - 1$	$\mu \sim \alpha_2 - 1$	$\mu \sim \alpha_3 - 1$
Case I	1	$\alpha_1 - 1$	$(\alpha_2 - 1)\alpha_1$	$\frac{\alpha_3-1}{\alpha_1}$
Case II	α_1	$\frac{\alpha_1-1}{\alpha_2}$	$\frac{(\alpha_2-1)\alpha_1}{\alpha_3}$	$(\alpha_3 - 1)\alpha_1$
Case III	$\frac{1}{\alpha_3}$	$(\alpha_1 - 1)\alpha_1$	$\frac{\alpha_2-1}{\alpha_3}$	$\frac{(\alpha_3-1)\alpha_2}{\alpha_3}$

Now we distinguish between two further cases: $B_1 = 0$ and $B_1 \neq 0$. In the case of $B_1 = 0$ we have

$$|\Lambda_1| = B_2\left(-\frac{1}{a} + \frac{1}{2a^2} + \frac{5}{3a^3} - \frac{11}{4a^4} + L\left(\frac{11.035}{a^5}\right)\right) + \log 2 - \frac{5}{a} - \frac{2}{a^2} + L\left(\frac{162.8341694}{a^3}\right) = L\left(\frac{220.1}{a^2}\right).$$

Solving this equation for B_2, we obtain

$$B_2 = a\log 2 + \frac{\log 2}{2} - 5 + L\left(\frac{233.7804338}{a}\right). \tag{22}$$

In the case of $B_1 \neq 0$ we similarly determine the quantity

$$\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{a\log 2 + \frac{\log 2}{2} - 5}{B_1} + L\left(\frac{46.920379\cdot\log a}{a}\right). \tag{23}$$

The results obtained in the other cases are listed in table 4.

Looking at table 4 we see that in the case of $B_1 = 0$ two different phenomena occur. In the cases I ($\mu \sim \alpha_3 - 1$), II ($\mu \sim 1$), II ($\mu \sim \alpha_1 - 1$), II ($\mu \sim \alpha_2 - 1$) and III ($\mu \sim \alpha_2 - 1$) the quantity B_2 is of the form constant plus some error term, while in the other cases B_2 is constant times $\log a$ plus lower terms. We are interested in the former cases. In case I ($\mu \sim \alpha_3 - 1$), II ($\mu \sim \alpha_2 - 1$) and III ($\mu \sim \alpha_2 - 1$) B_2 cannot be an integer if $a \ge 500$. However, by definition B_2 is an integer, so we have a contradiction. In the cases of II ($\mu \sim 1$) respectively II ($\mu \sim \alpha_1 - 1$) we have $B_2 = 1$ respectively $B_2 = 5$ provided $a \ge 500$. Therefore we have the following two linear systems:

$$\begin{aligned}-2b_1 + b_2 &= 0,\\ -b_1 - b_2 &= 1,\end{aligned} \qquad \text{and} \qquad \begin{aligned}-2b_1 + b_2 &= 0,\\ -b_1 - b_2 &= 5.\end{aligned}$$

Solving these systems we find $b_1 = -1/3, b_2 = -2/3$ and $b_1 = -5/3, b_2 = -10/3$. By definition b_1 and b_2 have to be integers, hence we have again a contradiction. Therefore we may exclude the cases I ($\mu \sim \alpha_3 - 1$), II ($\mu \sim 1$), II ($\mu \sim \alpha_1 - 1$), II ($\mu \sim \alpha_2 - 1$) and III ($\mu \sim \alpha_2 - 1$), if we assume $B_1 = 0$.

Table 4. The quantities B_2 and B_2/B_1.

Case I	$\mu \sim 1$	$B_2 = a\log 2 + \frac{\log 2}{2} - 1 + L\left(\frac{233.5726034}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{a\log 2 + \frac{\log 2}{2} - 1}{B_1} + L\left(\frac{46.89029255 \cdot \log a}{a}\right)$
	$\mu \sim \alpha_1 - 1$	$B_2 = a\log 4 + \log 2 - \frac{1}{2} + L\left(\frac{243.5541701}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{a\log 4 + \log 2 - \frac{1}{2}}{B_1} + L\left(\frac{48.33527238 \cdot \log a}{a}\right)$
	$\mu \sim \alpha_2 - 1$	$B_2 = a\log 2 + \frac{\log 2}{2} - 5 + L\left(\frac{233.7804338}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{a\log 2 + \frac{\log 2}{2} - 5}{B_1} + L\left(\frac{46.920379 \cdot \log a}{a}\right)$
	$\mu \sim \alpha_3 - 1$	$B_2 = -\frac{1}{2} + L\left(\frac{223.5783003}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} - \frac{1}{2B_1} + L\left(\frac{45.44346894 \cdot \log a}{a}\right)$
Case II	$\mu \sim 1$	$B_2 = 5 + L\left(\frac{225.5761744}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{5}{B_1} + L\left(\frac{45.7326909 \cdot \log a}{a}\right)$
	$\mu \sim \alpha_1 - 1$	$B_2 = 1 + L\left(\frac{221.7360355}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{1}{B_1} + L\left(\frac{45.17677378 \cdot \log a}{a}\right)$
	$\mu \sim \alpha_2 - 1$	$B_2 = \frac{15}{2} + L\left(\frac{231.7758252}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{15}{2B_1} + L\left(\frac{46.63018224 \cdot \log a}{a}\right)$
	$\mu \sim \alpha_3 - 1$	$B_2 = a\log 4 + \log 2 + \frac{9}{2} + L\left(\frac{248.3704756}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{a\log 4 + \log 2 + \frac{9}{2}}{B_1} + L\left(\frac{49.03250394 \cdot \log a}{a}\right)$
Case III	$\mu \sim 1$	$B_2 = a\log 2 + \frac{\log 2}{2} + 4 + L\left(\frac{237.8513408}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{a\log 2 + \frac{\log 2}{2} + 4}{B_1} + L\left(\frac{47.50970317 \cdot \log a}{a}\right)$
	$\mu \sim \alpha_1 - 1$	$B_2 = a\log 4 + \log 2 + \frac{1}{2} + L\left(\frac{244.3001410}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{a\log 4 + \log 2 + \frac{1}{2}}{B_1} + L\left(\frac{48.44326264 \cdot \log a}{a}\right)$
	$\mu \sim \alpha_2 - 1$	$B_2 = \frac{7}{2} + L\left(\frac{227.3839598}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{7}{2B_1} + L\left(\frac{45.99439458 \cdot \log a}{a}\right)$
	$\mu \sim \alpha_3 - 1$	$B_2 = a\log 2 + \frac{\log 2}{2} + 8 + L\left(\frac{242.3186056}{a}\right)$ $\frac{B_2}{B_1} = a\log a + \frac{\log a}{2} + \frac{a\log 2 + \frac{\log 2}{2} + 8}{B_1} + L\left(\frac{48.15640604 \cdot \log a}{a}\right)$

Next, we want to estimate the quantity c_3 and find a lower bound for $\log Y$. From (22) and (23) we find

$$B_2 = a\log 2 + \frac{\log 2}{2} - 5 + L\left(\frac{233.781}{a}\right) \geq 0.6883a \tag{24}$$

$$|B_2| = |B_1|\left(a\log a + \frac{\log a}{2}\right) + a\log 2 + \frac{\log 2}{2} - 5 + L\left(\frac{46.921 \cdot \log a}{a}\right) \geq 6.223a, \tag{25}$$

respectively. Let us estimate the quantity c_2. From (16) and (12) we find $c_2 \leq \frac{2.0006}{\log a}$. Now we are ready to estimate the quantity c_3. Put

$$\tilde{c} := 1 + \left| \frac{\log \left| \max_{i\neq j} \frac{|\gamma_i|}{|\sigma^i \mu|} \left(|\epsilon_j - \epsilon_i| + \frac{c_1}{|Y_0|^3} \right) \right|}{\log |Y_0|} \right|.$$

Using Lemma 3 together with the asymptotic expansions from Section 3 we obtain

$$\tilde{c} \leq 1 + \frac{0.5826}{\log a} - \frac{0.8405}{a\log a} + L\left(\frac{52.376}{a^3 \log a}\right)$$

and from the bound for c_2 we find

$$c_3 \leq \frac{2.006}{\log a} + \frac{1.1655}{(\log a)^2} - \frac{1.682}{a(\log a)^2} + L\left(\frac{104.782}{a^3(\log a)^2}\right) \leq \frac{2.169079894}{\log a}.$$

Since we have lower bounds for B_2, hence also for B, and upper bounds for c_3, using table 2 and inequality (16) we find that:

$$\log Y \geq 1.4612a \ \text{ if } \ B_1 = 0,$$
$$\log Y \geq 6.6053a \ \text{ if } \ B_1 \neq 0.$$

Computing again c_3 using this time instead of Lemma 3 the new bounds found for $\log Y$ we get "better" results. Iterating this procedure four times yields:

$$c_3 \leq \frac{2.00148}{\log a} \quad \text{and} \quad \log Y \geq 1.5836a \qquad \text{if } \ B_1 = 0, \ \text{respectively}$$
$$c_3 \leq \frac{2.0008}{\log a} \quad \text{and} \quad \log Y \geq 7.1609a \qquad \text{if } \ B_1 \neq 0.$$

The bounds for c_3 and $\log Y$ that are obtained in the other cases are listed in table 5 and table 6.

TABLE 5. Upper bounds for c_3.

$c_3 \leq$		$\mu \sim 1$	$\mu \sim \alpha_1 - 1$	$\mu \sim \alpha_2 - 1$	$\mu \sim \alpha_3 - 1$
Case I	$B_1 = 0$	$\frac{2.001471859}{\log a}$	$\frac{2.001035919}{\log a}$	$\frac{2.001474401}{\log a}$	/
	$B_1 \neq 0$	$\frac{2.000794053}{\log a}$	$\frac{2.000818748}{\log a}$	$\frac{2.000793370}{\log a}$	$\frac{2.002338185}{\log a}$
Case II	$B_1 = 0$	/	/	/	$\frac{2.009226921}{\log a}$
	$B_1 \neq 0$	$\frac{2.001760135}{\log a}$	$\frac{2.001759126}{\log a}$	$\frac{2.000705217}{\log a}$	$\frac{2.001890017}{\log a}$
Case III	$B_1 = 0$	$\frac{2.019731368}{\log a}$	$\frac{2.001472020}{\log a}$	/	$\frac{2.019611578}{\log a}$
	$B_1 \neq 0$	$\frac{2.002728648}{\log a}$	$\frac{2.000818944}{\log a}$	$\frac{2.002339951}{\log a}$	$\frac{2.002730579}{\log a}$

TABLE 6. Lower bounds for $\log Y$.

$\log Y \geq$		$\mu \sim 1$	$\mu \sim \alpha_1 - 1$	$\mu \sim \alpha_2 - 1$	$\mu \sim \alpha_3 - 1$
Case I	$B_1 = 0$	$1.5928a$	$3.1902a$	$1.5836a$	/
	$B_1 \neq 0$	$7.1563a$	$6.3575a$	$7.1609a$	$7.9477a$
Case II	$B_1 = 0$	/	/	/	$1.6026a$
	$B_1 \neq 0$	$11.915a$	$11.922a$	$13.112a$	$10.717a$
Case III	$B_1 = 0$	$0.7949a$	$1.5959a$	/	$0.8a$
	$B_1 \neq 0$	$7.1436a$	$6.3564a$	$7.9431a$	$7.1390a$

In the next step we use a powerful theorem on lower bounds for linear forms in two logarithms due to Laurent, Mignotte, and Nesterenko [8].

Lemma 5. *Let α_1 and α_2 be two multiplicatively independent elements in a number field of degree D over $\mathbb{Q}$. For $i = 1$ and $i = 2$, let $\log \alpha_i$ be any determination of the logarithm of α_i, and let $A_i > 1$ be a real number satisfying*

$$\log A_i \geq \max\{h(\alpha_i), |\log \alpha_i|/D, 1/D\},$$

where $h(\alpha_i)$ denotes the absolute logarithmic Weil height of α_i. Further, let b_1 and b_2 be two positive integers. Define

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1} \quad \text{and} \quad \log b = \max\left\{\log b', 21/D, \frac{1}{2}\right\}.$$

Then

$$|b_2 \log \alpha_2 - b_1 \log \alpha_1| \geq \exp\left(-30.9 D^4 (\log b)^2 \log A_1 \log A_2\right).$$

Before we apply this result we have to compute some heights:

Lemma 6. *Let h denote the absolute logarithmic Weil height, then*

$$h(\alpha_1) = h(\alpha_2) = h(\alpha_3) \leq \frac{\log a}{3} \tag{26}$$

and

$$h\left(\theta_{3,1,2}\frac{\sigma\mu}{\sigma^2\mu}\right) \leq \frac{4\log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.047}{a^3} \quad (a \geq 1000), \tag{27}$$

where $\mu = (\alpha_2 - 1)\alpha_1$. The estimations for $H := h\left(\theta_{i,j,k}\frac{\sigma^{k-1}\mu}{\sigma^{i-1}\mu}\right)$ in the other cases are given in table 7.

TABLE 7. Estimations for the absolute logarithmic Weil height $H := h\left(\theta_{i,j,k}\cdot\frac{\sigma^{k-1}\mu}{\sigma^{i-1}\mu}\right)$

Case I	$\mu \sim 1$	$H \leq \frac{4\log a}{3} + \frac{\log 2}{3} + \frac{1}{3a} + \frac{5}{3a^2} + \frac{90.0595}{a^3}$
	$\mu \sim \alpha_1 - 1$	$H \leq \frac{4\log a}{3} + \frac{\log 8}{3} - \frac{1}{3a} + \frac{9}{4a^2} + \frac{83.3557}{a^3}$
	$\mu \sim \alpha_2 - 1$	$H \leq \frac{4\log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.0466}{a^3}$
	$\mu \sim \alpha_3 - 1$	$H \leq \frac{4\log a}{3} + \frac{\log 4}{3} - \frac{4}{3a} + \frac{35}{12a^2} + \frac{146.3174}{a^3}$
Case II	$\mu \sim 1$	$H \leq \log a + \frac{\log 2}{3} + \frac{14.6473}{a^2}$
	$\mu \sim \alpha_1 - 1$	$H \leq \frac{5\log a}{3} + \frac{\log 8}{3} - \frac{1}{a} + \frac{23}{12a^2} + \frac{187.7049}{a^3}$
	$\mu \sim \alpha_2 - 1$	$H \leq \frac{4\log a}{3} + \frac{7}{6a} + \frac{39}{8a^2} + \frac{301.579}{a^3}$
	$\mu \sim \alpha_3 - 1$	$H \leq \frac{5\log a}{3} + \frac{\log 4}{3} + \frac{1}{2t} + \frac{37}{24a^2} + \frac{120.6103}{a^3}$
Case III	$\mu \sim 1$	$H \leq \log a + \frac{4}{3a} + \frac{10}{3a^2} + \frac{92.4204}{a^3}$
	$\mu \sim \alpha_1 - 1$	$H \leq \frac{4\log a}{3} + \frac{\log 8}{3} - \frac{5}{3a} + \frac{35}{12a^2} + \frac{97.6092}{a^3}$
	$\mu \sim \alpha_2 - 1$	$H \leq \frac{5\log a}{3} + \frac{5}{3a} + \frac{17}{6a^2} + \frac{101.7132}{a^3}$
	$\mu \sim \alpha_3 - 1$	$H \leq \frac{5\log a}{3} + \frac{\log 4}{3} + \frac{5}{3a} + \frac{25}{12a^2} + \frac{232.4536}{a^3}$

Proof. We start with the proof of (26). Since $\alpha_1, \alpha_2, \alpha_3$ are conjugate, we only have to check the last inequality.

$$h(\alpha_1) = \frac{1}{3}\left(\sum_{i=1,2,3} \max(0, \log|\alpha_i|)\right) =$$
$$\frac{1}{3}\left(\log a - \frac{1}{a} - \frac{3}{2a^2} + \frac{8}{3a^3} + L\left(\frac{2.27}{a^4}\right)\right) \leq \frac{\log a}{3},$$

therefore we obtain the first part of the lemma.

Since $\theta_{3,1,2}$ and $\frac{\sigma\mu}{\sigma^2\mu}$ are not integers in general we also have to compute their denominators, which can be estimated by

$$\Delta_\theta := \mathrm{N}_{K/\mathbb{Q}}\left(\gamma_1(\epsilon_2 - \epsilon_1)\right) = a^2 + 2a - 13 \quad \text{respectively,}$$
$$\Delta_\mu := \mathrm{N}_{K/\mathbb{Q}}(\alpha_2 - 1) = 2a + 1.$$

With this preliminary result we obtain

$$h\left(\theta_{3,1,2}\frac{\sigma\mu}{\sigma^2\mu}\right) \leq \frac{1}{3}\left(\log(\Delta_\theta\Delta_\mu) + \sum_{j=1,2,3} \max\left(0, \log\left|\sigma^j\left(\theta_{3,1,2}\frac{\sigma\mu}{\sigma^2\mu}\right)\right|\right)\right) =$$
$$\frac{4\log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + L\left(\frac{190.047}{a^3}\right)$$

□

Now we apply Lemma 5 to the linear form (19). We distinguish between the case of $B_1 = 0$ and $B_1 \neq 0$. In the case of $B_1 = 0$ we can apply Lemma 5 at once. In the notation of Lemma 5 we have

$$\begin{aligned} b' =& \frac{1}{\log a} + \frac{B_2}{4\log a + \log 2 - \frac{1}{a} + \frac{9}{a^2} + \frac{570.141}{a^3}} \\ \leq& \frac{1}{\log a} + \frac{a\log 2 + \frac{\log 2}{2} - 5 + \frac{233.781}{a}}{4\log a + \log 2 - \frac{1}{a} + \frac{9}{a^2} + \frac{570.141}{a^3}} \\ \leq& \frac{a}{\log a} 0.16898 \end{aligned}$$

Inserting the various bounds we obtain

$$\begin{aligned} \log|\Lambda_1| >& -834.3(\log a - \log\log a - 1.778)^2 \log a \\ & \times \left(\frac{4\log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.047}{a^3}\right). \end{aligned}$$

On the other hand we have from (18)

$$\begin{aligned} \log|\Lambda_1| < \log\frac{2c_4}{Y^3} <& \log(8.07a) - 0.99926 \\ & \times \left(a\log 2 + \frac{\log 2}{2} - 5 - \frac{233.781}{a}\right)\log a. \end{aligned}$$

Comparing the upper and lower bound for $\log|\Lambda_1|$ yields a contradiction for large a. In particular, if $a \geq 2529022.366$ we have a contradiction. Since a has to be an integer we know that we may have solutions with $|Y| \geq 2$ only if $a \leq a_0 := 2529022$.

Now we investigate the case $B_1 \neq 0$. In this case we do not have a linear form in two logarithms. But we can study the linear form

$$\Lambda_1 = \log\left(\alpha_1^{B_1}\theta_{3,1,2}\frac{\sigma\mu}{\sigma^2\mu}\right) + B_2\log\alpha_2.$$

Since $h(xy) \leq h(x) + h(y)$ we have $h\left(\alpha_1^{B_1}\theta_{3,1,2}\frac{\sigma\mu}{\sigma^2\mu}\right) \leq |B_1|h(\alpha_1) + h\left(\theta_{3,1,2}\frac{\sigma\mu}{\sigma^2\mu}\right)$ and because of Lemma 6 we choose

$$\begin{aligned} b' =& \frac{1}{\log a} + \frac{|B_2|}{|B_1|\log a + 4\log a + \log 2 - \frac{1}{a} + \frac{9}{a^2} + \frac{570.141}{a^3}} \\ \leq& \frac{1}{\log a} + \frac{|B_2|}{|B_1|\log a} \\ \leq& \frac{1}{\log a} + \frac{a\log a + \frac{\log a}{2} + a\log 2 + \frac{\log 2}{2} - 5 + \frac{46.921\cdot\log a}{a}}{\log a} \leq 1.10037a \end{aligned}$$

By Lemma 5 we find

$$\begin{aligned} \log|\Lambda_1| >& -834.3(\log a + 0.0957)^2\log a \\ & \times \left(\frac{|B_1|\log a}{3} + \frac{4\log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.05}{a^3}\right) \\ \geq& -834.3(\log a + 0.0957)^2\log a|B_2|\frac{|B_1|}{|B_2|} \\ & \times \left(\frac{5\log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.05}{a^3}\right) \\ >& -834.3\frac{(\log a + 0.0957)^2\log a|B_2|\left(\frac{5}{3}\log a + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.05}{a^3}\right)}{a\log a + \frac{\log a}{2} - a\log 2 - \frac{\log 2}{2} + 5 - \frac{46.921\cdot\log a}{a}} \end{aligned}$$

On the other hand

$$\begin{aligned}\log|\Lambda_1| &< \log 2c_4 - 3\log Y \le \log 8.07 + \log a - \frac{3B}{c_3}\\ &\le |B_2|\left(\frac{\log 8.07+\log a}{B_2} - \frac{1}{c_3}\right)\\ &\le |B_2|\left(\frac{\log 8.07+\log a}{a\log a + \frac{\log a}{2} - a\log 2 - \frac{\log 2}{2} + 5 - \frac{46.921\cdot\log a}{a}} - \frac{\log a}{2.000793370}\right)\end{aligned}$$

If we compare these bounds for $\log|\Lambda_1|$ we see that $|B_2|$ cancels, and we obtain an inequality which cannot hold for $a \ge 521855.0066$. That is, if there is a solution not found yet for this case, then $a \le a_0 := 521855$.

In table 8 one finds the other upper bounds a_0 of the parameter a for the remaining cases.

TABLE 8. Upper bounds a_0 for the parameter a.

		$\mu \sim 1$	$\mu \sim \alpha_1 - 1$	$\mu \sim \alpha_2 - 1$	$\mu \sim \alpha_3 - 1$
Case I	$B_1 = 0$	$a_0 = 2532736$	$a_0 = 1226494$	$a_0 = 2529022$	/
	$B_1 \ne 0$	$a_0 = 521904$	$a_0 = 579982$	$a_0 = 521855$	$a_0 = 487789$
Case II	$B_1 = 0$	/	/	/	$a_0 = 3259385$
	$B_1 \ne 0$	$a_0 = 229399$	$a_0 = 377086$	$a_0 = 270366$	$a_0 = 405414$
Case III	$B_1 = 0$	$a_0 = 4655030$	$a_0 = 3059080$	/	$a_0 = 8157825$
	$B_1 \ne 0$	$a_0 = 397229$	$a_0 = 579994$	$a_0 = 590044$	$a_0 = 651927$

By table 8 we have:

Proposition 1. *There are no other solutions to* (3) *than those listed in Theorem 1 if* $a > 8157825$.

6. THE METHOD OF MIGNOTTE

In this section we want to eliminate the case of $B_1 = 0$. We have already discussed the cases I ($\mu \sim \alpha_3 - 1$), II ($\mu \sim 1$), II ($\mu \sim \alpha_1 - 1$), II ($\mu \sim \alpha_2 - 1$) and III ($\mu \sim \alpha_2 - 1$). We know that B_2 has to be an integer therefore let us compute B_2 to a higher asymptotic order (in the remaining cases):

$$\begin{aligned}
B_2 =& a\log 2 - \frac{2-\log 2}{2} - \frac{54-23\log 2}{12a} + L\left(\frac{9.4241}{a^2} + 8.075a\, e^{-4.7784a}\right) \\
& \text{case I } (\mu \sim 1) \\
B_2 =& a\log 4 - \frac{1-\log 4}{2} - \frac{135-46\log 2}{24a} + L\left(\frac{8.528}{a^2} + 8.075a\, e^{-9.5706a}\right) \\
& \text{case I } (\mu \sim \alpha_1 - 1) \\
B_2 =& a\log 2 - \frac{10-\log 2}{2} - \frac{54-23\log 2}{12a} + L\left(\frac{11.4221}{a^2} + 8.075a\, e^{-4.7508a}\right) \\
& \text{case I } (\mu \sim \alpha_2 - 1) \\
B_2 =& a\log 2 + \frac{11+\log 2}{2} - \frac{27-46\log 2}{24a} + L\left(\frac{24.2511}{a^2} + 8.075a\, e^{-4.8078a}\right) \\
& \text{case II } (\mu \sim \alpha_3 - 1) \\
B_2 =& a\log 2 + \frac{8+\log 2}{2} - \frac{54-23\log 2}{12a} + L\left(\frac{13.9461}{a^2} + 8.075a\, e^{-2.3847a}\right) \\
& \text{case III } (\mu \sim 1) \\
B_2 =& a\log 4 + \frac{1+\log 4}{2} - \frac{135-46\log 2}{24a} + L\left(\frac{14.1731}{a^2} + 8.075a\, e^{-4.7877a}\right) \\
& \text{case III } (\mu \sim \alpha_1 - 1) \\
B_2 =& a\log 2 + \frac{16-\log 2}{2} - \frac{54-23\log 2}{12a} + L\left(\frac{15.9481}{a^2} + 8.075a\, e^{-2.4a}\right) \\
& \text{case III } (\mu \sim \alpha_3 - 1)
\end{aligned}$$

Since B_2 has to be an integer, for each case we have a criteria wether there exists a solution such that $B_1 = 0$ for one specific a. For example, the case I $(\mu \sim \alpha_2 - 1)$ yields following criteria:

Lemma 7. *Let $\|\cdot\|$ denote the distance to the nearest integer. If (3) has a solution, which is not found yet, that coresponds to the case I $(\mu \sim \alpha_2 - 1)$ such that $B_1 = 0$, then*

$$\left\| a\log 2 - \frac{10-\log 2}{2} - \frac{54-23\log 2}{12a} \right\| \leq \frac{11.4221}{a^2} + 8.075ae^{-4.7508a}.$$

The other cases yield similar criteria. Therefore, in the case of $B_1 = 0$ and I $(\mu \sim 1)$, I $(\mu \sim \alpha_1 - 1)$, I $(\mu \sim \alpha_2 - 1)$, II $(\mu \sim \alpha_3 - 1)$, III $(\mu \sim 1)$, III $(\mu \sim \alpha_1 - 1)$ or III $(\mu \sim \alpha_3 - 1)$ we check for each $1000 \leq a \leq a_0$ wether the corresponding criteria is fulfilled or not. A computation in MAGMA (see Section 8) yields:

Proposition 2. *If (X, Y) is a solution to (4) with $Y \geq 1$ which yields a solution to (3) that is not listed in Theorem 1, then $a \leq 651957$. Moreover the solution (X, Y) yields $B_1 \neq 0$ or $a < 1000$.*

Remark 1. This method is called Mignotte‘s method, because Mignotte [11] used a similar trick to solve the family of Thue equations

$$X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3 = 1$$

completely.

7. The method of Baker and Davenport

We cannot use the method described above to solve the case of $B_1 \neq 0$, because we have found an upper bound for the quantity $\frac{B_2}{B_1}$ but not for B_2 itself, which would be essential. So we are forced to use another method. We choose the method of Baker and Davenport [1]. In particular we adapt a lemma of Mignotte, Pethő and Roth [12] to our needs.

In order to use the method of Baker and Davenport, we have to find an absolute lower bound for B_2. Therefore we have to revise the linear forms Λ_1, Λ_2 and Λ_3. This time we do not consider them as linear combinations of two logarithms but as three logarithms. So we cannot use the theorem of Laurent, Mignotte and Nesterenko [8] and have to apply a result of Matveev [10]:

Lemma 8. *Denote by $\alpha_1, \dots, \alpha_n$ algebraic numbers, not 0 or 1, by $\log\alpha_1, \dots, \log\alpha_n$ determinations of their logarithms, by D the degree over $\mathbb{Q}$ of the number field $\mathbb{K} = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$, and by $b_1, \dots, b_n$ rational integers. Furthermore let $\kappa = 1$ if $\mathbb{K}$ is real and $\kappa = 2$ otherwise. Define*

$$\log A_i = \max\{Dh(\alpha_i), |\log\alpha_i|\} \quad (1 \le i \le n),$$

where $h(\alpha)$ denotes the absolute logarithmic Weil height of α and

$$B^* = \max\{1, \max\{|b_j| A_j/A_n : 1 \le j \le n\}\}.$$

Assume that $b_n \neq 0$ and $\log\alpha_1, \dots, \log\alpha_n$ are linearly independent over $\mathbb{Z}$; then

$$\log|\Lambda| \ge -C(n) C_0 W_0 D^2 \Omega,$$

with

$$\Omega = \log(A_1) \cdots \log(A_n),$$
$$C(n) = C(n,\kappa) = \frac{16}{n!\kappa} e^n (2n+1+2\kappa)(n+2)(4(n+1))^{n+1} \left(\frac{1}{2}en\right)^{\kappa},$$
$$C_0 = \log\left(e^{4.4n+7} n^{5.5} D^2 \log(eD)\right), \quad W_0 = \log(1.5eB^* D\log(eD)).$$

We already have computed all relevant heights in Lemma 6 respectively table 7. We combine Siegel‘s identity (17) with Matveev‘s lower bound (Lemma 8) and obtain for our standard case I ($\mu \sim \alpha_2 - 1$):

$$\frac{|B_2|\log a}{2.000793370} - \log 8.07 - \log a < 1.691497 \cdot 10^{11} (\log a)^2 \left(\frac{4\log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.047}{a^3}\right) \log(2.26688|B_2|). \tag{28}$$

The only not straightforward step is to compute B^*. Therefore let us rearrange the terms of Λ_j such that the term $\theta_{i,j,k}\frac{\sigma^{k-1}\mu}{\sigma^{i-1}\mu}$ is the last one. Since in any case $|B_2| > |B_1|$ and $|B_2| > a \ge 1000$ we have $B^* = |B_2| \frac{\log a}{4\log a + \log 2 + \cdots} \le \frac{|B_2|}{4}$. The inequality (28) yields a contradiction if $|B_2|$ is large, i.e. $|B_2| \ge c_5$, where c_5 is some quantity depending on a. In view of an absolute lower bound for $|B_2|$ the “worst” case occurs, if a is as large as possible. Therefore we insert a_0 instead of a into the inequality above and by solving this inequality we obtain $|B_2| > 8.93 \cdot 10^{15}$. The lower bounds for $|B_2|$ in the other cases can be found in table 9.

Table 9. Absolute lower bounds for $|B_2|$

$\lvert B_2\rvert >$	$\mu \sim 1$	$\mu \sim \alpha_1 - 1$	$\mu \sim \alpha_2 - 1$	$\mu \sim \alpha_3 - 1$
Case I	$8.92 \cdot 10^{15}$	$9.31 \cdot 10^{15}$	$8.92 \cdot 10^{15}$	$8.95 \cdot 10^{15}$
Case II	$3.88 \cdot 10^{15}$	$7.12 \cdot 10^{15}$	$5.22 \cdot 10^{15}$	$7.13 \cdot 10^{15}$
Case III	$6.33 \cdot 10^{15}$	$9.31 \cdot 10^{15}$	$1.12 \cdot 10^{16}$	$1.16 \cdot 10^{16}$

Now we find by the method of Baker and Davenport [1] criteria for which there are no solutions.

Lemma 9. *Suppose $1000 \le a \le a_0$ and put*

$$\delta_1 := \frac{\log\left|\theta_{i,j,k}\frac{\sigma^{k-1}\mu}{\sigma^{i-1}\mu}\right|}{\log|\alpha_2|} \quad \text{and} \quad \delta_2 := \frac{\log|\alpha_1|}{\log|\alpha_2|},$$

where i and k are chosen according to (19), (20) *and* (21). *Further let $\tilde{\delta}_1$ and $\tilde{\delta}_2$ be rationals such that*

$$|\delta_1 - \tilde{\delta}_1| < 10^{-60} \quad \text{and} \quad |\delta_2 - \tilde{\delta}_2| < 10^{-60}$$

and assume there exists a convergent p/q in the continued fraction expansion of δ_2, with $q \le 10^{30}$ and

$$q\|q\tilde{\delta}_1\| > 1.0001 + \frac{c_6}{a \log a},$$

then there is no solution for the case corresponding to j, μ and $B_1 \neq 0$. The quantities c_6 are listed in table 10.

TABLE 10. Absolute lower bounds for $|B_2|$

$c_6 =$	$\mu \sim 1$	$\mu \sim \alpha_1 - 1$	$\mu \sim \alpha_2 - 1$	$\mu \sim \alpha_3 - 1$
Case I	$1.9831 \cdot 10^{16}$	$2.329 \cdot 10^{16}$	$1.9818 \cdot 10^{16}$	$1.7907 \cdot 10^{16}$
Case II	$7.7806 \cdot 10^{15}$	$1.425 \cdot 10^{16}$	$9.5035 \cdot 10^{15}$	$1.5862 \cdot 10^{16}$
Case III	$1.4082 \cdot 10^{16}$	$2.2395 \cdot 10^{16}$	$2.2459 \cdot 10^{16}$	$2.5916 \cdot 10^{16}$

Proof. We give the details for our standard case I ($\mu \sim \alpha_2 - 1$). The other cases are similar.

Assume that there is a solution corresponding to case I ($\mu \sim \alpha_2 - 1$) such that $B_1 \neq 1$. From (18) we have

$$|\delta_1 + B_1\delta_2 + B_2| \le \frac{2c_4}{|Y_0|^3 \log |\alpha_2|} \le \frac{8.075a^2}{\exp(21.4827a)} < 10^{-1000}.$$

Multiplication by q yields

$$|q\tilde{\delta}_1 + q(\delta_1 - \tilde{\delta}_1) + B_1(\tilde{\delta}_2 q - p) + B_1 q(\delta_2 - \tilde{\delta}_2) + B_1 p + B_2 q| < 10^{-970}$$

and therefore

$$\|q\tilde{\delta}_1\| < 10^{-970} + q10^{-60} + |B_1||\tilde{\delta}_2 q - p| + |B_1|q10^{-60}.$$

By another multiplication with q we get

$$\begin{aligned} q\|q\tilde{\delta}_1\| &< 10^{-940} + q^2 10^{-60} + |B_1|q|\tilde{\delta}_2 q - p| + |B_1|q^2 10^{-60} \\ &< 1 + 10^{-940} + 2|B_1|. \end{aligned}$$

From table 4 and table 9 we obtain:

$$q\|q\tilde{\delta}_1\| < 1.0001 + \frac{2|B_2|}{0.8989002219a \log a} < 1.0001 + \frac{1.9818 \cdot 10^{16}}{a \log a}.$$

□

Using Lemma 9 we find:

Proposition 3. *There are no primitive solutions (X, Y) to* (4) *with $Y > 1$, provided $a \ge 1000$.*

Proof. In each case and each μ from table 3 we check by computer for each value of a in question whether the criteria given in Lemma 9 is fulfilled or not. Combining the result of this computer search with Proposition 2 we obtain the statement of the proposition. For more details on the implementation see Section 8. □

By part (5) of Lemma 2 and Proposition 3 it is left to solve the Thue equations

$$\begin{aligned} &X^3(a^2 + a + 7) - XY^2(a^2 + a + 7) - Y^3(2a + 1) = \pm 1, \\ &X^3(a^2 + a + 7) - XY^2(a^2 + a + 7) - Y^3(2a + 1) = \pm(2a + 1), \end{aligned}$$

for $0 \le a \le 999$. Solving these 3996 Thue equations with PARI yields no further solution. Therefore we have proved our main Theorem 1.

8. Computer Search

The computations needed to prove Proposition 2 via Lemma 7 and to prove Proposition 3 via Lemma 9 were implemented in MAGMA. The running times on an Intel Xeon PIII 700MHz processor are collected in table 11.

Finally, we have solved the corresponding equations in the case $0 \leq a \leq 999$ both in MAGMA and in PARI. For references concerning the computer algebra packages used in this work see [4], [15] and [13].

Table 11. Running times in seconds.

		$\mu \sim 1$	$\mu \sim \alpha_1 - 1$	$\mu \sim \alpha_2 - 1$	$\mu \sim \alpha_3 - 1$
Case I	$B_1 = 0$	4891	2363	4884	⁄
	$B_1 \neq 0$	5372	6020	5405	4879
Case II	$B_1 = 0$	⁄	⁄	⁄	6279
	$B_1 \neq 0$	2276	3764	2793	4192
Case III	$B_1 = 0$	8972	6097	⁄	15741
	$B_1 \neq 0$	4889	6627	5908	6766

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