

On the solutions of a class of iterated generalized Bers-Vekua equations in Clifford analysis

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Abstract

We consider functions with values in the Clifford algebra $Cl_{p,q}$ which are solutions of a certain class of the iterated generalized Bers-Vekua equation $D^m w = 0$ with $Dw = \partial w + c\bar{w}$ where $\partial = \sum_{j=0}^n e_j \partial/\partial x_j$ is the generalized Cauchy-Riemann operator. We prove that any such function w has a Almasi-type decomposition of the form $w = v_0 + x_0 v_1 + \dots + x_0^{m-1} v_{m-1}$ where $x = x_0 + x_1 e_1 + \dots + x_n e_n$, and the functions $v_j, j = 0, 1, \dots, m-1$, are solutions of the generalized Bers-Vekua equation $Dv = 0$.

Key words: iterated generalized Cauchy-Riemann operator; iterated generalized Bers-Vekua operator; polynomogenic functions; differential operators of Bauer-type

1 Introduction

Let $Cl_{p,q}$ with $p+q=n$ be the 2^n -dimensional Clifford algebra generated by the elements e_1, \dots, e_n , which obey the multiplication rules

$$e_j^2 = e_0, j = 1, \dots, p, \quad e_j^2 = -e_0, j = p+1, \dots, n \quad \text{and} \quad e_i e_j + e_j e_i = 0 \text{ for } i < j,$$

where e_0 denotes the identity of the algebra. An arbitrary element a of $Cl_{p,q}$ is given by

$$a = \sum_A \lambda_A e_A, \quad \lambda_A \in \mathbb{R}$$

with $e_A = e_{\alpha_1} e_{\alpha_2} \dots e_{\alpha_k}, \alpha_1, \dots, \alpha_k \in \{1, \dots, n\}$ and $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k \leq n$.

The conjugation is defined by $\bar{a} = \sum_A \lambda_A \bar{e}_A$ where $\bar{e}_A = \bar{e}_{\alpha_k} \dots \bar{e}_{\alpha_1}$ and $\bar{e}_0 = e_0, \bar{e}_j = -e_j, j = 1, \dots, n$, holds.

Now an element x in \mathbb{R}^{n+1} with the coordinates x_0, x_1, \dots, x_n is identified with

$$x = x_0 + x_1 e_1 + \dots + x_n e_n$$

and the conjugate of x is $\bar{x} = x_0 - x_1 e_1 - \dots - x_n e_n$.

In \mathbb{R}^{n+1} the generalized Cauchy-Riemann operator is defined by

$$\partial = \frac{\partial}{\partial x_0} + \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$$

and the conjugate operator $\bar{\partial}$ is given by

$$\bar{\partial} = \frac{\partial}{\partial x_0} - \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$$

These operators act on the space $C^1(\Omega, Cl_{p,q})$ where Ω denotes a domain in \mathbb{R}^{n+1} .

Using such a Clifford algebra in a multidimensional space, elliptic, hyperbolic and parabolic differential equations can be considered in an inductive way (cf. e.g. [1]).

A function u is said to be (left) Clifford holomorphic in Ω if it is a solution of the differential equation

$$\partial u = 0$$

Here we consider the generalized Bers-Vekua operator D defined by

$$Dv := \partial v + c \bar{v}$$

where c is a real valued function of x_0 .

In the algebra of complex quaternions H. Malonek [2] investigated some classes of generalized Bers-Vekua equations. B. Goldschmidt [3] presented regularity properties of generalized analytic vectors a particular subset of which is described by $Dv = 0$. In [4] a similar operator was used to describe pseudoanalytic functions in the space, whereas in a forthcoming paper of the author for $c(x_0) = k/x_0, k \in \mathbb{Z}$, a representation theorem for the solutions of $Dv = 0$ is proved using certain differential operators of Bauer-type acting on Clifford holomorphic functions.

Here we study the iterated operator D^m with $m \in \mathbb{N}$ and $D^m f = D(D^{m-1}f)$ and $D^0 f = f$. There exists a close connection between the solutions of the iterated generalized Bers-Vekua equation

$$D^m w = 0 \tag{1}$$

and the solutions of $Dv = 0$. This result generalizes the representation of the solutions of the iterated generalized Cauchy-Riemann equation

$$\partial^m u = 0 \tag{2}$$

(sometimes called (k) -monogenic or polymonogenic functions) given in [5] (see also [6]) and further the classical Almansi theorem [7] for polyharmonic functions.

Supposing that there exists a representation of the solutions of $Dv = 0$ by means of certain differential operators as it was discussed e.g. in the forthcoming article we prove a close connection between the solutions of (1) and the solutions of (2).

2 An Almansi-type decomposition

Proposition 1. *For any real valued, continuously differentiable function $\varphi(x_0)$ of the variable x_0 and for each function $\tilde{w} \in C^1(\Omega, Cl_{p,q})$ we have*

$$(i) \quad \partial(\varphi(x_0) \tilde{w}) = \varphi'(x_0) \tilde{w} + \varphi(x_0) (\partial \tilde{w})$$

$$(ii) \quad D(\varphi(x_0) \tilde{w}) = \varphi'(x_0) \tilde{w} + \varphi(x_0) (D \tilde{w})$$

(iii) *If $u \in C^1(\Omega, Cl_{p,q})$ is a solution of $Du = 0$ then αu with $\alpha \in \mathbb{R}$ is a solution of $Du = 0$ also.*

Proof. Using the relation $\tilde{w} \varphi(x_0) = \varphi(x_0) \tilde{w}$ which is true for any real valued function φ and all $\tilde{w} \in C(\Omega, Cl_{p,q})$ the assertions can be proved by direct calculation. \square

To find representations for the solutions of the iterated equation (1) we first choose a new function w_1 as

$$w_1 := D^{m-1}w - \frac{1}{x_0} D^{m-2}w$$

which obeys the differential equation

$$Dw_1 + \frac{1}{x_0} w_1 = 0 \tag{3}$$

This can be proved immediately using proposition 1(ii) and equation (1). Further we define the functions

$$w_k := D^{m-k}w - \frac{k}{x_0} D^{m-k-1}w, \quad k = 2, 3, \dots, m-1 \tag{4}$$

for which the relations

$$Dw_k + \frac{1}{x_0} w_k = w_{k-1}, \quad k = 2, 3, \dots, m-1 \tag{5}$$

hold. With $k = m-1$ equation (4) leads to

$$w_{m-1} = Dw - \frac{m-1}{x_0} w \tag{6}$$

Setting $w_m := w$ we get from (3), (5) and (6) a system of m differential equations for the functions $w_k, k = 1, \dots, m$. With the vector $W := (w_1, \dots, w_m)^t$ and the matrices $A = (a_{jk})$ with

$$a_{jk} = \begin{cases} 1/x_0 & \text{for } 1 \leq j = k \leq m-1 \\ (1-m)/x_0 & \text{for } j = k = m \\ -1 & \text{for } j = k+1, 1 \leq k \leq m-1 \\ 0 & \text{else} \end{cases}$$

and $B = (b_{jk})$ with

$$b_{jk} = \begin{cases} c & \text{for } 1 \leq j = k \leq m \\ 0 & \text{else} \end{cases}$$

this system can be given in matrix notation as

$$\partial W + AW + B\bar{W} = 0 \quad (7)$$

Writing ∂W means the application of the operator ∂ to each component of W .

Let $P = (p_{jk})$ denote a $m \times m$ -matrix the components of which are real valued, continuously differentiable functions of x_0 . By the transformation $W = PU$ in connection with proposition 1(i) equation (7) leads to the relation

$$P(\partial U) + (P' + AP)U + PB\bar{U} = 0$$

for the new unknown vector $U = (u_1, \dots, u_m)^t$.

The request $P' + AP = 0$ can be satisfied by the matrix P with

$$p_{jk} = \begin{cases} \frac{x_0^{j-k-1}}{(j-k)!} & \text{for } 1 \leq k \leq j, 1 \leq j \leq m-1 \\ 0 & \text{for } j < k \leq m, 1 \leq j \leq m-1 \\ -\frac{1}{k} \frac{x_0^{m-k-1}}{(m-k-1)!} & \text{for } j = m, 1 \leq k \leq m-1 \\ x_0^{m-1} & \text{for } j = k = m \end{cases}$$

which is nonsingular since $\det P = 1$. Thus for the vector U we have the simple system

$$\partial U + c\bar{U} = 0$$

which states that each component u_k of the vector U obeys the generalized Bers-Vekua equation

$$Du_k = \partial u_k + c\bar{u}_k = 0, \quad k = 1, \dots, m$$

The focal question is which form has the function $w = w_m$, the m -th component of the vector W . From $W = PU$ we have

$$w \equiv w_m = \sum_{k=1}^m p_{mk} u_k = \sum_{k=1}^{m-1} -\frac{1}{k} \frac{x_0^{m-k-1}}{(m-k-1)!} u_k + x_0^{m-1} u_m$$

Finally with proposition 1(iii) the function w can be written in the form

$$w = \sum_{k=0}^{m-1} x_0^k v_k \quad (8)$$

where the v_k are solutions of $Dv_k = 0$ and we have the following

Theorem 1. 1. Let the functions $v_k, k = 0, 1, \dots, m-1$, be solutions of the generalized Bers-Vekua equation $Dv_k = 0$ in Ω . Then the function w according to (8) represents a solution of the iterated generalized Bers-Vekua equation $D^m w = 0$ in Ω .

2. For each solution w of (1) defined in Ω there exist solutions $v_k, k = 0, 1, \dots, m-1$, of $Dv_k = 0$ in Ω such that w can be written in the form (8).

To bring out the connection between a solution w of (1) and the functions v_k in the representation (8) we first prove the relation

$$D^l w = \sum_{k=l}^{m-1} \frac{k!}{(k-l)!} x_0^{k-l} v_k, \quad l = 0, 1, \dots, m-1 \quad (9)$$

This can be considered as a linear system of m equations for the functions $v_k, k = 0, \dots, m-1$, which has the form

$$M V = \hat{W} \quad \text{with} \quad V = (v_0, \dots, v_{m-1})^t \quad \text{and} \quad \hat{W} = (w, Dw \dots, D^{m-1}w)^t$$

The coefficient matrix $M = (m_{ik})$ with

$$m_{ik} = \begin{cases} \frac{(k-1)!}{(k-i)!} x_0^{k-i} & \text{for } i \leq k \\ 0 & \text{for } i > k \end{cases}$$

is non singular, its inverse $M^{-1} = (\mu_{ik})$ is given by

$$\mu_{ik} = \begin{cases} \frac{(-1)^{i+k}}{(i-1)!(k-i)!} x_0^{k-i} & \text{for } i \leq k \\ 0 & \text{for } i > k \end{cases}$$

Thus the solution of system (9) can be written as

$$v_k = \sum_{l=0}^{m-k-1} \frac{(-1)^l}{k! l!} x_0^l D^{k+l} w, \quad k = 0, 1, \dots, m-1 \quad (10)$$

and we have the

Lemma 1. For each solution w of (1) in the form (8) the functions v_k are determined uniquely by (10).

In $Cl_{0,1}$ the operator ∂ reduces to the Cauchy-Riemann operator and eq. (1) to the iterated Bers-Vekua equation for which in [8] a corresponding representation theorem was proved. For the iterated Dirac operator $\tilde{\partial}^k$ with $\tilde{\partial} = \sum_{j=1}^n e_j \partial / \partial x_j$ a similar decomposition was proved in [9], whereas in [10] a unified approach to decomposing kernels of iterated operators was investigated.

3 Differential operators for the solutions

Now let us consider the case when the solutions of $Dv = 0$ can be represented by means of a suitable differential operator of Bauer-type acting on solutions of $\partial u = 0$. In the paper mentioned above a sufficient condition on the coefficient c in $Dv = 0$ was given for the existence of such a differential operator with which Clifford holomorphic functions g can be transformed into solutions v of $Dv = 0$ by

$$v = \sum_{j=0}^N a_j(x_0) (g \bar{\partial}^j) + \sum_{j=0}^{N-1} b_j(x_0) (\partial^j \bar{g}) , \quad N \in \mathbb{N}$$

In particular for the coefficient $c(x_0) = N/x_0, N \in \mathbb{Z}$, such a Bauer-type differential operator exists and the coefficients a_k and b_k can be given in an explicit form.

Let us assume that there exists a representation for the functions v_k in (8) of the form

$$v_k = \sum_{j=0}^N a_j(x_0) (g_k \bar{\partial}^j) + \sum_{j=0}^{N-1} b_j(x_0) (\partial^j \bar{g}_k) , \quad k = 0, \dots, m-1 \quad (11)$$

with suitable functions g_k which are solutions of $\partial g_k = 0, k = 0, \dots, m-1$. With the operator δ defined by

$$\delta u := (u \bar{\partial}) - (\partial u) , \quad \delta^{j+1} u := \delta(\delta^j u) , \quad \delta^0 u := u$$

by direct calculation we can prove the

Proposition 2.

(i) For any Clifford holomorphic function g_k the function u according to

$$u = \sum_{k=0}^{m-1} x_0^k g_k \quad (12)$$

represents a solution of $\partial^m u = 0$ (see also [5]).

(ii) For a function u given by (12) we have the relation

$$\delta^j u = \sum_{k=0}^{m-1} x_0^k (g_k \bar{\partial}^j)$$

Now from the representation (8) with the functions v_k given in (11) and with proposition 2(ii) we get the following form for the solutions of (1)

$$w = \sum_{j=0}^N a_j(x_0) (\delta^j u) + \sum_{j=0}^{N-1} b_j(x_0) \overline{(\delta^j u)} \quad (13)$$

Theorem 2. In the case of the existence of Bauer-type operators for the representation of the solutions of the generalized Bers-Vekua equation $Dv = 0$ the solutions of the iterated generalized Bers-Vekua equation $D^m w = 0$ can be given in terms of solutions of the iterated generalized Cauchy-Riemann equation $\partial^m u = 0$ by a differential operator in the form (13).

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