

ASYMPTOTIC NORMALITY OF b -ADDITIVE FUNCTIONS ON POLYNOMIAL SEQUENCES IN CANONICAL NUMBER SYSTEMS

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ABSTRACT. We consider b -additive functions f where b is the base of a canonical number system in an algebraic number field. In particular, we show that the asymptotic distribution of $f(p(z))$ with p a polynomial running through the integers of the algebraic number field is the normal law. This is a generalization of results of Bassily and Katai (for the integer case) and of Gittenberger and Thuswaldner (for the Gaussian integers).

1. INTRODUCTION

The objective of this paper is the consideration of additive functions in number systems. We start with a simple example of a q -additive function: Let s_q denote the sum-of-digits function in base q , where q is a positive integer. This function has been studied by several authors and we want to mention Delange [4]. He computed the average of the sum-of-digits function, *i.e.*,

$$\frac{1}{N} \sum_{n \leq N} s_q(n) = \frac{q-1}{2} \log_q(N) + \gamma_1(\log_q(N)),$$

where \log_q denotes the logarithm in base q and γ_1 is a continuous function of period 1.

A canonical question is the distribution into residue classes of the sum-of-digits functions. Thus we consider sets of the form

$$S_{r,m}(N) = \{n \leq N : s_q(n) \equiv r \pmod{m}\}.$$

Then Mauduit and Sárközy [23] were able to show the following.

Theorem. *Let $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$ with $N \in \mathbb{N}$. Then the estimate*

$$\left| \#\{(a, b) \in \mathcal{A} \times \mathcal{B} : a + b \in S_{r,m}(2N)\} - \frac{|\mathcal{A}| |\mathcal{B}|}{m} \right| = \mathcal{O}(N^\theta (|\mathcal{A}| |\mathcal{B}|)^{\frac{1}{2}})$$

holds, where $\theta < 1$ and the implied \mathcal{O} -constant is absolute.

An extension of the results above to general q -additive functions was given by Bassily and Katai [2]. Recall that a function f is said to be q -additive if it acts only on the q -adic digits, *i.e.*, $f(0) = 0$ and

$$f(n) = \sum_{k \geq 0} f(a_k(n)q^k) \quad \text{for } n = \sum_{k \geq 0} a_k(n)q^k,$$

where $a_k(n) \in \mathcal{N} := \{0, \dots, q-1\}$ are the *digits* of the q -adic expansion. Obviously, s_q is a special q -additive function.

The above mentioned distributional result by Bassily and Katai [2] reads as follows.

Theorem. *Let f be a q -additive function such that $f(aq^k) = \mathcal{O}(1)$ as $k \rightarrow \infty$ and $a \in \mathcal{N}$. Furthermore let*

$$m_{k,q} := \frac{1}{q} \sum_{a \in \mathcal{N}} f(aq^k), \quad \sigma_{k,q}^2 := \frac{1}{q} \sum_{a \in \mathcal{N}} f^2(aq^k) - m_{k,q}^2,$$

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and

$$M_q(x) := \sum_{k=0}^N m_{k,q}, \quad D_q^2(x) = \sum_{k=0}^N \sigma_{k,q}^2$$

with $N = \lfloor \log_q x \rfloor$. Assume that $D_q(x)/(\log x)^{1/3} \rightarrow \infty$ as $x \rightarrow \infty$ and let $p(x)$ be a polynomial with integer coefficients, degree d and positive leading term. Then, as $x \rightarrow \infty$,

$$\frac{1}{x} \# \left\{ n < x \mid \frac{f(p(n)) - M_q(x^d)}{D_q(x^d)} < y \right\} \rightarrow \Phi(y),$$

where Φ is the normal distribution function.

Results of similar type for different number systems related to substitution automaton were considered by Dumont and Thomas [8]. For distribution results of additive functions defined over number systems based on linear recurrences we refer the reader to Drmota and Gajdosik [5].

In this paper we focus on generalizations of number systems to algebraic number fields. These number systems are called canonical number systems (CNS). We shortly recall their definition following [26, 30].

Definition 1.1. Let

$$m(x) := b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0 \in \mathbb{Z}[x]$$

be such that $n \geq 1$ and $b_n = 1$. Furthermore let $\mathcal{N} = \{0, 1, \dots, |b_0| - 1\}$ and \mathcal{R} be the quotient ring

$$\mathcal{R} = \mathbb{Z}[x]/p(x)\mathbb{Z}[x].$$

- We say that a $g \in \mathcal{R}$ has a *finite* representation if it admits a representation of the form

$$g = a_0 + a_1 x + \cdots + a_h x^h$$

with $a_i \in \mathcal{N}$ for $0 \leq i \leq h$ and $a_h \neq 0$ for $h \neq 0$.

- The numbers $a_i = a_i(g)$ are called the *digits* of g with respect to $(m(x), \mathcal{N})$.
- The pair $(m(x), \mathcal{N})$ is called a *canonical number system* (CNS) in \mathcal{R} , if each $g \in \mathcal{R}$ has a *finite* representation. Furthermore we call \mathcal{N} its *digit set*.
- If $m(x)$ is *irreducible*, then let b be one of its zeros. In this case \mathcal{R} is isomorphic to $\mathbb{Z}[b]$. Therefore we may replace x by b in the above expansions. In this case, we simplify the notation $(m(x), \mathcal{N})$ to (b, \mathcal{N}) and b is called the *base* of this CNS.

Throughout this paper we are mainly interested in number systems in algebraic number fields. Let K be an algebraic number field and denote by \mathcal{O}_K its ring of integers. Furthermore let N and Tr denote the *norm* and *trace* of an element of K over \mathbb{Q} , respectively. Then for $b \in \mathcal{O}_K$ we set $\mathcal{N} := \{0, 1, \dots, N(b) - 1\}$ and get that (b, \mathcal{N}) is a number system if the minimal polynomial $m_b(x)$ of b over \mathbb{Q} satisfies the conditions of $m(x)$ above.

When considering number systems one is interested in certain properties, first of all, which are the possible bases. This has been answered in a series of papers [1, 3, 14, 15, 16]. Secondly one has to face effects such as amenability, *i.e.*, there exist two different expansions of one number. In fact, one can construct a graph which characterizes all the amenable expansions. This has been done by Müller *et al.* [25] (with a direct approach) and by Scheicher and Thuswaldner [29] (consideration of the odometer).

Another view on number systems are normal numbers. These are numbers in which expansion every possible block occurs asymptotically equally often. Constructions of such numbers have been considered by Dumont *et al.* [7] and the author in [21, 22].

In this paper we mainly concentrate on additive functions. These are functions that act only on the digits of an expansion. Thus we define additive functions in these number systems as follows.

Definition 1.2. Let (b, \mathcal{N}) be a CNS over \mathcal{O}_K . A function f is called *b-additive* if $f(0) = 0$ and for $g \in \mathcal{O}_K$

$$(1.1) \quad f(g) = \sum_{k \geq 0} f(a_k(g)b^k), \quad \text{for } g = \sum_{k \geq 0} a_k(g)b^k \quad (a_k(g) \in \mathcal{N}).$$

As indicated above the simplest version of an additive function is the sum-of-digits function s_b defined by

$$s_b(g) := \sum_{k \geq 0} a_k(g).$$

A first step towards CNS is the consideration of b -additive function in the field of Gaussian rationals. First a generalization of the result by Delange was given by Grabner *et al.* in [12].

Theorem. *Let $b = -n \pm i$ be a base of a CNS in $\mathbb{Z}[i]$. Then we have*

$$\sum_{\substack{N(z) \leq N \\ z \in \mathbb{Z}[i]}} s_b(z) = \frac{N(b) - 1}{2} \pi N \log_{N(b)} N + N \gamma_2 \left(\log_{N(b)} N \right) + \mathcal{O} \left(\sqrt{N} \log_{N(b)} N \right)$$

where γ_2 is a continuous function of period 1.

Also the result by Bassily and Katai was generalized to number systems over the Gaussian rationals. Gittenberger and Thuswaldner [11] gained the following distribution result.

Theorem. *Let (b, \mathcal{N}) be a canonical number system over $\mathbb{Z}[i]$. Let f be a b -additive function such that $f(cb^k) = \mathcal{O}(1)$ as $k \rightarrow \infty$ and $c \in \mathcal{N}$. Furthermore let*

$$m_{k,b} := \frac{1}{N(b)} \sum_{c \in \mathcal{N}} f(cb^k), \quad \sigma_{k,b}^2 := \frac{1}{N(b)} \sum_{c \in \mathcal{N}} f^2(cb^k) - m_{k,b}^2,$$

and

$$M_b(x) := \sum_{k=0}^L m_{k,b}, \quad D_b^2(x) = \sum_{k=0}^L \sigma_{k,b}^2$$

with $L = \lfloor \log_{N(b)} x \rfloor$. Assume that $D_b(x)/(\log x)^{1/3} \rightarrow \infty$ as $x \rightarrow \infty$ and let $p(x)$ be a polynomial of degree d with coefficients in $\mathbb{Z}[i]$. Then, as $N \rightarrow \infty$,

$$\frac{1}{\#\{z \in \mathbb{Z}[i] | N(z) < N\}} \# \left\{ N(z) < N \mid \frac{f(p(z)) - M_b(N^d)}{D_b(N^d)} < y \right\} \rightarrow \Phi(y),$$

where Φ is the normal distribution function and z runs over the Gaussian integers.

This build the base for further considerations of b -additive functions over CNS in general. First the result of Delange was further generalized to arbitrary number fields by Thuswaldner [32]. Furthermore the moments of the sum-of-digits function in algebraic number fields were considered by Gittenberger and Thuswaldner [10].

Theorem. *Let K be a number field of degree n and \mathcal{O}_K its ring of integers. Furthermore, let b be a base of a CNS. Then*

$$\begin{aligned} \sum_{z \in P(N)} (s_b(z))^d &= c_b \left(\frac{|N(b)| - 1}{2} \right)^d N \log_{|N(b)|}^d N + N \sum_{j=0}^{d-1} \log_{|N(b)|}^j N \gamma_j(\log_{|N(b)|} N) \\ &\quad + \mathcal{O} \left(N^{\frac{n-1}{n}} \log_{|N(b)|}^d N \right), \end{aligned}$$

where c_b is a constant depending on K and b and the γ_j s are continuous functions of period 1.

In the same vain the above mentioned result by Mauduit and Sárközy was generalized to arbitrary number fields by Thuswaldner [33]. Therefore we write

$$U_{r,m}(M(T)) = \{z \in M(T) : s_b(z) \equiv r \pmod{m}\},$$

where $M(T)$ is the set described below in (2.2). Then his result reads as follows.

Theorem. *Let K be a number field of degree n with ring of integers \mathcal{O}_K . Let b be the base of a canonical number system in \mathcal{O}_K and $m_b(x) = x^n + \dots + b_1 x + b_0$ the minimal polynomial of b . If $(m_b(1), m) = 1$, then*

$$\left| \# \{(a, b) \in \mathcal{A} \times \mathcal{B} : a + b \in S_{r,m}(M(2T))\} - \frac{|\mathcal{A}| |\mathcal{B}|}{m} \right| = \mathcal{O}(|M(T)|^\theta (|\mathcal{A}| |\mathcal{B}|)^{\frac{1}{2}})$$

holds for any two sets $\mathcal{A}, \mathcal{B} \subset M(T)$. Furthermore $\theta < 1$ and the implied \mathcal{O} -constant is absolute.

Despite of these considerations of the sum-of-digits function and other b -additive functions, we also want to mention Katai and Liardet [17], who could show a Delange type result for b -multiplicative functions. Finally there has also been work on the generalization of Waring's Problem restricted to sets of the form $S_{r,m}$ defined above. Here we want to mention Pethő and Tichy [27] (counting the number of solutions for a S -unit equation) and Thuswaldner and Tichy [34] (counting the number of solutions of Waring's Problem with digital restrictions).

In the present paper our aim is to extend the result of Bassily and Katai and Gittenberger and Thuswaldner to arbitrary algebraic number fields. The main problem we have to face is the different setting for number systems. Especially the fact that the conjugates of a number need not to have equal absolute value will play a role. This difference occurs essentially in the estimation of the involved Weyl sums. Known estimates deal with Waring's problem and its generalization. Since in these generalizations the leading coefficients of the involved polynomial go through the whole unit interval and its approximations therefore only targets the splitting into major and minor arcs, we need a different approach. In Section 3 we will develop a more general estimation of these sums.

2. DEFINITIONS AND RESULTS

In the following paragraphs we will define the tools we need in order to properly estimate the distribution. These definitions are standard in the area and the reader may refer to Ribenboim [28] or Wang [35].

Throughout the rest of the paper we fix an algebraic number field K and denote by \mathcal{O}_K its ring of integers. Then let $K^{(\ell)}$ ($1 \leq \ell \leq r_1$) be the real conjugates of K , while $K^{(m)}$ and $K^{(m+r_2)}$ ($r_1 < m \leq r_1 + r_2$) are the complex conjugates of K , where $r_1 + 2r_2 = n$. Throughout this paper the indices ℓ and m are always over the sets of integers cited here. Furthermore we set $r = r_1 + r_2$ and call the pair (r_1, r_2) the signature of K .

If not stated otherwise an upper case letter will always denote a real number, a lower case letter an element of \mathcal{O}_K and a Greek letter an element of \overline{K} , the completion of K . Furthermore sums are always extended over rational or algebraic integers, respectively.

For $\gamma \in K$ we denote by $\gamma^{(i)}$ ($1 \leq i \leq n$) the conjugates of γ . In order to extend the term of conjugation to the completion \overline{K} of K we define for $\gamma_j \in K$ and $x_j \in \mathbb{R}$ ($1 \leq j \leq n$) $\lambda = \sum_{1 \leq j \leq n} x_j \gamma_j$ and $\lambda^{(i)} := \sum_{1 \leq j \leq n} x_j \gamma_j^{(i)}$. We recall that for $\lambda \in \overline{K}$

$$N(\lambda) = \prod_{1 \leq i \leq n} \lambda^{(i)}, \quad \text{Tr}(\lambda) = \sum_{1 \leq i \leq n} \lambda^{(i)}, \quad |\overline{\lambda}| = \max_{1 \leq i \leq n} |\lambda^{(i)}|$$

are the *norm*, *trace* and *house* of an element of \overline{K} over \mathbb{Q} , respectively. Furthermore for $\lambda \in \overline{K}$ let

$$(2.1) \quad e(x) := \exp(2\pi i x) \quad \text{and} \quad E(\lambda) = e(\text{Tr}(\lambda)).$$

Let δ be the *different* and D the absolute value of the *discriminant* of K . We need some geometry of numbers and therefore let $\omega_1, \dots, \omega_n$ be an integral basis of \mathcal{O}_K and let ρ_1, \dots, ρ_n be the corresponding basis of δ^{-1} such that

$$\text{Tr}(\rho_i \omega_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Let Λ be the lattice constructed by the n linear forms

$$L_i = \sum_{j=1}^n \rho_j^{(i)} x_j.$$

Then we denote by λ_1 the first successive minimum of the convex body $B := \{z \in \mathbb{R}^n : |z| \leq 1\}$ with respect to the lattice Λ .

We call a number totally non-negative if $\lambda^{(i)} \geq 0$ for $1 \leq i \leq n$. As we will successively extend the maximum length of the expansions we define $M(T_1, \dots, T_r)$ to be the set

$$(2.2) \quad M(T_1, \dots, T_r) := \left\{ \eta \in \mathcal{O}_K : \left| \eta^{(i)} \right| \leq T_i, 1 \leq i \leq r \right\}.$$

Furthermore we will write for short $M(T) := M(T, \dots, T)$.

Now we can state our main result.

Theorem 2.1. *Let (b, \mathcal{N}) be a CNS and f be a b -additive function such that $f(ab^k) = \mathcal{O}(1)$ as $k \rightarrow \infty$ and $a \in \mathcal{N}$. Furthermore let*

$$m_{k,b} := \frac{1}{N(b)} \sum_{a \in \mathcal{N}} f(ab^k), \quad \sigma_{k,b}^2 := \frac{1}{N(b)} \sum_{a \in \mathcal{N}} f^2(ab^k) - m_{k,b}^2,$$

and

$$M_b(x) := \sum_{k=0}^L m_{k,b}, \quad D_b^2(x) = \sum_{k=0}^L \sigma_{k,b}^2$$

with $L = \lfloor \log_{N(b)} x \rfloor$. Assume that there exists an $\varepsilon > 0$ such that $D_b(x)/(\log x)^\varepsilon \rightarrow \infty$ as $x \rightarrow \infty$ and let $p \in \mathcal{O}_K[X]$ be a polynomial of degree d . Then, as $T \rightarrow \infty$,

$$\frac{1}{\#M(T)} \# \left\{ z \in M(T) : \left| \frac{f(p(z)) - M_b(T^d)}{D_b(T^d)} < y \right\} \rightarrow \Phi(y),$$

where Φ is the normal distribution function.

We will show this theorem in five steps.

- (1) In Section 3 we start with the estimation of the Weyl sums which will occur in the proof. Therefore we use tools which come from generalizations of Waring's Problem to algebraic number fields together with Hua's method of estimating Weyl sums. The main difference will be the approximation of the coefficients we use.
- (2) For counting the number of occurrences of a certain digit in the expansion of the integer values of the polynomial p we need an Urysohn function. This function is developed in Section 4 where we consider the fundamental domain of the CNS.
- (3) The error term when counting the number of occurrences of a digit comes from those near the border of the fundamental domain. Therefore we estimate the number of elements which are in a small tube around the border in Section 5.
- (4) In Section 6 we have to count the number of elements within the fundamental domain itself, which gives the proof of the central proposition.
- (5) Finally we show Theorem 2.1 by truncation and two applications of the Fréchet-Shohat Theorem.

3. WEYL SUMS

In this section we want to consider and estimate the exponential sums which will occur in the following sections. We will begin by giving some background on how exponential sums are defined in number fields.

Throughout this section let

$$g(x) = \alpha_d x^d + \dots + \alpha_1 x$$

be a polynomial of degree d . Let a boldface letter always denote a vector. Then for $\mathbf{T} = (T_1, \dots, T_r)$ we set $M(\mathbf{T}) = M(T_1, \dots, T_r)$. The sum

$$S(g, \mathbf{T}) = \sum_{x \in M(\mathbf{T})} E(g(x)),$$

where E is as in (2.1), is called a *Weyl sum*.

The main tool in order to estimate these sums is Weyl's differentiation method. Therefore we need a generalization of Dirichlet's theorem on rational approximation, which is provided by Siegel (cf. [31]). Since in our case the T_i are not all equal (see Lemma 6.1) we have to slightly modify Siegel's original theorem in order to cope with this new situation.

Lemma 3.1. *Let N_1, \dots, N_r be real numbers and let $N = \sqrt[r]{N_1 \cdots N_{r_1} (N_{r_1+1} \cdots N_{r_1+r_2})^2}$ be their geometric mean. Suppose that $N > D^{\frac{1}{n}}$, then, corresponding to any $\xi \in K$, there exist $q \in \mathcal{O}_K$ and $a \in \delta^{-1}$ such that*

$$\begin{aligned} |q^{(i)} \xi^{(i)} - a^{(i)}| &< N_i^{-1}, \quad 0 < |q^{(i)}| \leq N_i, \quad 1 \leq i \leq r, \\ \max \left(N_i |q^{(i)} \xi^{(i)} - a^{(i)}|, |q^{(i)}| \right) &\geq D^{-\frac{1}{2}}, \quad 1 \leq i \leq r, \end{aligned}$$

and

$$N((q, a\delta)) \leq D^{\frac{1}{2}}.$$

Proof. This easily follows from the appropriate modifications in the proof of Theorem 3.1 of [35]. \square

Now we can state the main estimate of the exponential sum above.

Proposition 3.2. *Let*

$$T = \sqrt[r]{T_1 \cdots T_{r_1} (T_{r_1+1} \cdots T_{r_1+r_2})^2}$$

be the geometric mean of the T_i ($1 \leq i \leq r$). Suppose that α_k satisfies the hypotheses from Lemma 3.1 with $N_i = T_i^d (\log T)^{-\sigma_1}$ and

$$(\log T)^{\sigma_1} \leq \alpha_k^{(i)} \leq T_i^d (\log T)^{-\sigma_1} \quad 1 \leq i \leq r,$$

then

$$S(g, \mathbf{T}) \ll T^n (\log T)^{-\sigma_0}$$

with $\sigma_1 \geq 2^{d-1} (\sigma_0 + r2^{2d})$.

The proof consists of modifications of the proof of Theorem 3.2 of Wang [35], since in our considerations the T_i need not all be equal and we essentially need the $(\log T)^{-\sigma_0}$ term. We start with the main tools needed for the proof of Proposition 3.2.

Lemma 3.3 ([35, Lemma 3.2]). *Let T_i ($1 \leq i \leq r$) be positive integers and set $T_{r_1+r_2+i} = T_{r_1+i}$ for ($1 \leq i \leq r_2$). Then*

$$M(T) = \frac{2^r \pi^{r_2}}{\sqrt{D}} T_1 \cdots T_n + \mathcal{O}(T_0^{n-1}),$$

where $T_0 = \max(1, (T_1 \cdots T_n)^{1/n})$.

Since in the classical case the T_i are all equal we have to rewrite the proof of Wang's Lemma 3.6. Therefore we need the following adoption of Lemma 3.5 of Wang [35].

Lemma 3.4. *Let T_i and $N_i \geq 0$ for $1 \leq i \leq s$ be integers. Then denote by M the set of all points $(t_1, \dots, t_s) \in \mathbb{Z}^s$ such that*

$$T_i \leq t_i < T_i + N_i \quad 1 \leq i \leq s.$$

Let M_0 be a subset of M and define

$$S = \sum_{\mathbf{t} \in M_0} \min \left(\frac{N_1}{t_1}, \dots, \frac{N_s}{t_s} \right).$$

Then

$$S \ll N (\#M_0)^{1-\frac{1}{s}} (\log(N+2)),$$

where N is the geometric mean of the N_i .

Proof. This proof mainly follows that of Lemma 3.5 of Wang [35]. In the same way we start by setting M_ν to be the subset of M_0 such that $\frac{t_\nu}{N_\nu} \geq \frac{t_i}{N_i}$ for $1 \leq i \leq s$. Furthermore we denote by $A_\nu = \#M_\nu$ its number of elements and by

$$S_\nu = \sum_{\mathbf{t} \in M_\nu} \min \left(\frac{N_1}{t_1}, \dots, \frac{N_s}{t_s} \right)$$

the restriction of the sum S to elements of M_ν .

Then it suffices to show that

$$S_\nu \ll NA_\nu^{1-\frac{1}{s}} \log(N_\nu + 2),$$

which together with $S \leq S_1 + \dots + S_s$ proves the lemma.

Without loss of generality we show this for $\nu = 1$. For $t > 0$ let $D(t)$ be the subset of $\mathbf{u} \in \mathbb{R}^s$ such that

$$t \geq u_1 > 0, \quad \frac{u_1}{N_1} \geq \frac{u_i}{N_i} \geq 0, \quad 2 \leq i \leq s.$$

Let $M(t) := D(t) \cap \mathbb{Z}^s$ be the integer points in $D(t)$ and denote by $n(t)$ their number. In the same manner let $M_0(t) := \{\mathbf{t} \in M_0 : t \geq t_1\}$ and denote by $n_0(t) = \#M_0(t)$ its cardinality.

Now let t_0 be an integer such that

$$n(t_0) \leq A_1 = n_0(N_1) < n(t_0 + 1).$$

Then

$$S_1 \leq \sum_{\mathbf{t} \in M(t_0+1)} \frac{N_1}{t_1} \leq N_1 \prod_{i=2}^s \frac{N_i}{N_1} (t_0 + 2) \sum_{t_1 \leq t_0+1} \frac{1}{t_1} \leq \frac{N^s (t_0 + 2)^{s-1}}{N_1^{s-1}} \log(t_0 + 2)$$

and

$$A_1 \geq n(t_0) \geq \sum_{t=1}^{t_0} \prod_{i=2}^s \frac{N_i}{N_1} t = \frac{N^s}{N_1^s} \sum_{t=1}^{t_0} t^{s-1} \geq c(s) \frac{N^s}{N_1^s} t_0^s.$$

Putting these together proves the lemma. \square

Now we state our modified version of Lemma 3.6 of Wang [35]. In its original version this lemma essentially goes back to Mitsui [24].

Lemma 3.5. *Let A, B_i ($1 \leq i \leq r$), N be positive numbers satisfying $A \geq 1$ and $N \gg 1$. Furthermore let B be the geometric mean of the B_i and let N_i ($1 \leq i \leq r$) be in the same ration to N as the B_i are to B , i.e.,*

$$B := \sqrt[r]{B_1 \cdots B_r (B_{r_1+1} \cdots B_r)^2} \quad \text{and} \quad N_i := N \frac{B_i}{B} \quad 1 \leq i \leq r.$$

Suppose that $1 \leq B \ll N$, then, for any ξ ,

$$\sum_{m \in M(\mathbf{B})} \min_{1 \leq i \leq n} \left(A, |1 - E(\xi m \omega_i)|^{-1} \right) \ll AB^n \left(\frac{1}{|N(q)|} + \frac{1}{B} + \frac{N \log N}{AB} + \frac{\log N}{A} \right),$$

where q denotes an integer satisfying the conditions in Lemma 3.1 with ξ and N_i .

Proof. Since this is a modification of the proof of Lemma 3.6 in Wang [35] we only sketch the proof and mainly follow the lines there.

First let \mathbf{X} be the Minkowski embedding, i.e., for $\xi \in K$

$$\mathbf{X}(\xi) = (X_1(\xi), \dots, X_n(\xi)) := (\xi^{(1)}, \dots, \xi^{(r_1)}, \Re(\xi^{(r_1+1)}), \dots, \Im(\xi^{(r_1+r_2)})).$$

Then we write with rational integers e_i and $-\frac{1}{2} \leq d_i \leq \frac{1}{2}$

$$\text{Tr}(\xi m \omega_i) = e_i + d_i \quad (1 \leq i \leq n) \quad \text{and set} \quad \zeta = \sum_{i=1}^n d_i \rho_i.$$

One easily checks that

$$S = \sum_{m \in M(\mathbf{B})} \min_{1 \leq i \leq n} \left(A, |1 - E(\xi m \omega_i)|^{-1} \right) \ll \sum_{m \in M(\mathbf{B})} \min_{1 \leq i \leq n} \left(A, |X_i(\zeta)|^{-1} \right) = S^*.$$

Now we assign to every m in $M(\mathbf{B})$ its corresponding ζ and a vector $y(m)$ defined by

$$y(m) = (R_1 X_1(\zeta), \dots, R_n X_n(\zeta)) \quad \text{with} \quad R_i = 2D^{\frac{1}{n}} \left| q^{(i)} \right|.$$

We divide the set $\{1, 2, \dots, n\}$ into three parts

$$\begin{aligned} J_1 &:= \left\{ 1 \leq i \leq n : \frac{B_i}{N_i} D^{\frac{1}{n}} \geq 2c_{11} \left| q^{(i)} \right| \right\}, \\ J_2 &:= \left\{ 1 \leq i \leq n : \frac{1}{2} \geq 2c_{11} \left| q^{(i)} \right| > \frac{B_i}{N_i} D^{\frac{1}{n}} \right\}, \\ J_3 &:= \left\{ 1 \leq i \leq n : 2c_{11} \left| q^{(i)} \right| > \frac{1}{2} \right\}. \end{aligned}$$

Furthermore we set

$$\tau_i = 2 \frac{B_i}{N_i} D^{\frac{1}{n}} \text{ for } i \in J_1, \quad \tau_i = 4c_{11} \left| q^{(i)} \right| \text{ for } i \in J_2.$$

For the rest we set τ_i such that $\prod_{i=1}^n \tau_i = 2^{-2n}$. Then we divide the set of possible vectors $y(m)$ by defining for every vector $\mathbf{g} \in \mathbb{Z}^n$

$$B(\mathbf{g}) := \left\{ \mathbf{x} : \tau_i \left(g_i - \frac{1}{2} \right) \leq x_i < \tau_i \left(g_i + \frac{1}{2} \right), 1 \leq i \leq n \right\}.$$

By the same lines as in the proof of Wang we get that if there are two m and m_1 such that $m, m_1 \in B(\mathbf{g})$ then $m - m_1 \in \mathfrak{a}$ for a certain ideal \mathfrak{a} with

$$|N(q)| \leq D^{\frac{1}{n}} N(\mathfrak{a}).$$

Finally we denote by $W(\mathbf{g})$ the number of $m \in M(\mathbf{B})$ such that $y(m) \in B(\mathbf{g})$. Thus following the lines of the proof of Wang we get that

$$W(\mathbf{g}) \ll 1 + W_0 = 1 + \prod_{i \in J_1} N_i \prod_{j \in J_2 \cup J_3} \frac{B_j}{|q^{(j)}|} = 1 + B^n \prod_{i \in J_1} \frac{N_i}{B_i} \prod_{j \in J_2 \cup J_3} \left| q^{(j)} \right|^{-1}.$$

Now we split the sum S^* up into two parts where S_1 consists of all elements $m \in M(\mathbf{B})$ and $y(m) \in B(\mathbf{0})$ and S_2 is the rest.

We start with the estimation of S_1 and distinguish two cases according to whether $J_1 \cup J_2 = \emptyset$ or not.

- For $J_1 \cup J_2 = \emptyset$ we get as in the proof of Wang that

$$(3.1) \quad S_1 \ll A + \frac{AB^n}{|N(q)|}.$$

- For $J_1 \cup J_2 \neq \emptyset$ we rewrite the sum and get

$$S_1 \ll \sum_{\substack{m \in \mathfrak{a} \\ m \in M(2\mathbf{B})}} \min_{j \in J_1 \cup J_2} \left(A, \frac{N_j}{|X_j(m + \xi_0)|} \right).$$

We again divide the area of possible \mathbf{X} . For $\mathbf{t} \in \mathbb{Z}^n$ we define

$$B^*(\mathbf{t}) = \left\{ \mathbf{x} : \frac{N(\mathfrak{a})}{3} \frac{B_i}{B} \left(t_i - \frac{1}{2} \right) \leq x_i < \frac{N(\mathfrak{a})}{3} \frac{B_i}{B} \left(t_i + \frac{1}{2} \right) \right\}.$$

We get that there is at most one element $m \in M(2\mathbf{B})$ for every $\mathbf{t} \in \mathbb{Z}^n$. By noting our definition of N_i we rewrite the sum to get

$$S_1 \ll \sum_{\mathbf{t}} \min_{j \in J_1 \cup J_2} \left(A, \frac{N}{|t_j| N(\mathfrak{a})^{\frac{1}{n}}} \right) \ll AB^n \left(\frac{1}{B} + \frac{N \log N}{AB} \right),$$

where we used Wang's estimations since the sum is the same.

Together with the estimation (3.1) we get for S_1 that

$$(3.2) \quad S_1 \ll AB^n \left(\frac{1}{|N(q)|} + \frac{1}{B} + \frac{N \log N}{AB} \right).$$

Now we are left with estimating S_2 . Therefore we distinguish the two cases according to whether $W_0 > 1$ or not.

- For the first case ($W_0 > 1$) we get by following the lines of the proof of Wang that

$$(3.3) \quad S_2 \ll B^n \log N.$$

- In the case of $W_0 \leq 1$ we reach the estimate

$$S_2 \ll \sum_{\mathbf{g}_i \in G_0} \min_{i \in J_3} \left(\frac{N_i}{|q^{(i)}|} \right),$$

where G_0 is the set of all $g_i, i \in J_3$ such that

$$W(\mathbf{g}) \neq 0, \quad \{g_i\} \neq \mathbf{0}.$$

In the same manner as in Wang's proof we get that the value $|g_i|$ in G_0 does not exceed N_i . Thus by an application of Lemma 3.4 we get that

$$S_2 \ll NB^{n-1} \log N$$

Together with the estimation (3.3) this yields

$$(3.4) \quad S_2 \ll AB^n \left(\frac{\log N}{A} + \frac{N \log N}{AB} \right).$$

Putting the estimates of S_1 and S_2 in (3.2) and (3.4) together proves the lemma. \square

Lemma 3.6 ([35, Lemma 3.8]). *Let $m \in \mathcal{O}_K$ and T be the geometric mean of the T_i , i.e.,*

$$T = \sqrt[n]{T_1 \cdots T_{r_1} (T_{r_1+1} \cdots T_{r_1+r_2})^2}$$

Then

$$\sum_{h \in M(\mathbf{T})} E(\xi m h) \ll T^{n-1} \min_{1 \leq i \leq n} (T, |1 - E(\xi m \omega_i)|^{-1}),$$

where ω_i ($1 \leq i \leq n$) is an integral basis.

Lemma 3.7 ([35, Lemma 3.9]). *Suppose that $1 \leq t \leq d-1$ and T is the geometric mean of the T_i , then we have*

$$|S(g, \mathbf{T})|^{2^t} \ll T^{(2^t - t - 1)n} \sum_{h_1, \dots, h_t \in M(2T)} \sum_{h \in M(\mathbf{T})} E(h_1 \cdots h_t \bar{g}(h, h_1, \dots, h_t) \xi),$$

where

$$\bar{g}(h, h_1, \dots, h_t) = d(d-1) \cdots (d-t+1) \alpha_d h^{d-t} + \cdots$$

is a polynomial of degree $d-t$ in h, h_1, \dots, h_t .

Now we consider the divisor function in more detail. This idea essentially goes back to Hua [13].

Lemma 3.8 ([35, Lemma 3.7]). *For $a \in \mathcal{O}_k$ and $T \geq 0$ let $d_k(a, T)$ be the number of solutions of the equation*

$$u_1 \cdots u_k = a, \quad \text{where } u_1, \dots, u_k \in M(T) \text{ for } i = 1, \dots, k.$$

Then

$$d_k(a, T) \ll |N(a)|^\varepsilon (\log T)^{(r-1)(k-1)}.$$

We write for short $d(a, T) := d_2(a, T)$. Now we have collected all our tools in order to estimate sums involving divisor function, which will occur in our proof of Proposition 3.2.

Lemma 3.9. *Let t be a non-negative integer. Furthermore let $\mathbf{T} = (T_1, \dots, T_r)$ and suppose that $T_i \ll T_0$ for $1 \leq i \leq r$, then*

$$\sum_{m \in M(\mathbf{T})} \frac{(d(m, T_0))^t}{N(m)} \ll (n(\log T_0)^r)^{2^t}.$$

Proof. For simplicity we set $T_{r_1+r_2+i} = T_{r_1+i}$ for $1 \leq i \leq r_2$ and continue by induction on t . For $t = 0$ we get that $m \in M(\mathbf{T})$ implies that $|\mathbf{N}(m)| \leq \prod_{i=1}^n T_i \ll T_0^n$. Thus by Lemma 3.8

$$\sum_{m \in M(\mathbf{T})} \frac{1}{|\mathbf{N}(m)|} \ll \sum_{N \leq T_0^n} \frac{(\log T_0)^{r-1}}{N} \ll n(\log T_0)^r.$$

Now we assume that the lemma holds for $t-1$. Then

$$\begin{aligned} \sum_{m \in M(\mathbf{T})} \frac{(d(m, T_0))^t}{|\mathbf{N}(m)|} &= \sum_{m \in M(\mathbf{T})} \frac{(d(m, T_0))^{t-1}}{|\mathbf{N}(m)|} \sum_{\substack{uv=m \\ u, v \in M(T_0)}} 1 \leq \sum_{u \in M(T_0)} \sum_{\substack{uv=m \\ m \in M(\mathbf{T})}} \frac{(d(m, T_0))^{t-1}}{|\mathbf{N}(m)|} \\ &= \sum_{u \in M(\mathbf{T})} \sum_{v \in M(T_1|u^{(1)}|^{-1}, \dots, T_r|u^{(r)}|^{-1})} \frac{(d(uv, T_0))^{t-1}}{|\mathbf{N}(u \cdot v)|} \\ &\ll (n \cdot (\log T_0)^r)^{2^{t-1}} (n \cdot (\log T_0)^r)^{2^{t-1}} = (n \cdot (\log T_0)^r)^{2^t}. \end{aligned}$$

□

By the last lemma we can estimate a divisor sum which occurs in the estimation of our Weyl sum.

Lemma 3.10. *Let t be a non-negative integer and let $\mathbf{T} = (T_1, \dots, T_r)$. Furthermore set $T_{r_1+r_2+i} = T_{r_1+i}$ for $1 \leq i \leq r_2$ and suppose that $T_i \ll T_0$ for $1 \leq i \leq n$. Then*

$$\sum_{m \in M(\mathbf{T})} (d(m, T_0))^t \ll T_1 \cdots T_n (n(\log T_0)^r)^{2^{t-1}}.$$

Proof. We show this by induction on t . For $t = 0$ this essentially is Lemma 3.3. Now we assume that the lemma holds for $t-1$, then by an application of Lemma 3.9

$$\begin{aligned} \sum_{m \in M(\mathbf{T})} (d(m, T_0))^t &= \sum_{m \in M(\mathbf{T})} (d(m, T_0))^{t-1} \sum_{\substack{uv=m \\ u, v \in M(T_0)}} 1 = \sum_{u \in M(\mathbf{T})} \sum_{\substack{uv=m \\ u \in M(\mathbf{T})}} (d(m, T_0))^{t-1} \\ &= \sum_{u \in M(\mathbf{T})} \sum_{v \in M(T_1|u^{(1)}|^{-1}, \dots, T_r|u^{(r)}|^{-1})} (d(uv, T_0))^{t-1} \\ &\ll \sum_{u \in M(\mathbf{T})} (d(u, T_0))^{t-1} \frac{T_1 \cdots T_n}{|\mathbf{N}(u)|} (n(\log T_0)^r)^{2^{t-1}-1} \\ &\ll T_1 \cdots T_n (n(\log T_0)^r)^{2^t-1}. \end{aligned}$$

□

Finally we need a Lemma whose idea essentially goes back to Hua (c.f. Hilfssatz 6.1 of [13]) in order to get a better estimation of the Weyl sum.

Lemma 3.11. *Let t be a non-negative integer and let $\mathbf{T} = (T_1, \dots, T_r)$. Suppose that $T_i \ll T_0$ for $1 \leq i \leq n$. Then, for any $\sigma_2 \geq 2^{3t-1}$ we get*

$$\sum'_{m \in M(\mathbf{T})} (d(m, T_0))^t \ll T_0^n (n(\log T_0)^r)^{-\sigma_2}$$

where \sum' stands for the sum over all m such that

$$(n(\log T_0)^r)^{\sigma_2} \ll (d(m, T_0))^t.$$

Proof.

$$\begin{aligned} (n(\log T_0)^r)^{2\sigma_2} \sum'_{m \in M(\mathbf{T})} (d(m, T_0))^t &\ll \sum_{m \in M(\mathbf{T})} (d(m, T_0))^{3t} \\ &\ll T_0^n (n(\log T_0)^r)^{2^{3t}-1} \ll T_0^n (n(\log T_0)^r)^{\sigma_2}. \end{aligned}$$

□

Proof of Proposition 3.2. We set $G = 2^{d-1}$ and get by an application of Lemma 3.7

$$|S(f, \mathbf{T})|^G \ll T^{(G-d)n} \sum_{h_1, \dots, h_{d-1} \in M(2T)} \left| \sum_{h \in M(\mathbf{T})} E(\alpha_d m h) \right|,$$

where

$$(3.5) \quad m = d!h_1 \cdots h_{d-1}.$$

Now we denote by $A(m)$ the number of solutions of (3.5). Noting that $d_k(m, T) \leq d(m, T)^k$ we get that

$$A(m) \ll \begin{cases} T^{(d-2)n} & \text{if } m = 0, \\ d(m, T)^{d-1} & \text{if } m \neq 0. \end{cases}$$

Putting everything together yields

$$|S(f, \mathbf{T})|^G \ll T^{(G-2)n} + T^{(G-d)n} \sum_{m \in M(d!2^q T^{d-1})} d(m, T)^{d-1} \left| \sum_{h \in M(\mathbf{T})} E(\alpha_d m h) \right|.$$

Now we distinguish two cases for m according to the hypotheses of Lemma 3.11, *i.e.*, whether $(n(\log T)^r)^{\sigma_2} \ll d(m, T)$ or not. Thus by an application of Lemma 3.5 and Lemma 3.6

$$\begin{aligned} |S(f, \mathbf{T})|^G &\ll T^{(G-2)n} + T^{(G-d)n} \left(T^n (n(\log T)^r)^\sigma + \sum'_m (n(\log T)^r)^\sigma \left| \sum_h E(\alpha_d m h) \right| \right) \\ &\ll T^{(G-2)n} + T^{(G-d+1)n} (\log T)^{r\sigma} \\ &\quad + T^{(G-d)n} (\log T)^{r\sigma} \sum'_m T^{n-1} \min_i \left(T, |1 - E(\alpha_d m h)|^{-1} \right) \\ &\ll T^{(G-2)n} + T^{(G-d+1)n} (\log T)^{r\sigma} + T^{Gn} (\log T)^{r\sigma} \left(\frac{1}{N(\alpha_d)} + \frac{1}{T} + (\log T)^{-\sigma_1} \right) \\ &\ll T^{(G-2)n} + T^{(G-d+1)n} (\log T)^{r\sigma} + T^{Gn} (\log T)^{r\sigma - \sigma_1}. \end{aligned}$$

□

4. FUNDAMENTAL DOMAIN

In this section we want to construct the Urysohn function for indicating the elements starting with a certain digit. The following definitions are standard in that area and we mainly follow Gittenberger and Thuswaldner [11]. We need the fundamental domain, which is defined as the set of all numbers whose integer part is zero, *i.e.*,

$$\mathcal{F}' := \left\{ z \in \mathbb{C} \left| z = \sum_{k \geq 1} a_k b^{-k}, a_k \in \mathcal{N} \right. \right\}.$$

It is more convenient to consider the embedding of the fundamental domain in \mathbb{R}^n . This idea is based on a result by Kovács (*cf.* Lemma 1 of [18]), that if (b, \mathcal{N}) is a CNS then $\{1, b, b^2, \dots, b^{n-1}\}$ is an integral basis for K . Thus we define the embedding ϕ by

$$\phi : \begin{cases} \overline{K} & \rightarrow \mathbb{R}^n, \\ \alpha_0 + \alpha_1 b + \cdots + \alpha_{n-1} b^{n-1} & \mapsto (\alpha_0, \dots, \alpha_{n-1}). \end{cases},$$

where \overline{K} is the completion of K .

Now let $m_b(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ be the minimal polynomial of b , then we define the corresponding matrix B by

$$B = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & -a_0 \\ 1 & 0 & \cdots & \cdots & 0 & \vdots \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

One easily checks that

$$\phi(b \cdot z) = B \cdot \phi(z).$$

By this we define the embedding of the fundamental domain by

$$\mathcal{F} = \phi(\mathcal{F}') = \left\{ z \in \mathbb{R}^n \left| z = \sum_{k \geq 1} B^{-k} a_k, a_k \in \phi(\mathcal{N}) \right. \right\}.$$

The rest of this section follows the ideas of Gittenberger and Thuswaldner [11] and therefore we only show the results and left the proofs to the reader. For the construction of the Urysohn function the border of the fundamental domain is of special interest. We can approximate \mathcal{F} with help of the sets

$$Q_0 := \{z \in \mathbb{R}^2 \mid \|z\|_\infty \leq \tfrac{1}{2}\},$$

$$Q_k := \bigcup_{a \in \mathcal{N}} B^{-1}(Q_{k-1} + \phi(a)).$$

This approximation satisfies $d(\partial Q_k, \partial \mathcal{F}) \ll |b|^{-k}$, where $d(\cdot, \cdot)$ denotes the Hausdorff metric. By consulting [25, 29], we get that the Q_k consists of $|\mathcal{N}|^k$ parallelograms. Furthermore there exists a μ with $1 < \mu < |\mathcal{N}(b)|$ such that $\mathcal{O}(\mu^k)$ of these parallelograms intersect the boundary of Q_k .

Now we define the fundamental domain consisting of all numbers whose first digit equals $a \in \mathcal{N}$, i.e.,

$$\mathcal{F}_a = B^{-1}(\mathcal{F} + \phi(a)).$$

Imitating the proof of Lemma 3.1 of [11] we get the following.

Lemma 4.1. *For all $a \in \mathcal{N}$ and all $k \in \mathbb{N}$ there exists an axe-parallel tube $P_{k,a}$ with the following properties:*

- $\partial \mathcal{F}_a \subset P_{k,a}$ for all $k \in \mathbb{N}$,
- the Lebesgue measure of $P_{k,a}$ is a $\mathcal{O}\left(\frac{\mu^k}{|\mathcal{N}(b)|^k}\right)$,
- $P_{k,a}$ consists of $\mathcal{O}(\mu^k)$ axe-parallel rectangles, each of which has Lebesgue measure $\mathcal{O}(|\mathcal{N}(b)|^k)$,

where λ denotes the Lebesgue measure.

As in the proof of Lemma 3.1 of [11] we can construct for each pair (k, a) an axe-parallel polygon $\Pi_{k,a}$ and the corresponding tube

$$P_{k,a} := \left\{ z \in \mathbb{R}^n \mid \|z - \Pi_{k,a}\|_\infty \leq 2c_p |b|^{-k} \right\}.$$

Furthermore we denote by $I_{k,a}$ the set of all points inside $\Pi_{k,a}$. Now we define our Urysohn function u_a by

$$(4.1) \quad u_a(x_1, \dots, x_n) = \frac{1}{\Delta^n} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \cdots \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \psi_a(x_1 + y_1, \dots, x_n + y_n) dy_1 \cdots dy_n,$$

where

$$(4.2) \quad \Delta := 2c_u |b|^{-k}$$

and

$$\psi_a(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } (x_1, \dots, x_n) \in I_{k,a} \\ \frac{1}{2} & \text{if } (x_1, \dots, x_n) \in \Pi_{k,a} \\ 0 & \text{otherwise.} \end{cases}$$

Thus u_a is the desired Urysohn function which equals 1 for $z \in I_{k,a} \setminus P_{k,a}$, 0 for $z \in \mathbb{R}^n \setminus (I_{k,a} \cup P_{k,a})$, and linear interpolation in between.

We now do a Fourier transform of u_a and estimate the coefficients in the same way as in Lemma 3.2 of [11].

Lemma 4.2. *Let $u_a(x_1, \dots, x_n) = \sum_{(m_1, \dots, m_n) \in \mathbb{Z}^n} c_{m_1, \dots, m_n} e(m_1 x_1 + \dots + m_n x_n)$ be the Fourier expansion of u_a . Then the Fourier coefficients c_{m_1, \dots, m_n} can be estimated by*

$$c_{0, \dots, 0} = \frac{1}{|\mathbf{N}(b)|}, \quad c_{m_1, \dots, m_n} \ll \mu^k \prod_{i=1}^n \frac{1}{r(m_i)}$$

with

$$r(m_i) = \begin{cases} \Delta m_i & m_i \neq 0, \\ 1 & m_i = 0. \end{cases}$$

5. ESTIMATION OF THE BORDER

Before we proof the central proposition in the next chapter, we have to consider the error term, which mainly comes from the number of points within the tube $P_{k,a}$ defined in the previous chapter. Therefore we fix a positive integer k , which we will choose later, and define

$$(5.1) \quad F_j := \# \left\{ z \in M(\mathbf{T}) \left| \phi \left(\frac{p(z)}{b^{j+1}} \right) \in \bigcup_{a \in \mathcal{N}} P_{k,a} \pmod{B^{-1}\mathbb{Z}^n} \right. \right\}.$$

The main target of this section is the estimation of F_j .

Proposition 5.1. *Let $\mu < |\mathbf{N}(b)|$ be as in section 4 and C_l and C_u be sufficiently large positive reals. Suppose that j is a positive integer such that*

$$(5.2) \quad C_l \log \log T \leq j \leq d \log_{\mathbf{N}(b)} T - C_u \log \log T.$$

Then for any positive σ_3

$$F_j \ll \mu^k T^n \left(|\mathbf{N}(b)|^{-k} + (\log T)^{-\sigma_3} \right).$$

In order to estimate F_j we apply the Erdős-Turan-Koksma Inequality.

Lemma 5.2 ([6, Theorem 1.21]). *Let x_1, \dots, x_L be points in the n -dimensional real vector space \mathbb{R}^n and H an arbitrary positive integer. then the discrepancy $D_L(x_1, \dots, x_L)$ fulfills the inequality*

$$D_L(x_1, \dots, x_L) \ll \frac{2}{H+1} + \sum_{0 < \|\mathbf{h}\|_\infty \leq H} \frac{1}{r(\mathbf{h})} \left| \frac{1}{L} \sum_{\ell=1}^L e(\mathbf{h} \cdot x_\ell) \right|,$$

where $\mathbf{h} \in \mathbb{Z}^n$ and $r(\mathbf{h}) = \prod_{i=1}^n \max(1, |h_i|)$.

Moreover we need that all conjugates of b have absolute value greater than one, which is established by the following lemma.

Lemma 5.3 ([19, Theorem 3]). *If (b, \mathcal{N}) is a canonical number system, then*

$$|b^{(i)}| > 1 \quad \text{for } 1 \leq i \leq n.$$

Now we are ready to prove the proposition.

Proof of Proposition 5.1. In order to apply the Erdős-Turán-Koksma Inequality we have to consider rectangles. Recall that the tube $P_{k,a}$ consists of rectangles as mentioned in Lemma 4.1. We split the tube $P_{k,a}$ into this family of rectangles and let R_a be one of them. Then our goal is to estimate

$$F_j(R_a) := \left\{ z \in M(\mathbf{T}) \left| \phi \left(\frac{p(z)}{b^{j+1}} \right) \in R_a \bmod B^{-1}\mathbb{Z}^n \right. \right\}.$$

We set $x_z := \phi(p(z)/b^{j+1})$ and get by the definition of the discrepancy (cf. [6, Definition 1.5]) that

$$(5.3) \quad F_j(R_a) \ll T^n (\lambda(R_a) + D_L(\{x_z\})),$$

where L is the number of elements in $M(\mathbf{T})$ and T is the geometric mean of the T_i . Thus we get by Lemma 3.3 that

$$(5.4) \quad L = \frac{2^r \pi^{r_2}}{\sqrt{D}} T^n + \mathcal{O}(T^{n-1}).$$

In order to estimate the discrepancy we use Lemma 5.2 to get

$$(5.5) \quad D_L(\{x_z\}) \ll \frac{2}{H+1} + \sum_{0 < \|\mathbf{h}\|_\infty \leq H} \frac{1}{r(\mathbf{h})} \left| \frac{1}{L} \sum_{z \in M(\mathbf{T})} e(\mathbf{h} \cdot x_z) \right|.$$

Our aim is the application of Proposition 3.2. Since E is defined in (2.1) as $E = e \circ \text{Tr}$ we have to rewrite the exponential sum with scalar multiplication into one involving the trace. It is easy to see that the following function suffices our purpose.

$$(5.6) \quad \tau(z) := (\text{Tr}(z), \text{Tr}(bz), \dots, \text{Tr}(b^{n-1}z)) = \Xi \phi(z),$$

where $\Xi = VV^T$ and V is the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ b & b^{(2)} & \dots & b^{(n)} \\ \vdots & \vdots & & \vdots \\ b^{n-1} & (b^{(2)})^{n-1} & \dots & (b^{(n)})^{n-1} \end{pmatrix}.$$

Thus we get

$$(5.7) \quad \mathbf{h} \cdot x_z = \mathbf{h} \cdot \phi \left(\frac{p(z)}{b^{j+1}} \right) = \mathbf{h} \Xi^{-1} \tau \left(\frac{p(z)}{b^{j+1}} \right) = \text{Tr} \left(\sum_{i=1}^n \tilde{h}_i b^{i-1} \frac{p(z)}{b^{j+1}} \right),$$

where we have set $(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) := \mathbf{h} \Xi^{-1}$.

Plugging (5.4), (5.5) and (5.7) into (5.3) and applying Lemma 4.1 yields

$$(5.8) \quad F_j(R_a) \ll \lambda(R_a) T^n + \frac{T^n}{H+1} + \sum_{0 < \|\mathbf{h}\|_\infty \leq H} \frac{1}{r(\mathbf{h})} \sum_{z \in M(\mathbf{T})} E \left(\sum_{i=1}^n \tilde{h}_i b^{i-1} \frac{p(z)}{b^{j+1}} \right),$$

where

$$(5.9) \quad H = (\log T)^{\sigma_4}.$$

Now we want to apply Proposition 3.2 for the last sum. Since $\sum_{i=1}^n \tilde{h}_i b^{i-1} p(z)$ is a polynomial of degree d we write ξ for its leading coefficient for short. Then we apply Lemma 3.1 to get $a \in \delta^{-1}$ and $q \in \mathcal{O}_K$ such that

$$\left| \frac{\xi^{(i)}}{(b^{(i)})^{j+1}} q^{(i)} - a^{(i)} \right| \leq T_i^{-d} (\log T)^{\sigma_1} \quad \text{and} \quad 0 < q^{(i)} \leq T_i^d (\log T)^{-\sigma_1} \quad \text{for } 1 \leq i \leq r.$$

Now we distinguish three cases according to the size of $\lceil q \rceil$.

- **Case 1**, $\lceil q \rceil \geq (\log T)^{\sigma_1}$: By Proposition 3.2 we get

$$\sum_{z \in M(\mathbf{T})} E \left(\sum_{i=1}^n \tilde{h}_i b^{i-1} \frac{p(z)}{b^{j+1}} \right) \ll T^n (\log T)^{-\sigma_0}.$$

- **Case 2**, $2 \leq \lceil q \rceil < (\log T)^{\sigma_1}$: Let $1 \leq i \leq n$ be such that $|q^{(i)}| \geq 2$. Then by noting $|q^{(i)}| \leq \lceil q \rceil$ we get

$$\left| \frac{\xi^{(i)}}{(b^{(i)})^{j+1}} q^{(i)} - a^{(i)} \right| < \frac{(\log T)^{\sigma_1}}{T_i^d} \leq \frac{1}{|q^{(i)}|}$$

and thus (using our first successive minimum λ_1)

$$(5.10) \quad \left| \frac{\xi^{(i)}}{(b^{(i)})^{j+1}} \right| \geq \left| \frac{a^{(i)}}{q^{(i)}} \right| - \frac{1}{|q^{(i)}|^2} \geq \lambda_1 \left(\frac{1}{|q^{(i)}|} - \frac{1}{|q^{(i)}|^2} \right) \geq \lambda_1 \frac{1}{2|q^{(i)}|} \gg (\log T)^{-\sigma_1}.$$

Therefore by noting Lemma 5.3 and our definition of H in (5.9) we have

$$|b^{(i)}|^{j+1} \ll |\xi^{(i)}| (\log T)^{\sigma_1} \ll nH(\log T)^{\sigma_1} \ll (\log T)^{\sigma_1 + \sigma_4},$$

which yields

$$j \ll \frac{(\sigma_1 + \sigma_2)}{\log |b|} \log \log T$$

contradicting the lower bound for j in (5.2) for sufficiently large C_l .

- **Case 3**, $0 < \lceil q \rceil < 2$: We have to consider two further cases according to whether $a \neq 0$ or $a = 0$.

- **Case 3.1**, $\left| \frac{\xi}{b^{j+1}} q \right| \geq \frac{\lambda_1}{2}$: Let $1 \leq i \leq n$ be such that $\left| \frac{\xi^{(i)}}{(b^{(i)})^{j+1}} q^{(i)} \right| \geq \frac{\lambda_1}{2}$. Then we get with H as in (5.9)

$$|b^{(i)}|^{j+1} \ll H = (\log T)^{\sigma_4}$$

which again contradicts the lower bound of j in (5.2).

- **Case 3.2**, $\left| \frac{\xi}{b^{j+1}} q \right| < \frac{\lambda_1}{2}$: Since λ_1 is the first successive minima for the sphere with respect to the lattice of δ^{-1} we get that $a = 0$. Thus for $1 \leq i \leq n$

$$\left| \frac{\xi^{(i)}}{(b^{(i)})^{j+1}} q^{(i)} \right| < \frac{(\log T)^{\sigma_1}}{T_i^d}.$$

Talking the norm we get

$$N \left(\frac{\xi}{b^{j+1}} q \right) = \frac{N(\xi)}{N(b^{j+1})} N(q) < \frac{(\log T)^{n\sigma_1}}{T^{nd}}.$$

This implies

$$j+1 \geq nd \log_{|N(b)|} T - \frac{n\sigma}{\log |N(b)|} \log \log T + C$$

contradicting the upper bound of j in (5.2) for sufficiently large C_u .

Thus we get for all j satisfying (5.2), that

$$\sum_{z \in M(\mathbf{T})} E \left(\sum_{i=1}^n \tilde{h}_i b^{i-1} \frac{f(z)}{b^{j+1}} \right) \ll T^n (\log T)^{-\sigma_0}.$$

Plugging this into (5.8) together with (5.9) yields

$$\begin{aligned} F_j(R_a) &\ll T^n \lambda(R_a) + \frac{T^n}{(\log T)^{\sigma_4}} + T^n (\log T)^{-\sigma_0} \sum_{0 < \|\mathbf{h}\|_\infty \leq H} \frac{1}{r(\mathbf{h})} \\ &\ll T^n \lambda(R_a) + \frac{T^n}{(\log T)^{\sigma_4}} + T^n (\log T)^{-\sigma_0} (\log \log T)^n. \end{aligned}$$

Finally setting $\sigma_4 := \sigma_0/2$ and summing over all rectangles R_a yields

$$F_j \ll \mu^k T^n \left(|N(b)|^{-k} + (\log T)^{-\sigma_0/2} \right).$$

By setting $\sigma_3 = \sigma_0/2$ the proposition is proven. \square

6. THE CENTRAL PROPOSITION

Before we can state the central proposition we need a lemma estimating the length of the expansion of every integer. This will provide us with an upper bound for the scope of interest (namely for the T_i in $M(\mathbf{T})$) and further establish us with the tools needed in order to prove the Theorem 2.1.

Lemma 6.1 ([20, Theorem]). *Let $\ell(\gamma)$ be the length of the expansion of γ to the base b . Then*

$$\left| \ell(\gamma) - \max_{1 \leq i \leq n} \frac{\log |\gamma^{(i)}|}{\log |b^{(i)}|} \right| \leq C.$$

We fix a T and set T_i for $1 \leq i \leq n$ such that

$$(6.1) \quad \log T_i = \log T \frac{\log |b^{(i)}|^n}{\log |N(b)|}.$$

Thus we get in view of Lemma 6.1 that the expansions of the elements of $M(\mathbf{T})$ have the same maximum length. Now we are in the position to state the central proposition in the proof of Theorem 2.1.

Proposition 6.2. *Let $T \geq 0$ and T_i for $1 \leq i \leq n$ be defined as in (6.1). Let L be the maximum length of the b -adic expansion of $z \in M(\mathbf{T})$ and let C_l and C_u be sufficiently large. Then for*

$$(6.2) \quad C_l \log L \leq l_1 < l_2 < \dots < l_h \leq dL - C_u \log L$$

we have,

$$\Theta := \# \{z \in M(\mathbf{T}) \mid a_{l_j}(f(z)) = b_j, j = 1, \dots, h\} = \frac{2^r \pi^{r_2}}{\sqrt{D} |N(b)|^h} T^n + \mathcal{O}(T^n (\log T)^{-\sigma_0})$$

uniformly for $T \rightarrow \infty$, where $b_j \in \mathcal{N}$ are given and σ_0 is an arbitrary positive constant.

Proof. We recall our Urysohn function u_a (defined in (4.1)) and set for $\nu \in \mathbb{R}^n$

$$t(\nu) = u_{b_1}(B^{-l_1-1}\nu) \dots u_{b_h}(B^{-l_h-1}\nu).$$

Now we want to apply the Fourier transformation, which we developed in Lemma 4.2. Therefore we set

$$\mathcal{M} := \{M = (\mu_1, \dots, \mu_h) \mid \mu_j = (m_{j1}, \dots, m_{jn}) \in \mathbb{Z}^n, \text{ for } j = 1, \dots, h\}.$$

An application of Lemma 4.2 yields

$$t(\nu) = \sum_{M \in \mathcal{M}} T_M e \left(\sum_{j=1}^h \mu_j B^{-l_j-1} \nu \right),$$

where $T_M = \prod_{j=1}^h c_{m_{j1}, \dots, m_{jn}}$. Combining these results we get

$$(6.3) \quad \left| \Theta - \sum_{z \in M(\mathbf{T})} t(\phi(p(z))) \right| \leq F_{l_1} + \dots + F_{l_h}.$$

Using the function τ defined in (5.6) we get

$$\sum_{z \in M(\mathbf{T})} t(\phi(p(z))) = \sum_{M \in \mathcal{M}} T_M \sum_{z \in M(\mathbf{T})} e \left(\sum_{j=1}^h \mu_j B^{-l_j-1} \Xi^{-1} \tau(p(z)) \right).$$

Setting

$$\tilde{\mu}_j = (\tilde{m}_{j1}, \dots, \tilde{m}_{jn}) := \mu_j B^{-l_j-1} \Xi^{-1} \quad (j = 1, \dots, h)$$

yields

$$\sum_{z \in M(\mathbf{T})} t(\phi(p(z))) = \sum_{M \in \mathcal{M}} T_M \sum_{z \in M(\mathbf{T})} E \left(\sum_{j=1}^h \sum_{i=1}^n \frac{\tilde{m}_{ji} p(z)}{b^{l_j+1}} \right).$$

We want to apply the same ideas as in the proof of Proposition 5.1. For $1 \leq j \leq h$ we set ξ_j to be the leading coefficient of $\sum_{i=1}^n \tilde{m}_{ji} p(z)$. Then we apply Lemma 3.1 with $N_i = T_i^d (\log T)^{-\sigma_1}$ for $1 \leq i \leq r$ in order to get that there exist $a \in \delta^{-1}$ and $q \in \mathcal{O}_K$ such that

$$\left| \sum_{j=1}^h \frac{\xi_j^{(i)}}{(b^{(i)})^{l_j+1}} q^{(i)} - a^{(i)} \right| < \frac{(\log T)^{\sigma_1}}{T_i^d} \quad \text{and} \quad 0 < |q^{(i)}| < \frac{T_i^d}{(\log T)^{\sigma_1}} \quad \text{for } 1 \leq i \leq n.$$

Now we again distinguish several cases.

- **Case 1**, $\lceil q \rceil \geq (\log T)^{\sigma_1}$: We apply Proposition 3.2 and get

$$\sum_{z \in M(\mathbf{T})} E \left(\sum_{j=1}^h \sum_{i=1}^n \frac{\tilde{m}_{ji} p(z)}{b^{l_j+1}} \right) \ll T^n (\log T)^{-\sigma_0}.$$

- **Case 2**, $2 \leq \lceil q \rceil < (\log T)^{\sigma_1}$: In the same manner as in (5.10) we get

$$(\log T)^{\sigma_1} \leq \left| \sum_{j=1}^h \frac{\xi_j^{(i)}}{(b^{(i)})^{l_j+1}} \right| \leq \frac{\sum_{j=1}^h |\xi_j^{(i)}|}{|b^{(i)}|^{l_1+1}},$$

$$l_1 + 1 \ll \log \log T.$$

Together with the definition of L this contradicts the lower bound of l_1 for sufficiently large C_l in (6.2).

- **Case 3**, $0 < \lceil q \rceil < 2$: We have to consider two sub cases
 - **Case 3.1**, $\left| \sum_{j=1}^h \frac{\xi_j}{b^{l_j+1}} q \right| \geq \frac{\lambda_1}{2}$: Let $1 \leq i \leq n$ be such that

$$\frac{\lambda_1}{2} \leq \left| \sum_{j=1}^h \frac{\xi_j^{(i)}}{(b^{(i)})^{l_j+1}} q^{(i)} \right| \leq \frac{\sum_{j=1}^h |\xi_j^{(i)}|}{|b^{(i)}|^{l_1+1}} |q^{(i)}|,$$

then

$$l_1 + 1 \ll \log \log T$$

again contradicts the lower bound of l_1 for sufficiently large C_l in (6.2).

- **Case 3.2**, $\left| \sum_{j=1}^h \frac{\xi_j}{b^{l_j+1}} q \right| < \frac{\lambda_1}{2}$: By Minkowski's theorem we get that $a = 0$. Thus for $1 \leq i \leq n$

$$\left| \sum_{j=1}^h \frac{\xi_j^{(i)}}{(b^{(i)})^{l_j+1}} q^{(i)} \right| = \left| \frac{1}{(b^{(i)})^{l_h+1}} \sum_{j=1}^h \xi_j^{(i)} (b^{(i)})^{l_h-l_j} q^{(i)} \right| \leq \frac{(\log T)^{\sigma_1}}{T_i^d}$$

which implies (taking the norm of the left side)

$$l_h + 1 \geq nd \log_{|N(b)|} T - c(\log \log_{|N(b)|} T)$$

contradicting the upper bound for sufficiently large C_u .

After these considerations it is clear, that **Case 1** is the only possible case which suffices the hypotheses in (6.2). Plugging this into (6.3) and applying Lemma 4.2 and Lemma 3.3 for the coefficient $c_{0,\dots,0}$ yields

$$\Theta = \frac{2^r \pi^{r^2}}{\sqrt{D} |N(b)|^h} T^n + \mathcal{O} \left(T^n (\log T)^{-\sigma_0} \sum_{0 \neq M \in \mathcal{M}} T_M \right) + \mathcal{O} \left(\sum_{j=1}^h F_j \right).$$

An application of Proposition 5.1 with $k \ll \log \log T$ and the observation that

$$\sum_{M \in \mathcal{M}} |T_M| \ll \Delta^{-2h} \ll |b|^{2hk} \ll (\log T)^{\sigma_0/2},$$

where we used the definition of Δ in (4.2), proves the proposition. \square

7. PROOF OF THEOREM 2.1

At this point we will need the Fréchet-Shohat Theorem which we state for completeness.

Lemma 7.1 ([9, Lemma 1.43]). *Let $(F_n(z))_{n \geq 1}$ be a sequence of distribution functions. For each non-negative integer k let*

$$\alpha_k = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} z^k dF_n(z)$$

exist.

Then there is a subsequence of $F_{n_j}(z)$ ($n_1 < n_2 < \dots$), which converges weakly to a limiting distribution $F(z)$ for which

$$\alpha_k = \int_{-\infty}^{\infty} z^k dF(z) \quad (k = 0, 1, \dots).$$

Moreover, if the set of moments α_k determine $F(z)$ uniquely, then as $n \rightarrow \infty$ the distribution $F_n(z)$ converge weakly to $F(z)$.

For the proof of Theorem 2.1 we mainly follow the proof of Theorem 1.2 of Gittenberger and Thuswaldner [11]. In the same manner as in their proof we set $C := \max(C_l, C_u)$, $A := \lceil C \log L \rceil$ and $B := L - A$, where L , C_l and C_u are defined in the statement of Proposition 6.2. Furthermore we define the truncated function f' to be

$$f'(p(z)) = \sum_{j=A}^B f(a_j(p(z))b^j).$$

By the definition of A and $f(ab^j) \ll 1$ with $a \in \mathcal{N}$ we get that $f'(p(z)) = f(p(z)) + \mathcal{O}(\log L)$. In the same manner we define the truncated mean and standard deviation

$$M'_b(T) := \sum_{j=A}^B m_j \quad \text{and} \quad D'^2_b(T) := \sum_{j=A}^B \sigma_j^2.$$

Since $M_b(T) - M'_b(T) = \mathcal{O}(\log L)$ and $D_b^2(T) - D'^2_b(T) = \mathcal{O}(\log L)$ we get that it suffices to show that

$$\frac{1}{\#M(T)} \# \left\{ z \in M(T) \left| \frac{f'(p(z)) - M'_b(T^d)}{D'_b(T^d)} < y \right. \right\} \longrightarrow \Phi(y).$$

By Lemma 7.1 this holds true if and only if the moments

$$\xi_k(T) := \frac{1}{\#M(T)} \sum_{z \in M(T)} \left(\frac{f'(p(z)) - M'_b(T^d)}{D'_b(T^d)} \right)^k$$

converge to the moments of the normal law for $T \rightarrow \infty$. We will show the last statement by comparing the moments ξ_k with

$$\eta_k(T) := \frac{1}{\#M(T^d)} \sum_{z \in M(T^d)} \left(\frac{f'(z) - M'_b(T^d)}{D'_b(T^d)} \right)^k.$$

An application of Proposition 6.2 gives that

$$\xi_k(T) - \eta_k(T) \rightarrow 0 \quad \text{for } T \rightarrow \infty.$$

Furthermore we get by Lemma 6.1 that these sums consist of independently identically distributed random variables (with possible $2C$ exceptions). By the central limit theorem we get that their distribution converges to the normal law. Thus the $\eta_k(T)$ converge to the moments of the normal law. This yields

$$\lim_{T \rightarrow \infty} \xi_k(T) = \lim_{T \rightarrow \infty} \eta_k(T) = \int x^k d\Phi.$$

We apply the Fréchet-Shohat Theorem again to prove the theorem.

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