CONVERGENCE PROPERTIES OF THE q-DEFORMED BINOMIAL DISTRIBUTION

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ABSTRACT. We consider the q-deformed binomial distribution introduced by Jing 1994 and Chung et al. 1995 and establish several convergence results involving the Euler and the exponential distribution; some of them are q-analogues of classical results.

1. INTRODUCTION

The q-deformed binomial distribution was introduced by Jing [5] in connection with the q-deformed boson oscillator and by Chung et al. [2]. Its probabilites are given by

$$\mathbb{P}(X_{QD} = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \tau^x (\tau; q)_{n-x}, \qquad 0 \le x \le n, \ 0 \le \tau \le 1,$$

where

$$\begin{bmatrix} n \\ x \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_x (q; q)_{n-x}} \qquad \text{and} \qquad (z; q)_n = \prod_{i=0}^{n-1} (1 - zq^i)$$

are the q-binomial coefficient and the q-shifted Pochhammer symbol; an introduction to the q-calculus and basic hypergeometric series can be found in Gasper and Rahman [3]. A random variable X which is q-deformed binomial distributed let us denote by $X \sim QD(n, \tau, q)$. Elementary and asymptotic properties are given in Jing [5], Jing and Fan [6], Kemp [10, 11] and in the encyclopaedic book Johnson, Kemp and Kotz [7]. Chung et al. [2] and Kupershmidt [12] gave a representation of the q-deformed binomial distribution as a sum of dependent and not identically distributed random variables. One important property is the convergence for fixed paramter τ and $n \to \infty$ to an Euler distribution with parameter $\lambda = \tau/(1-q)$, i.e.

$$\mathbb{P}(X_E = x) = \frac{\lambda^x}{(q; q)_x} (-\lambda; q)_\infty.$$

This is a q-analogue of the Poisson distribution since $E((1-q)\lambda) \to P(\lambda)$ for $q \to 1$. For properties of this distribution we refer to [7], Kemp [8, 9, 11] and Charalambides and Papadatos [1]. The q-deformed binomial distribution is a q-analogue of the classical binomial distribution, since in the limit $q \to 1$ the q-deformed binomial distribution with paramter (n, τ, q) reduces to the binomial distribution with parameters (n, τ).

This paper is devoted to study sequences of random variables X_n, where the X_n are $QD(n, \tau_n, q)$-distributed and τ_n converges to a limit $c \in [0, 1]$. Section 2 deals with the case $c \in [0, 1)$. In particular we show that there is a q-analogue of the

convergence of the classical binomial distribution with constant mean to a Poisson distribution. In Section 3 we examine the case $c = 1$. Here the limit distribution can be - depending on the growth rate of the parameters τ_n - degenerate, truncated exponential-like or exponential.

2. Parameter sequences with limit < 1

In the present section we study sequences of random variables X_n which are $QD(n, \tau_n, q)$-distributed, where the parameters τ_n converge to a $c \in [0, 1)$. In particular we prove a q-analogue of the convergence of the classical binomial distribution with constant mean to a Poisson distribution.

As noted above the sequence converges in the case of constant parameters $\tau_n = \tau$ to an Euler distribution with parameter $\tau/(1-q)$. The following theorem shows that the Euler distribution is the limit distribution for every convergent parameter sequence τ_n with limit in $[0, 1)$.

Theorem 2.1. *Let $X_n \sim QD(n, \tau_n, q)$. Then, for $n \to \infty$,*

$$X_n \to E(\tau, q)$$

if $\tau_n \to \tau$ and $0 \leq \tau < 1$.

Proof. Observe that [4, Lemma 2.1] still remains true for $\theta > -1$. Then the theorem follows immediately from the facts $e_q(x) = 1/E_q(-x)$ and

$$\begin{bmatrix} n \\ x \end{bmatrix}_q \to \frac{1}{(q; q)_x}$$

for $n \to \infty$. $\square$

We are now interested in special choices of the parameters τ_n such that the limit $X(q)$ of the sequence $X_n(q)$ converges to a Poisson distribution for $q \to 1$. From the previous theorem we conclude immediately

Theorem 2.2. *Let $X_n \sim QD(n, \tau_n(q), q)$ with $\tau_n(q) \to \frac{\lambda}{n}$ for $q \to 1$ and $\tau_n(q) \to \tau(q)$ for $n \to \infty$ with the additional property $\frac{\tau(q)}{1-q} \to \lambda$ in the limit $q \to 1$. Then the following diagram is commutative:*

$$
\begin{array}{ccc}
QD(n, \tau_n, q) & \xrightarrow{\ n \to \infty\ } & E(\tau(q), q) \\
{\scriptstyle q \to 1} \downarrow & & \downarrow {\scriptstyle q \to 1} \\
B\left(n, \frac{\lambda}{n}\right) & \xrightarrow[n \to \infty]{} & P(\lambda)
\end{array}
$$

One very natural way to choose the parameters is to set $\tau_n = \frac{\lambda}{[n]_q}$.

Our next goal is to establish a convergence result, which is analogous to the convergence of the classical binomial distribution with constant mean to a Poisson distribution and reduces in the limit $q \to 1$ to that theorem. For this purpose we start with an elementary fact.

Lemma 2.3. *Let $f_n(x)$, $n \in \mathbb{N}$, be a sequence of continuous functions which converge pointwise to a continuous limit $f(x)$. Assume that for each n the function $f_n(x)$ has a single root $\hat{x}_n$, and $f(x)$ has a single root $\hat{x}$. Then $\hat{x}_n \to \hat{x}$.*

Proof. For given $\varepsilon > 0$ choose a $\delta(\varepsilon) < \min(f(\hat{x}+\varepsilon), -f(\hat{x}-\varepsilon))$. Then there exists a $N = N(\delta(\varepsilon))$ such that for all $n \geq N$ we have $|f_n(\hat{x} + \varepsilon) - f(\hat{x} + \varepsilon)| < \delta(\varepsilon)$. Therefore $f_n(\hat{x} + \varepsilon) > 0$. Moreover there exists a $M = M(\delta(\varepsilon))$ such that for all $n \geq M$ we have $|f_n(\hat{x} - \varepsilon) - f(\hat{x} - \varepsilon)| < \delta(\varepsilon)$. Therefore $f_n(\hat{x} - \varepsilon) < 0$. Hence, by continuity, for all $n \geq \max(N, M)$ we have $|\hat{x} - \hat{x}_n| < 2\varepsilon$. $\square$

The essential key to apply this lemma is the following representation of the means $\mu_n(\tau, q)$, which allows us to extract important properties of the means easily.

Lemma 2.4. *The means $\mu_n(\tau, q)$ have the alternative representation*

$$\mu_n(\tau, q) = \sum_{j=1}^{n} (q; q)_{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \tau^j.$$

Proof. By definition, the means are given by

$$\mu_n(\tau, q) = \sum_{x=1}^{n} x\tau^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau; q)_{n-x}.$$

Using the well-known formula (eg. see [3, p. 20])

$$(1) \qquad (z; q)_k = \sum_{i=0}^{k} (-1)^i \begin{bmatrix} k \\ i \end{bmatrix}_q q^{i(i-1)/2} z^i$$

we obtain

$$\mu_n(\tau) = \sum_{x=1}^{n} x\tau^x \begin{bmatrix} n \\ x \end{bmatrix}_q \sum_{i=0}^{n-x} (-1)^i \begin{bmatrix} n-x \\ i \end{bmatrix}_q q^{i(i-1)/2} \tau^i.$$

Extracting coefficients in this formula we get

$$[\tau^j]\mu_n(\tau) = \sum_{x=1}^{j} x \begin{bmatrix} n \\ x \end{bmatrix}_q (-1)^{j-x} \begin{bmatrix} n-x \\ j-x \end{bmatrix}_q q^{(j-x)(j-x-1)/2}$$

$$(2) \qquad = \frac{[n]_q!}{[n-j]_q!} \sum_{x=1}^{j} x(-1)^{j-x} q^{(j-x)(j-x-1)/2} \frac{1}{[x]_q![j-x]_q!}.$$

Now we show by induction that

$$S(j) := \sum_{x=1}^{j} x(-1)^{j-x} q^{(j-x)(j-x-1)/2} \frac{[j]_q!}{[x]_q![j-x]_q!} = (q; q)_{j-1}.$$

For $j = 1$ this is true. Now suppose that this formula is shown for $1 \leq i \leq j-1$. Using the recursion formula for the binomial coefficients and splitting sums we get

$$S(j) = j + \sum_{x=1}^{j-1} x(-1)^{j-x} q^{(j-x)(j-x-1)/2} \left(\begin{bmatrix} j-1 \\ x \end{bmatrix}_q q^x + \begin{bmatrix} j-1 \\ x-1 \end{bmatrix}_q \right)$$

$$= j + \sum_{x=1}^{j-1} x(-1)^{j-x} q^{(j-x-1)(j-x-2)/2 + j-x-1+x} \begin{bmatrix} j-1 \\ x \end{bmatrix}_q$$

$$+ \sum_{x=2}^{j-1} (x-1)x(-1)^{j-x} q^{(j-x)(j-x-1)/2} \begin{bmatrix} j-1 \\ x-1 \end{bmatrix}_q + \sum_{x=1}^{j-1} (-1)^{j-x} q^{(j-x)(j-x-1)/2} \begin{bmatrix} j-1 \\ x-1 \end{bmatrix}_q.$$

Notice that the first sum equals $q^{j-1}S(j-1)$. Shifting indices in the other sums we obtain

$$S(j) = -q^{j-1}(q; q)_{j-2} + (j-1) + \sum_{x=1}^{j-2} x(-1)^{j-x-1} q^{(j-x-1)(j-x-2)/2} \begin{bmatrix} j-1 \\ x \end{bmatrix}_q$$

$$+ 1 + \sum_{x=0}^{j-2} (-1)^{j-x-1} q^{(j-x-1)(j-x-2)/2} \begin{bmatrix} j-1 \\ x \end{bmatrix}_q.$$

Now combine the second and third term as well as the fourth and the fifth term to

$$S(j) = (1 - q^{j-1})(q;q)_{j-2} + \sum_{x=0}^{j-1} (-1)^{j-x-1} q^{(j-x-1)(j-x-2)/2} \begin{bmatrix} j-1 \\ x \end{bmatrix}_q$$

$$= (q;q)_{j-1} + (1;q)_{j-1} = (q;q)_{j-1},$$

where we used formula (1). $\square$

Remark 2.5. *An alternative way to prove this lemma is to use Kemp's* [10, p. 300] *representation of the probability generating function, to differentiate and to manipulate the sum.*

Using the monotonicity of the q-binomial coefficients in n we immediately get

Theorem 2.6. *The means $\mu_n(\tau, q)$ are strictly increasing in n (for $\tau > 0$) and τ.*

Now we turn to the convergence result:

Theorem 2.7. *Fix $\mu > 0$. Then the following commutative diagram is valid:*

$$
\begin{array}{ccc}
QD(n, \tau_n(q), q) & \xrightarrow{\ n \to \infty\ } & E(\tau(q), q) \\
{\scriptstyle q \to 1} \downarrow & & \downarrow {\scriptstyle q \to 1} \\
B\left(n, \frac{\mu}{n}\right) & \xrightarrow[\ n \to \infty\ ]{} & P(\mu)
\end{array}
$$

with $\tau_n(q)$ determined by $\mu_n(\tau, q) = \mu$ and $\tau(q) = \lim_{n \to \infty} \tau_n(q)$.

Proof. First we check, that for given $\mu > 0$ and fixed q there exists a unique sequence $(\tau_n(q))_{n \geq N}$ with $\mu_n(\tau_n(q), q) = \mu$. The function $\mu_n(\tau, q)$ is continuous and strictly increasing in n and τ by the previous theorem. Moreover, we have $\lim_{\tau \to 0} \mu_n(\tau, q) = 0$. Choosing τ_n in such a way that $\tau_n^n \to 1$, then $\mu_n(\tau_n, q)$ becomes arbitrary large. Consequently there is a unique solution of $\mu_n(\tau, q) = \mu$. By Lemma 2.3 the sequence $\tau_n(q)$ converges to a limit $\tau(q)$ where $\tau(q)$ is the unique solution of $\mu_E(\tau, q) = \mu$, where $\mu_E(\tau, q)$ is the mean of an Euler-distribution mit parameters τ and q. This mean can be written as

$$\mu_E(\tau, q) = \sum_{i=0}^{\infty} \frac{q^i \tau}{1 - q^i \tau},$$

see [8] or take the limit $n \to \infty$ in Lemma 2.4 and manipulate the sum.

Again be Lemma 2.3 we get that $\tau(q) \to \mu/n$. It remains to check that $\tau_n(q)/(1 - q)$ converges to μ in the limit $q \to 1$. But this is again a consequence of Lemma 2.3 since $\tau(q)/(1 - q)$ is the unique solution of $\mu_E((1 - q)\tau, q) = \mu$ and $\mu_E((1 - q)\tau, q)$ tends to τ for $q \to 1$. $\square$

3. Parameter sequences with limit 1

In this section we investigate in sequences X_n of random variables, where X_n is $QD(n, \tau_n, q)$-distributed and the parameters τ_n converge to 1. The behaviour of the sequences X_n depends on the growth rate of τ_n. Therefor we will distinguish three cases: Firstly we examine the case $\tau_n^n \to 1$, where it will turn out that the limit distribution is degenerate. Then we study the case $\tau_n^n \to c$ with $0 < c < 1$. Here the limit distribution will depend only on c and be like a truncated exponential distribution. Finally we turn to the case $\tau_n^{f(n)} \to c$ where $0 < c < 1$ and $f(n) = o(n)$; this will lead to an exponential distribution.

Consider sequences of random variables $X_n \sim QD(n, \tau_n, q)$ with $\tau_n \to 1$ and additionally $\tau_n^n \to 1$ first. Then we have the following theorem:

Theorem 3.1. *Let $X_n \sim QD(n, \tau_n, q)$ with $\tau_n \to 1$ and $\tau_n^n \to 1$. Then $n - X_n$ converges to the point measure at 0.*

Proof. The probability that $Y_n = n - X_n$ is equal to 0 is given by

$$\mathbb{P}(Y_n = 0) = \tau_n^n$$

which converges to 1 by assumption. $\square$

Now let us investigate in sequences $X_n \sim QD(n, \tau_n, q)$, where $\tau_n \to 1$ and $\tau_n^n \to c$ for a $c \in (0, 1)$. Before we can establish the distribution of the limit of such a sequence, we start with several lemmas, which allow us to compute the asymptotic behaviour of certain sums of probabilities of $QD(n, \tau_n, q)$-distributed random variables and its means and variances.

The first lemma is an analogue to Lemma 2.4 and gives an alternative representation of the variance:

Lemma 3.2. *The second moment of $X_n(\tau, q)$ can be written as*

$$\sum_{x=1}^{n} x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q \tau^x (q; q)_{n-x} = \sum_{j=1}^{n} a_j \tau^j$$

with

$$a_j = \begin{bmatrix} n \\ j \end{bmatrix}_q (q; q)_{j-1} \left(1 + 2 \sum_{i=1}^{j-1} \frac{1}{1 - q^{j-i}} \right).$$

Proof. Starting as in the proof of Lemma 2.4 we obtain for the coefficients of τ^j in $\mathbb{E}(X^2)$ the formula

$$[\tau^j]\mathbb{E}(X^2) = \begin{bmatrix} n \\ j \end{bmatrix}_q \sum_{x=1}^{j} x^2 (-1)^{j-x} q^{(j-x)(j-x-1)/2} \begin{bmatrix} j \\ x \end{bmatrix}_q.$$

It remains to show that

$$T(j) := \sum_{x=1}^{j} x^2 (-1)^{j-x} q^{(j-x)(j-x-1)/2} \begin{bmatrix} j \\ x \end{bmatrix}_q = (q; q)_{j-1} \left(1 + 2 \sum_{i=1}^{j-1} \frac{1}{1 - q^{j-i}} \right).$$

For this purpose we use induction. For $j = 1$ this formula is true. Now suppose that it has been shown for $i \le j - 1$. We use similar ideas as before and write

$$T(j) = j^2 + \sum_{x=1}^{j-1} x^2 (-1)^{j-x} q^{(j-x)(j-x-1)/2} \left(\begin{bmatrix} j-1 \\ x \end{bmatrix}_q q^x + \begin{bmatrix} j-1 \\ x-1 \end{bmatrix}_q \right)$$

$$= j^2 - q^{j-1} T(j-1) + \sum_{x=2}^{j-1} (x-1)^2 (-1)^{j-x} q^{(j-x)(j-x-1)/2} \begin{bmatrix} j-1 \\ x-1 \end{bmatrix}_q$$

$$+ \sum_{x=1}^{j-1} (2x-1)(-1)^{j-x} q^{(j-x)(j-x-1)/2} \begin{bmatrix} j-1 \\ x-1 \end{bmatrix}_q.$$

Shifting index in the first sum and adding $(j-1)^2$ we obtain

$$T(j) = \left(1 - q^{j-1} \right) T(j-1) + (2j-1) + \sum_{x=1}^{j-1} (2x-1)(-1)^{j-x} q^{(j-x)(j-x-1)/2}$$

$$= \left(1 - q^{j-1} \right) T(j-1) + 2(j-1) + 2 \sum_{x=0}^{j-2} x(-1)^{j-x-1} q^{(j-x-1)(j-x-2)/2} \begin{bmatrix} j-1 \\ x \end{bmatrix}_q$$

$$+ 1 + \sum_{x=0}^{j-2} (-1)^{j-x-1} q^{(j-x-1)(j-x-2)/2} \begin{bmatrix} j-1 \\ x \end{bmatrix}_q.$$

Now we use the proof of Lemma 2.4 again and induction hypothesis to get

$$T(j) = \left(1 - q^{j-1}\right) T(j-1) + 2S(j-1)$$

$$= \left(1 - q^{j-1}\right)(q;q)_{j-2}\left(1 + 2\sum_{i=1}^{j-2}\frac{1}{1-q^{j-i-1}}\right) + 2(q;q)_{j-2}$$

$$= (q;q)_{j-1}\left(1 + 2\sum_{i=1}^{j-2}\frac{1}{1-q^{j-i-1}} + \frac{2}{1-q^{j-1}}\right)$$

$$= (q;q)_{j-1}\left(1 + 2\sum_{i=1}^{j-1}\frac{1}{1-q^{j-i}}\right),$$

which completes the proof. □

The next three lemmas are devoted to the asymptotic behaviour of sums of powers of θ_n, where $0 < \theta_n < 1$ and $\theta_n \to 1$.

Lemma 3.3. *If* $f(n) \to \infty$ *for* $n \to \infty$ *and* $\theta_n \le 1$ *such that* $\theta_n^{f(n)} \to c$ *with* $0 < c < 1$, *then*

$$\sum_{i=0}^{\infty}\theta_n^i \sim \frac{-f(n)}{\log c}, \qquad n \to \infty.$$

Proof. We use Euler-MacLaurin's formula and obtain

$$\sum_{i=0}^{\infty}\theta_n^i = \int_0^{\infty}\theta_n^x \mathrm{d}x + \frac{1}{2} + \log\theta_n\int_0^{\infty}\left(\{x\} - \frac{1}{2}\right)\theta_n^x \mathrm{d}x$$

$$= \frac{-1}{\log\theta_n} + \frac{1}{2} + \log\theta_n\int_0^{\infty}\left(\{x\} - \frac{1}{2}\right)\theta_n^x$$

$$= \frac{-f(n)}{\log\theta_n^{f(n)}} + \frac{1}{2} + \log\theta_n\int_0^{\infty}\left(\{x\} - \frac{1}{2}\right)\theta_n^x \sim \frac{-f(n)}{\log c},$$

since the term with the integral is bounded. □

Lemma 3.4. *For* $\theta_n \le 1$ *and* $\theta_n \to 1$, $\theta_n^{f(n)} \to c$ *for a* $c \in (0,1)$ *and* $g(n)/f(n) \sim \beta$, $g(n) \le n$ *we have*

$$\sum_{i=0}^{\lfloor g(n)\rfloor}\theta_n^i \sim \frac{c^\beta - 1}{\log c}f(n)$$

and

$$\sum_{i=0}^{\lfloor g(n)\rfloor}\begin{bmatrix}n\\i\end{bmatrix}_q\theta_n^i \sim e_q(q)\frac{c^\beta - 1}{\log c}f(n)$$

as $n \to \infty$.

Proof. For the first sum, we rewrite this sum as

$$\sum_{i=0}^{\lfloor g(n)\rfloor}\theta_n^i = \frac{1 - \theta_n^{\lfloor g(n)\rfloor+1}}{1 - \theta_n}.$$

The growth of the denominator is given in Lemma 3.3, and the numerator tends to $1 - c^\beta$, since $\theta_n^{\lfloor g(n)\rfloor} = \theta_n^{g(n)-\{g(n)\}} \to c^\beta$ because of $\theta_n \to 1$.

To get the asymptotic of the second sum we write

$$\sum_{i=0}^{\lfloor g(n)\rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i = \sum_{i=0}^{\lfloor \sqrt{g(n)}\rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i + \sum_{\lfloor \sqrt{g(n)}\rfloor +1}^{\lfloor g(n)-\sqrt{g(n)}\rfloor -1} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i \quad + \sum_{\lfloor g(n)-\sqrt{g(n)}\rfloor}^{\lfloor g(n)\rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i.$$

The first and the third sum on the right-hand side are $\mathcal{O}(\sqrt{g(n)})$ and therefore asymptotically negligible. The second sum ist bounded by

$$\frac{(q;q)_n}{(q;q)_{\lfloor \sqrt{g(n)}\rfloor +1}(q;q)_{n-\lfloor \sqrt{g(n)}\rfloor -1}} \sum_{\lfloor \sqrt{g(n)}\rfloor +1}^{\lfloor g(n)-\sqrt{g(n)}\rfloor -1} \theta_n^i \leq \sum_{\lfloor \sqrt{g(n)}\rfloor +1}^{\lfloor g(n)-\sqrt{g(n)}\rfloor -1} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i$$

$$\leq \frac{(q;q)_n}{(q;q)_{\lfloor n/2\rfloor}^2} \sum_{\lfloor \sqrt{g(n)}\rfloor +1}^{\lfloor g(n)-\sqrt{g(n)}\rfloor -1} \theta_n^i.$$

By the first part of this lemma the lower and the upper bound has the asserted asymptotic. $\qquad\square$

Lemma 3.5. *If $\theta_n \leq 1$ and $\theta_n \to 1$ with $\theta_n^n \to c$ for $0 < c < 1$, then*

$$\sum_{i=0}^{n} i\theta_n^i \sim \frac{1-c+c\log c}{\log^2 c} n^2$$

and

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q i\theta_n^i \sim e_q(q)\frac{1-c+c\log c}{\log^2 c} n^2$$

as $n \to \infty$.

Proof. To estimate this sum we use Lemma 3.3 again and the identity

$$\sum_{i=0}^{n} it^i = \frac{t(1-t^n - nt^n(1-t))}{(1-t)^2}.$$

Hence, setting $t = \theta_n$,

$$\sum_{i=0}^{n} i\theta_n^i \sim (1-c-n\theta_n^n(1-\theta_n))\frac{n^2}{\log^2 c} \sim (1-c+c\log c)\frac{n^2}{\log^2 c}.$$

Here we used that under the assumption $\tau_n^n \to c$ we have $(1-\tau_n)n \to -\log c$. To see this, write

$$\tau_n = c^{1/n} + f(n).$$

Then

$$\tau_n^n = c + \sum_{i=1}^{n} \binom{n}{i} f(n)^i c^{\frac{n-i}{n}} =: c + s_n.$$

Now suppose for a moment $f(n) \geq 0$. The sum s_n must tend to 0 and we have $|s_n| = s_n \geq f(n)nc$. Therefore $f(n)n \to 0$. Similarly, $f(n)n \to 0$ for negative f. Consequently $f(n)n \to 0$ for arbitrary f. Now consider

$$(1-\tau_n)n = n(1-c^{1/n}) + nf(n) \to -\log c.$$

The asymptotic for the sum with the q-binomial coefficient is obtained as in Lemma 3.4. $\qquad\square$

Now we are ready to establish the essential key in proving the convergence result: we give the asymptotic behaviour of sums of probabilities and the means and variances of $QD(n,\tau_n,q)$-distributed random variables.

Lemma 3.6. *Let X_n be $QD(n, \tau_n, q)$-distributed and denote with $\mu_n(\tau_n, q)$ and $\sigma_n^2(\tau_n, q)$ the corresponding mean and variance. If $\tau_n \to 1$ and $\tau_n^n \to c$ with $0 < c < 1$ and $f(n) \sim \beta n$, $f(n) < n$, then*

$$\sum_{x=0}^{\lfloor f(n) \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau_n; q)_{n-x} \sim 1 - c^\beta$$

$$\mu_n(\tau_n, q) \sim \frac{c-1}{\log c} n$$

$$\sigma_n^2(\tau_n, q) \sim \frac{1 + 2c \log c - c^2}{(\log c)^2} n^2$$

as $n \to \infty$.

Proof. We start the first assertion. Since $f(n) < n$ we can write

$$S_n := \sum_{x=0}^{\lfloor f(n) \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau_n; q)_{n-x} = (1 - \tau_n) \sum_{x=0}^{\lfloor f(n) \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q \prod_{i=1}^{n-x-1} (1 - \tau_n q^i).$$

The summands are bounded by $e_q(q)^2$, hence

$$S_n \sim (1 - \tau_n) \sum_{x=\lfloor \sqrt{n} \rfloor}^{\lfloor f(n) \rfloor - \lfloor \sqrt{n} \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q \prod_{i=1}^{n-x-1} (1 - \tau_n q^i) =: \hat{S}_n.$$

Estimating the product and using again the boundedness of the summands yields

$$\hat{S}_n \le (1 - \tau_n)(\tau_n; q)_{n - \lfloor f(n) \rfloor + \lfloor \sqrt{n} \rfloor - 1} \sum_{x=\lfloor \sqrt{n} \rfloor}^{\lfloor f(n) \rfloor - \lfloor \sqrt{n} \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q$$

$$\sim (1 - \tau_n)(\tau_n; q)_{n - \lfloor f(n) \rfloor + \lfloor \sqrt{n} \rfloor - 1} \sum_{1}^{n} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q =: \hat{\hat{S}}_n.$$

Since [4, Lemma 2.1] remains true for $\theta_n > \varepsilon > -1$ and with use of Lemma 3.4 (with $g(n) := f(n)$ and $f(n) := n$) we obtain

$$\hat{\hat{S}}_n \sim (1 - \tau_n) \frac{1}{e_q(q)} e_q(q) \frac{c^\beta - 1}{\log c} n \sim 1 - c^\beta.$$

In an analogous way we find a lower bound of $\hat{S}_n$ that is asymptotic equivalent to $1 - c^\beta$.

Now we prove the second proposition of the lemma: Use Lemma 2.4, easy estimates of the q-Pochhammer symbol and the asymptotics given in Lemma 3.4 to obtain

$$\mu_n(\tau_n, q) \le \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} \frac{(q; q)_n}{(q; q)_{\lceil n/2 \rceil}^2} + (q; q)_{\lfloor \sqrt{n} \rfloor} \sum_{j=\lfloor \sqrt{n} \rfloor}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \sim \frac{1}{e_q(q)} e_q(q) \frac{c-1}{\log c} n$$

and

$$\mu_n(\tau_n, q) \ge (q; q)_n \sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \sim \frac{c-1}{\log c} n.$$

Similarly we proceed for the second moments of $X_n(\tau_n, q)$ and estimate with use of Lemma 3.5

$$\mathbb{E}(X_n^2) \ge \sum_{j=1}^{n} (q; q)_{j-1} (1 + 2(j-1)) \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \ge 2(q; q)_n \sum_{j=1}^{n} (j-1) \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \sim 2 \frac{1 - c + c \log c}{(\log c)^2} n^2.$$

To bound the second moment from above we do the following:

$$\mathbb{E}(X_n^2) \leq \sum_{j=1}^{\lfloor\sqrt{n}\rfloor} \frac{(q;q)_n}{(q;q)_{\lceil n/2\rceil}^2}\left(1+\frac{2n}{1-q}\right) + \sum_{j=\lfloor\sqrt{n}\rfloor}^n (q;q)_{j-1}\left(1+2\sum_{i=1}^{j-1}\frac{1}{1-q^{j-i}}\right)\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j$$

$$= o(n^2) + \sum_{j=\lfloor\sqrt{n}\rfloor}^n (q;q)_{j-1}\left(1+2\sum_{i=\lfloor\sqrt{j}\rfloor}^{j-1}\frac{1}{1-q^{j-i}}\right)\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j$$

$$+ \sum_{j=\lfloor\sqrt{n}\rfloor}^n (q;q)_{j-1}\left(1+2\sum_{i=1}^{\lfloor\sqrt{j}\rfloor}\frac{1}{1-q^{j-i}}\right)\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j$$

$$\leq o(n^2) + 2(q;q)_{\lfloor\sqrt{n}\rfloor}\sum_{j=\lfloor\sqrt{n}\rfloor}^n j\frac{1}{1-q^{j-\lfloor\sqrt{j}\rfloor-1}}\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j$$

$$\leq o(n^2) + 2(q;q)_{\lfloor\sqrt{n}\rfloor}\frac{1}{1-q^{n-\sqrt{n}-1}}\sum_{j=1}^n j\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j$$

$$\sim 2\frac{1-c+c\log c}{(\log c)^2}n^2.$$

Thus

$$\mathbb{E}(X_n^2(\tau_n,q)) \sim 2\frac{1-c+c\log c}{(\log c)^2}n^2.$$

Hence

$$\sigma_n^2(\tau_n,q) = \mathbb{E}(X_n^2(\tau_n,q)) - \mu_n(\tau,q)^2 \sim \left(2\frac{1-c+c\log c}{(\log c)^2} - \left(\frac{c-1}{\log c}\right)^2\right)n^2$$

$$\sim \frac{1+2c\log c - c^2}{(\log c)^2}n^2.$$

$\square$

After this anaysis of the means and variances it is now easy to obtain the limiting distribution of the sequence X_n.

Theorem 3.7. *Let* $Y_n \sim QD(n,q,\tau_n)$ *with* $\tau_n \to 1$ *and* $\tau_n^n \to c$ *with* $0 < c < 1$. *Then the sequence of the normalised random variables* $X_n = (Y_n - \mu_n)/\sigma_n$ *converges to a limit* X *with*

$$\mathbb{P}(X \leq x) = 1 - e^{c-1}e^{-\sqrt{1+2c\log c - c^2}x}$$

for

$$x \in \left[-\frac{1-c}{\sqrt{1+2c\log c - c^2}}, \frac{c-\log c - 1}{\sqrt{1+2c\log c - c^2}}\right)$$

and

$$\mathbb{P}(x) = 1 \qquad for \qquad x = \frac{c-\log c - 1}{\sqrt{1+2c\log c - c^2}}.$$

Proof. The support of X is given by

$$\left[\lim_{n\to\infty} -\frac{\mu_n(\tau_n,q)}{\sigma_n(\tau_n,q)}, \lim_{n\to\infty}\frac{n-\mu_n(\tau_n,q)}{\sigma_n(\tau_n,q)}\right].$$

Using Lemma 3.6 the stated support follows immediately.

Computing the distribution function of X yields with use of Lemma 3.6

$$\mathbb{P}(X_n \leq x) = \sum_{0\leq y\leq \sigma_n x+\mu_n}\tau_n^y\begin{bmatrix}n\\y\end{bmatrix}_q(\tau_n;q)_{n-y} \sim 1 - c^\alpha$$

with

$$\alpha = \frac{\sqrt{1 + 2c \log c - c^2}}{-\log c} x + \frac{c-1}{\log c}$$

for

$$x < \frac{c - \log c - 1}{\sqrt{1 + 2c \log c - c^2}}.$$

Simplifying c^α yields the theorem. $\qquad\square$

Now we turn to the third case, which treats with sequences of random variables $X_n \sim QD(n, \tau_n, q)$ where $\tau_n \to 1$ and $\tau_n^{f(n)} \to c$ for a $c \in (0, 1)$ and $f(n) = o(n)$. This case is very similar to the previous one, and so we start with an analogue of Lemma 3.5

Lemma 3.8. *Let* $f(n) \to \infty$, $f(n) = o(n)$, $\theta_n^{f(n)} \to c$ *with* $0 < c < 1$. *Then*

$$\sum_{i=0}^{n} i \theta_n^i \sim \frac{f(n)^2}{\log^2 c} \qquad \text{and} \qquad \sum_{i=0}^{n} i \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i \sim e_q(q) \frac{f(n)^2}{\log^2 c}$$

as $n \to \infty$.

Proof. Follow the proof of Lemma 3.5 and observe that $n \theta_n^n (1 - \theta_n)$ tends to zero. $\qquad\square$

Following the proof of Lemma 3.6 and using Lemma 3.8 instead of Lemma 3.5 we obtain

Lemma 3.9. *If* $\tau_n \to 1$ *and* $\tau_n^{f(n)} \to c$ *with* $0 < c < 1$ *and* $f(n) = o(n)$, $g(n) \sim \beta f(n)$, *then*

$$\sum_{x=0}^{\lfloor g(n) \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau_n; q)_{n-x} \sim 1 - c^\beta$$

$$\mu_n(\tau_n, q) \sim \frac{-f(n)}{\log c}$$

$$\sigma_n^2(\tau_n, q) \sim \frac{f(n)^2}{(\log c)^2}$$

as $n \to \infty$.

As an immediate consequence we get the distribution of the limit of X_n, which is an exponential distribution and is again independent of q.

Theorem 3.10. *Let* $Y_n \sim QD(n, q, \tau_n)$ *with* $\tau_n \to 1$ *and* $\tau_n^{f(n)} \to c$ *with* $0 < c < 1$ *and* $f(n) = o(n)$. *Then the sequence of the normalised random variables* $X_n = (Y_n - \mu_n)/\sigma_n$ *converges to a normalised exponential distribution with parameter* 1, *i.e.*

$$\mathbb{P}(X \leq x) = 1 - e^{-x-1}, \qquad x \geq -1.$$

Proof. Lemma 3.9 yields immediately that the support of the limit distribution is $[-1, \infty)$. Computing the distribution function gives

$$\mathbb{P}(X \leq x) = \sum_{0 \leq y \leq \sigma_n x + \mu_n} \tau_n^y \begin{bmatrix} n \\ y \end{bmatrix}_q (\tau_n; q)_{n-y} \sim 1 - c^{\frac{x+1}{-\log c}} = 1 - e^{-x-1}.$$

$\qquad\square$

Comparing this result with Theorem 3.7 we see that this corresponds to taking the limit $c \to 0$.

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