BROWNIAN MOTION AND HARMONIC FUNCTIONS ON $\mathsf{Sol}(\mathsf{p},\mathsf{q})$

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Abstract. The Lie group $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ is the semidirect product induced by the action of $\mathbb{R}$ on $\mathbb{R}^2$ which is given by $(x,y) \mapsto (e^{\mathsf{p}z}x, e^{-\mathsf{q}z}y)$, $z \in \mathbb{R}$. Viewing $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ as a 3-dimensional manifold, it carries a natural Riemannian metric and Laplace-Beltrami operator. We add a linear drift term in the z-variable to the latter, and study the associated Brownian motion with drift. We derive a central limit theorem and compute the rate of escape. Also, we introduce the natural geometric compactification of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ and explain how Brownian motion converges almost surely to the boundary in the resulting topology. We also study all positive harmonic functions for the Laplacian with drift, and determine explicitly all minimal harmonic functions. All this is carried out with a strong emphasis on understanding and using the geometric features of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$, and in particular the fact that it can be described as the horocyclic product of two hyperbolic planes with curvatures $-\mathsf{p}^2$ and $-\mathsf{q}^2$, respectively.

1. Introduction

$\mathsf{Sol}(\mathsf{p},\mathsf{q})$ is the group of all matrices of the form

$$\mathfrak{g} = \begin{pmatrix} e^{\mathsf{p}z} & x & 0 \\ 0 & 1 & 0 \\ 0 & y & e^{-\mathsf{q}z} \end{pmatrix}, \quad x,y,z \in \mathbb{R}\,. \tag{1.1}$$

The parameters p and q are positive real numbers. It will be useful to think separately of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ as a Lie group and as a manifold. In the latter situation, we shall often write $\mathfrak{z} = (x,y,z)$ or also $\mathfrak{x}$ or $\mathfrak{y}$ for its elements, instead of $\mathfrak{g}$. Its length element is

$$ds^2 = d_{\mathsf{p},\mathsf{q}}s^2 = e^{-2\mathsf{p}z}\,dx^2 + e^{2\mathsf{q}z}\,dy^2 + dz^2\,,$$

which is invariant under the left action of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ on itself as an isometry group. If we identify the element $\mathfrak{g}$ of (1.1) with (x,y,z), then $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ is $\mathbb{R}^3$ topologically (but of course not metrically). In those coordinates, the group product is

$$(a,b,c)\cdot(x,y,z) = \bigl(e^{\mathsf{p}c}x + a, e^{-\mathsf{q}c}y + b, c+z\bigr)\,. \tag{1.2}$$

The purpose of this case study is to describe the behaviour of Brownian motion in space and time, and to determine all positive harmonic functions on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ with respect to its Laplace-Beltrami operator and the variant where a "vertical" drift term (in z) is added to the latter. More precisely, we shall derive a central limit theorem for Brownian motion with drift, describe convergence of this process to the natural geometric boundary at infinity, and we shall determine all positive eigenfunctions of those Laplacians. The experienced reader will know how intimately such stochastic and potential theoretic features are linked with each other.

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Before we can explain the results, we need some details. To start, let $\mathbb{H} = \{x + \mathfrak{i}\,w : x \in \mathbb{R}\,,\ w > 0\}$ be hyperbolic upper half plane with the standard length element $w^{-2}(dx^2 + dw^2)$. We can pass to the logarithmic model by substituting $z = \log w$, and in those coordinates the length element becomes $e^{-2z}dx^2 + dz^2$. Now we also change curvature by modifying the length element into

$$ds^2 = d_{\mathsf{p}}s^2 = e^{-2\mathsf{p}z}\,dx^2 + dz^2\,.$$

We write $\mathbb{H}(\mathsf{p})$ for the hyperbolic plane with this parametrization and metric. Then we have the natural projections

$$\begin{aligned}\pi_1 &: \mathsf{Sol}(\mathsf{p},\mathsf{q}) \to \mathbb{H}(\mathsf{p})\,, \quad (x,y,z) \mapsto (x,z)\\ \pi_2 &: \mathsf{Sol}(\mathsf{p},\mathsf{q}) \to \mathbb{H}(\mathsf{q})\,, \quad (x,y,z) \mapsto (y,-z)\,.\end{aligned} \tag{1.3}$$

The *horocycle at level* z in $\mathbb{H}(\mathsf{p})$ is the set $\{(x,z) : x \in \mathbb{R}\}$, and we write $\widetilde{\pi}(x,z) = z$. Thus, we get another natural projection $\widetilde{\pi} : \mathbb{H}(\mathsf{p}) \to \mathbb{R}$. We also consider $\widetilde{\pi}$ as a projection of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ onto $\mathbb{R}$, where $\widetilde{\pi}(x,y,z) = z$. We shall write d for each of the metrics induced by the respective length elments; it will usually be evident from the context to which of the underlying spaces this refers – or else, that space will appear in the index. (On $\mathbb{R}$ we then have $\mathsf{d}_{\mathbb{R}}(z_1,z_2) = |z_1 - z_2|$.) Note that our projections preserve distances in the following sense:

$$\begin{aligned}\mathsf{d}_{\mathsf{Sol}}\big((x,y_1,z_1),(x,y_2,z_2)\big) &= \mathsf{d}_{\mathbb{H}(\mathsf{q})}\big((y_1,-z_1),(y_2,-z_2)\big)\,,\\ \mathsf{d}_{\mathsf{Sol}}\big((x_1,y,z_1),(x_2,y,z_2)\big) &= \mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((x_1,z_1),(x_2,z_2)\big)\,, \quad \text{and}\\ \mathsf{d}_{\mathsf{Sol}}\big((x,y,z_1),(x,y,z_2)\big) &= |z_1 - z_2|\,.\end{aligned} \tag{1.4}$$

A main structural feature is that the manifold $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ is made up by two hyperbolic planes (with respective curvatures $-\mathsf{p}^2$ and $-\mathsf{q}^2$) that are glued together by identifying opposite horocycles: it can be seen as the *horocyclic product* of $\mathbb{H}(\mathsf{p})$ and $\mathbb{H}(\mathsf{q})$,

$$\mathsf{Sol}(\mathsf{p},\mathsf{q}) = \{(\mathfrak{u},\mathfrak{v}) \in \mathbb{H}(\mathsf{p}) \times \mathbb{H}(\mathsf{q}) : \widetilde{\pi}(\mathfrak{u}) + \widetilde{\pi}(\mathfrak{v}) = 0\}\,, \tag{1.5}$$

with its metric arising naturally from those two hyperbolic planes.

We remark here that there are various different types of horocyclic products. $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ has two sister structures. One is the *Diestel-Leader graph* $\mathsf{DL}(\mathsf{p},\mathsf{q})$, which is the horocyclic product of two regular trees with degrees $\mathsf{p}+1$ and $\mathsf{q}+1$, respectively, where $\mathsf{p},\mathsf{q} \geq 2$ are integer. One of its interesting features is that when $\mathsf{p} = \mathsf{q}$, it is a Cayley graph of the *lamplighter group* $(\mathbb{Z}/\mathsf{p}\mathbb{Z}) \wr \mathbb{Z}$. Random walks and harmonic functions on $\mathsf{DL}(\mathsf{p},\mathsf{q})$ have been studied intensively by Bertacchi [6], Woess [32], Bartholdi and Woess [3] and Brofferio and Woess [8], [9]. The other sister structure is *treebolic space* $\mathsf{HT}(\mathsf{p},\mathsf{q})$, which is the horocyclic product of $\mathbb{H}(\mathsf{p})$ and the tree with degree $\mathsf{q}+1$, where $\mathsf{p} > 0$ (real) and $\mathsf{q} \geq 2$ (integer). When $\mathsf{p} = \mathsf{q}$, the Baumslag-Solitar group $\langle a, b \mid ab = b^{\mathsf{q}}a\rangle$ acts on $\mathsf{HT}(\mathsf{q},\mathsf{q})$ with compact quotient. The study of potential theory and Brownian motion on treebolic space is harder than on Sol and on DL (where random walk replaces Brownian motion), first of all because of the conceptual and technical difficulty in constructing the right Laplacian(s) on the 2-dimensional complex HT. This is ongoing work of Bendikov, Saloff-Coste, Salvatori and Woess [4], [5].

Brownian motion and random walks on $\mathsf{Sol}(1,1)$ made a brief appearance in the work of Lyons and Sullivan [27]. Harmonic functions for random walks on $\mathsf{Sol}(1,1)$ also appear in Raugi [29, Exemple 2, p. 677].

HT, DL and Sol are also objects of great interest in relation with geometric group theory. Quasi-isometries of those spaces have been studied by Farb and Mosher [15] (for $\mathsf{HT}(\mathsf{p},\mathsf{p})$) and by Eskin, Fisher and Whyte [13], [14] (for DL and Sol). The last two papers also contain a good description of several aspects of the geometry of Sol.

The *Laplace operator* with *vertical drift parameter* $\mathsf{a} \in \mathbb{R}$ on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ is

$$\mathfrak{L}_{\mathsf{a}} = \mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})} = \frac{1}{2}\left(e^{2\mathsf{p}z}\,\frac{\partial^2}{\partial x^2} + e^{-2\mathsf{q}z}\,\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \mathsf{a}\,\frac{\partial}{\partial z}\,. \tag{1.6}$$

The Laplace-Beltrami operator arises for $\mathsf{a} = (\mathsf{q}-\mathsf{p})/2$.

As a matter of fact, this involves a small abuse of terminology: in differential geometry, the "true" Laplace-Beltrami operator would be *twice* the one which we are using. Here, we are following the probabilistic habits: with the factor $\frac{1}{2}$, in the standard Euclidean situation, the Laplacian is the infinitesimal generator of standard Brownian motion. The situation is similar here.

Under the projection π_1, the operator $\mathfrak{L}_a$ projects onto the operator on $\mathbb{H}(\mathsf{p})$ given by

$$\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})} = \frac{1}{2}\left(e^{2\mathsf{p}z}\,\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) + \mathsf{a}\,\frac{\partial}{\partial z}\,. \tag{1.7}$$

By "projects" we mean that for a C^2-function f_1 on $\mathbb{H}(\mathsf{p})$, one (obviously) has $\mathfrak{L}_{\mathsf{a}}(f_1 \circ \pi_1) = (\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})} f_1) \circ \pi_1$. This is the Laplace-Beltrami operator on $\mathbb{H}(\mathsf{p})$ when $\mathsf{a} = -\mathsf{p}/2$.

Analogously, under the projection π_2 (where the sign of z is changed), $\mathfrak{L}_{\mathsf{a}}$ projects onto the operator on $\mathbb{H}(\mathsf{q})$ given by

$$\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})} = \frac{1}{2}\left(e^{2\mathsf{q}z}\,\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \mathsf{a}\,\frac{\partial}{\partial z}\,. \tag{1.8}$$

And finally, $\mathfrak{L}_{\mathsf{a}}$ projects under $\widetilde{\pi}$ onto the operator on $\mathbb{R}$ given by

$$\widetilde{\mathfrak{L}}_{\mathsf{a}} = \frac{1}{2}\,\frac{d^2}{dz^2} + \mathsf{a}\,\frac{d}{dz}\,. \tag{1.9}$$

Coming back to the outline of the contents of this paper, some basic preliminaries are laid out in §2. Our first, probabilistic object of study is then Brownian motion with drift $\mathfrak{Z}_t = (X_t, Y_t, Z_t)_{t>0}$ on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$, i.e., the diffusion process whose infinitesimal generator is $\mathfrak{L}_{\mathsf{a}}$. The projections of $\mathfrak{Z}_t$ on $\mathbb{H}(\mathsf{p})$, $\mathbb{H}(\mathsf{q})$ and $\mathbb{R}$ are the diffusions whose infinitesimal generators are the respective projected operators defined above.

In §3, we describe this process in terms of stochastic integrals and first derive a central limit theorem for (X_t, Y_t, Z_t). Combining this with estimates from §2 for the metric of Sol, we also obtain a central limit theorem for $\mathsf{d}(\mathfrak{Z}_t, \mathfrak{o})$. Its form for the case $\mathsf{a} = 0$ is somewhat different from what happens for $\mathsf{a} \neq 0$. As a corollary, we get the linear rate of escape:

$$\frac{\mathsf{d}(\mathfrak{Z}_t, \mathfrak{o})}{t} \to |\mathsf{a}| \quad \text{almost surely, as } t \to \infty\,,$$

where $\mathfrak{o} = (0, 0, 0)$. This is the same as the rate of escape for the projected ("vertical") Brownian motion with drift $(Z_t)_{t>0}$ on $\mathbb{R}$, so that the lateral motion in the x-and y-variables does not contribute to that rate.

Since the Sol-group has exponential growth, our process is always transient, that is, with probability 1 it eventually leaves each compact set. §4 adds more details to the description of how our process tends to infinity in space. Namely, $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$ has a natural geometric compactification: since $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$ is a subset of the product of two hyperbolic planes (or equivalently, hyperbolic disks), it embeds naturally into the product of two closed unit disks, and the closure of $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$ in this bi-disk is the compactification. Topologically, the resulting boundary at infinity has the shape of a filled number "8", that is, two full closed disks glued together at a single (glueing) point . It is not a "visibility" boundary: neither the glueing point nor any of the interior points of the two disks come up as a limit of some geodesic ray in Sol; all the other boundary points *are* limits of geodesic rays.

It is a rather straighforward, but nevertheless informative task to verify that Brownian motion tends almost surely in the topology of that compactification to a limit random variable that lives on the boundary at infinity of $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$. If $\mathsf{a} = 0$ then this limit is the glueing point deterministically. Otherwise, that limit random variable lies on one of the two circles that make up the "8" (not their interiors) and its distribution is continuous. Thus, when $\mathsf{a} \neq 0$, we (almost surely) have the geodesic ray from the origin to the random limit point. If $\boldsymbol{\gamma} = \big(\boldsymbol{\gamma}(t)\big)_{t\geq 0}$ is that limit geodesic, then we show that for $\mathsf{a} \neq 0$, the deviation of $\mathfrak{Z}_t$ from that ray is at most logarithmic, that is, there is $c > 0$ such that

$$\limsup_{t\to\infty} \mathsf{d}(Z_t, \boldsymbol{\gamma})/\log t \leq c \quad \text{almost surely.}$$

This result comprises the analogous one for Brownian motion with drift on the hyperbolic plane. For the latter, we are not aware of a proof that has appeared in print, but there is a correponding theorem for random walks on free groups, resp. trees, that was first shown by LEDRAPPIER [25]; see WOESS [33, Thm. 9.59] for a general and simple proof.

The second main body of this work concerns positive harmonic functions. These are the positive C^2-functions that are anihilated by the respective Laplacian. We can also handle positive eigenfunctions.

We start in §5 by displaying some of the potential theoretic, resp. analytic ingredients that are needed. Then we prove in §6 that every positive $\mathfrak{L}_{\mathsf{a}}$-eigenfunction on $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$ has the form

$$h(x, y, z) = h_1(x, z) + h_2(y, -z)\,,$$

where h_1 is a non-negative $\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})}$-eigenfunction on $\mathbb{H}(\mathsf{p})$ and h_2 is non-negative $\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})}$-eigenfunction on $\mathbb{H}(\mathsf{q})$, both with the same eigenvalue as h.

This decomposition is not unique, but we can also see where non-uniqueness comes from, namely, harmonic functions that only depend on the "height" z. What we do is indeed to describe all minimal positive eigenfunctions, based on ideas from the discrete setting of Diestel-Leader graphs, see [9].

Since the positive $\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}}(\mathsf{p})$-eigenfunctions are known explicitly as integrals of modified Poisson kernels, the above result leads to a complete description of all positive $\mathfrak{L}_{\mathsf{a}}$-eigenfunctions on $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$. Thus, the positive eigenfunctions of the Laplacian on $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$

can be described fully in terms of (modified) Poisson kernels on each of the two hyperbolic planes that make up our space.

The computations undertaken here are related with the study of Martin compactifications of symmetric spaces, although the group $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ embeds into this context only when $\mathsf{p}=\mathsf{q}$. The reader is referred to the book by Guivarc'h, Ji and Taylor [17] and the survey by Kaimanovich [19] plus the references given there. In particular, we get close to answering the question of Lyons and Sullivan [27] to determine the Martin boundary of Sol; we find the minimal boundary and have a clear idea what the Martin compactification has to be.

We want to underline that the main spirit of this paper is to study the outlined issues via strong use of the geometry of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ in terms of the two hyperbolic planes and their horoycyclic product.

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2. Basic facts

The first part of this section contains some basic facts regarding $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ that are quite straightforward. They are included here for the sake of the completeness of the picture; most proofs are omitted.

(2.1) Lemma. *The Riemannian volume element of the* Sol*-manifold is*

$$d\mathfrak{z} = e^{(\mathsf{q}-\mathsf{p})z}\,dx\,dy\,dz\,.$$

This is also the left Haar measure of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ *as a group. The modular function on this group is* $\Delta(\mathfrak{g}) = e^{(\mathsf{q}-\mathsf{p})\,\widetilde{\pi}(\mathfrak{g})}$, *where* $\mathfrak{g}$ *is parametrized by* (x,y,z) *as in* (1.1) *and* $\widetilde{\pi}(\mathfrak{g}) = z$. *The group is unimodular if and only if* $\mathsf{p}=\mathsf{q}$.

Next, consider the group $\mathsf{Aff}(\mathsf{p})$ of all matrices of the form

$$\begin{pmatrix} e^{\mathsf{p}z} & x \\ 0 & 1 \end{pmatrix}, \quad x,z\in\mathbb{R}\,. \tag{2.2}$$

This is nothing but the group of orientation preserving *affine transformations* of hyperbolic plane, again parametrized by the logarithmic model and substituting the habitual upper left term e^z with $e^{\mathsf{p}z}$. We can identify the group $\mathsf{Aff}(\mathsf{p})$ with the surface $\mathbb{H}(\mathsf{p})$ in the same way as we identified Sol as a group with Sol as a manifold. By left multiplication, $\mathsf{Aff}(\mathsf{p})$ acts isometrically on $\mathbb{H}(\mathsf{p})$. We recall the following.

(2.3) Lemma. (a) *The Riemannian area element of* $\mathbb{H}(\mathsf{p})$ *is* $e^{-\mathsf{p}z}\,dx\,dz$. *This is also left Haar measure on the group* $\mathsf{Aff}(\mathsf{p})$, *and the modular function on* $\mathsf{Aff}(\mathsf{p})$ *is* $\Delta(\mathfrak{g}) = e^{\mathsf{p}\,\widetilde{\pi}(\mathfrak{g})}$, *where* $\mathfrak{g} = \left(\begin{smallmatrix} e^{\mathsf{p}z} & x \\ 0 & 1\end{smallmatrix}\right)$ *and* $\widetilde{\pi}(\mathfrak{g}) = z$.

We can interpret the projections π_1 and π_2 of (1.3) as homomorphisms from the group $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ onto $\mathsf{Aff}(\mathsf{p})$ and $\mathsf{Aff}(\mathsf{q})$, respectively. In the same way, $\widetilde{\pi}$ is a homomorphism onto the additive group $\mathbb{R}$.

(2.4) Lemma. (a) *The Laplacian* $\mathfrak{L}_{\mathsf{a}}$ *on* $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ *is reversible (self-adjoint) with respect to the measure*

$$\mathbf{m}_{\mathsf{a}}(d\mathfrak{z}) = e^{(2\mathsf{a}+\mathsf{p}-\mathsf{q})z}\, d\mathfrak{z} = e^{2\mathsf{a}z}\, dx\, dy\, dz\,.$$

(b) *The Laplacian* $\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})}$ *on* $\mathbb{H}(\mathsf{p})$ *is reversible with respect to the measure*

$$e^{2\mathsf{a}z}\, dx\, dz\,.$$

(c) *The Laplacian* $\widetilde{\mathfrak{L}}_{\mathsf{a}}$ *on* $\mathbb{R}$ *is reversible with respect to the measure*

$$e^{2\mathsf{a}z}\, dz\,.$$

Proof (hint). For proving (a) one has to show that for compactly supported C^2-functions f, g on Sol, one has

$$\iiint f(x,y,z)\ \mathfrak{L}_{\mathsf{a}} g(x,y,z)\, e^{2\mathsf{a}z}\, dx\, dy\, dz = \iiint \mathfrak{L}_{\mathsf{a}} f(x,y,z)\ g(x,y,z)\, e^{2\mathsf{a}z}\, dx\, dy\, dz\,.$$

This is straightforward by partial integration. (b) and (c) are analogous. □

Our Laplacian is invariant under the group action of Sol. Let $\mathfrak{g}_0 = (a,b,c)$ be a group element, and define the translate of a function f on Sol as $\tau_{\mathfrak{g}_0} f(\mathfrak{g}) = f(\mathfrak{g}_0\mathfrak{g})$, that is,

$$\tau_{\mathfrak{g}_0} f(x,y,z) = f\big(e^{\mathsf{p}c}x + a,\, e^{-\mathsf{q}c}y + b,\, c+z\big)\,. \tag{2.5}$$

(2.6) Lemma. *For any* $\mathfrak{g}_0 \in \mathsf{Sol}(\mathsf{p},\mathsf{q})$,

$$\mathfrak{L}_{\mathsf{a}}(\tau_{\mathfrak{g}_0} f) = \tau_{\mathfrak{g}_0}(\mathfrak{L}_{\mathsf{a}} f)\,.$$

The proof is completely elementary, using (2.5).

We shall need the following observations on the metric. Regarding our hyperbolic planes in the logarithmic model, let us remark here that the metric of $\mathbb{H}(\mathsf{p})$ is linked with the standard one of $\mathbb{H} = \mathbb{H}(1)$ by the formula

$$\mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((x,z)\,,\,(x',z')\big) = \frac{1}{\mathsf{p}}\,\mathsf{d}_{\mathbb{H}(1)}\big((\mathsf{p}x,\mathsf{p}z)\,,\,(\mathsf{p}x',\mathsf{p}z')\big). \tag{2.7}$$

While for Diestel-Leader graphs, there is an explicit formula for the graph metric in terms of the two underlying trees [6], we do not have such a formula on Sol. However, we have at least the following distance estimates.

(2.8) Proposition. *For all* $\mathfrak{z} = (x, y, z) \in \mathsf{Sol}(\mathsf{p}, \mathsf{q})$, *with* $x, y \neq 0$ *in* (iv),

(i) $\mathsf{d}_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}) \geq |z|$,

(ii) $\mathsf{d}_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}) \geq 2\,\dfrac{\log|x|}{\mathsf{p}} + 2\,\dfrac{\log|y|}{\mathsf{q}} - |z| - \left(\dfrac{1}{\mathsf{p}} + \dfrac{1}{\mathsf{q}}\right) \log \mathsf{d}_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z})$,

(iii) $\mathsf{d}_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}) \leq \mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((x, z)\,,\, (0, 0)\big) \;+\; \mathsf{d}_{\mathbb{H}(\mathsf{q})}\big((y, -z)\,,\, (0, 0)\big) - |z|$,

(iv)
$$\mathsf{d}_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}) \leq c_{\mathsf{p}} + c_{\mathsf{q}} + \left|\frac{\log|x|}{\mathsf{p}} + \frac{\log|y|}{\mathsf{q}}\right| + \min\left\{\left|\frac{\log|x|}{\mathsf{p}}\right| + \left|\frac{\log|y|}{\mathsf{q}} + z\right|\,,\, \left|\frac{\log|x|}{\mathsf{p}} - z\right| + \left|\frac{\log|y|}{\mathsf{q}}\right|\right\},$$

where $c_{\mathsf{p}},\ c_{\mathsf{q}} > 0$.

Proof. Inequality (i) is clear.

For (ii), Let $\mathfrak{z}(t) = \big(x(t), y(t), z(t)\big)_{t\in[0,d]}$ be a geodesic path in Sol from $\mathfrak{o}$ to $\mathfrak{z}$, where $d = \mathsf{d}_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z})$. Let

$$M = \max\{z(t) : t \in [0, d]\} \quad \text{and} \quad m = \min\{z(t) : t \in [0, d]\}\,,$$

so that $M \geq 0$ and $m \leq 0$. Then

$$\begin{aligned} d_{\mathsf{Sol}}(\mathfrak{z}\,,\, \mathfrak{o}) &= \int_0^b \sqrt{e^{-2\mathsf{p}z(t)}\dot{x}(t)^2 + e^{2\mathsf{q}z(t)}\dot{y}(t)^2 + \dot{z}(t)^2}\, dt \\ &\geq e^{-\mathsf{p}M} \int_0^b \sqrt{\dot{x}(t)^2 + \dot{z}(t)^2}\, dt \geq e^{-\mathsf{p}M}\sqrt{x^2 + z^2} \end{aligned}$$

Thus

$$\mathsf{p}M \geq \log\sqrt{x^2 + z^2} - \log d_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z})\,, \quad \text{and analogously} \quad -\mathsf{q}m \geq \log\sqrt{y^2 + z^2} - \log d_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z})\,.$$

Now let $\mathfrak{z}_M$ and $\mathfrak{z}_m$ be points on the geodesic from 0 to $\mathfrak{z}$ with heights M and m, respectively. Then, according to which of the two "comes first", (i) yields that either

$$d_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}) = d_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}_M) + d_{\mathsf{Sol}}(\mathfrak{z}_M, \mathfrak{z}_m) + d_{\mathsf{Sol}}(\mathfrak{z}_m, \mathfrak{z}) \geq M + (M - m) + (z - m)\,, \quad \text{or}$$
$$d_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}) = d_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}_m) + d_{\mathsf{Sol}}(\mathfrak{z}_m, \mathfrak{z}_M) + d_{\mathsf{Sol}}(\mathfrak{z}_M, \mathfrak{z}) \geq -m + (M - m) + (M - z)\,.$$

We see that

$$d_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}) \geq 2(M - m) - |z|\,,$$

and combining this with the above, we obtain (ii).

For proving (iii), we may suppose without loss of generality that $z \geq 0$.

Note that in the logarithmic model of $\mathbb{H}(\mathsf{p})$, any geodesic arc is either vertical (i.e., of the form $t \mapsto (x_0, t)$, where x_0 is fixed and t varies in an interval), or else it can be realised as $t \mapsto \big(t, z(t)\big)$, where $z(t)$ is a strictly concave function of t varying in an interval.

Let (x', z) be the first ("leftmost") point on the geodesic arc from $(0, 0)$ to (x, z) in $\mathbb{H}(\mathsf{p})$ with second coordinate z, and let $(y', 0)$ be the last ("rightmost") point on the geodesic arc from $(0, 0)$ to $(y, -z)$ in $\mathbb{H}(\mathsf{q})$ with second coordinate $-z$.

We may have $x' = x$ or $y' = 0$, but in any case, the geodesic arc from $(0, z)$ to (x', z) in $\mathbb{H}(\mathsf{p})$ is strictly increasing in both coordinates, while the geodesic arc from $(y', 0)$ to $(y, -z)$

in $\mathbb{H}(\mathsf{q})$ is strictly inreasing in the first and strictly decreasing in the second variable. That is, these two arcs can be parametrised, respectivley, as

$$t \mapsto \big(x(t), t\big) \quad \text{and} \quad t \mapsto \big(y(t), -t\big)\,,$$

where $t \in [0\,,\,z]$ and $\dot{x}(t), \dot{y}(t) > 0$. Now we can "synchronise" the two in order to get the curve

$$t \mapsto \big(x(t), y(t), t\big)\,, \quad t \in [0\,,\,z]\,,$$

that connects $(0, y', 0)$ with (x', y, z) in $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$. The length of this curve majorises the distance between these two points in $\mathsf{Sol}(\mathsf{p}, \mathsf{q})$ and is

$$\begin{aligned}
&\int_0^z \sqrt{e^{-2\mathsf{p}t}\,\dot{x}(t)^2 + e^{2\mathsf{q}t}\,\dot{y}(t)^2 + 1}\;dt \\
&\quad \le \int_0^z \Big(\sqrt{e^{-2\mathsf{p}t}\,\dot{x}(t)^2 + 1} + \sqrt{e^{2\mathsf{q}t}\,\dot{y}(t)^2 + 1} - 1\Big)\,dz \\
&\quad = \mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((0,0)\,,\,(x',z)\big) + \mathsf{d}_{\mathbb{H}(\mathsf{q})}\big((y',0)\,,\,(y,-z)\big) - z\,.
\end{aligned}$$

Now, by (1.4),

$$\mathsf{d}_{\mathsf{Sol}}(\mathfrak{o}\,,\,\mathfrak{z}) \le \underbrace{\mathsf{d}_{\mathsf{Sol}}\big((0,0,0)\,,\,(0,y',0)\big)}_{=\,\mathsf{d}_{\mathbb{H}(\mathsf{q})}((0,0)\,,\,(y',0))} + \mathsf{d}_{\mathsf{Sol}}\big((0,y',0)\,,\,(x',y,z)\big) + \underbrace{\mathsf{d}_{\mathsf{Sol}}\big((x',y,z)\,,\,(x,y,z)\big)}_{=\,\mathsf{d}_{\mathbb{H}(\mathsf{p})}((x',z)\,,\,(x,z))}$$

We insert the upper bound for the middle term that we derived above. Since

$$\begin{aligned}
\mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((0,0)\,,\,(x',z)\big) + \mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((x',z)\,,\,(x,z)\big) &= \mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((0,0)\,,\,(x,z)\big) \quad \text{and} \\
\mathsf{d}_{\mathbb{H}(\mathsf{q})}\big((0,0)\,,\,(y',-z)\big) + \mathsf{d}_{\mathbb{H}(\mathsf{q})}\big((y',-z)\,,\,(y,-z)\big) &= \mathsf{d}_{\mathbb{H}(\mathsf{q})}\big((0,0)\,,\,(y,-z)\big),
\end{aligned}$$

the proposed inequality follows.

For proving (iv), first note that for all $x \ne 0$,

$$\mathsf{d}_{\mathbb{H}(\mathsf{p})}\Big(\big(0, \tfrac{\log|x|}{\mathsf{p}}\big)\,,\,\big(x, \tfrac{\log|x|}{\mathsf{p}}\big)\Big) = c_{\mathsf{p}}$$

depends only on p. Then, using (1.4),

$$\begin{aligned}
\mathsf{d}_{\mathsf{Sol}}(\mathfrak{o}\,,\,\mathfrak{z}) &\le \mathsf{d}_{\mathsf{Sol}}\Big(\mathfrak{o}\,,\,\big(0,0,\tfrac{\log|x|}{\mathsf{p}}\big)\Big) + \mathsf{d}_{\mathsf{Sol}}\Big(\big(0,0,\tfrac{\log|x|}{\mathsf{p}}\big)\,,\,\big(x,0,\tfrac{\log|x|}{\mathsf{p}}\big)\Big) \\
&\quad + \mathsf{d}_{\mathsf{Sol}}\Big(\big(x,0,\tfrac{\log|x|}{\mathsf{p}}\big)\,,\,\big(x,0,-\tfrac{\log|y|}{\mathsf{q}}\big)\Big) \\
&\quad + \mathsf{d}_{\mathsf{Sol}}\Big(\big(x,0,-\tfrac{\log|y|}{\mathsf{q}}\big)\,,\,\big(x,y,-\tfrac{\log|y|}{\mathsf{q}}\big)\Big) + \mathsf{d}_{\mathsf{Sol}}\Big(\big(x,y,-\tfrac{\log|y|}{\mathsf{q}}\big)\,,\,\mathfrak{z}\Big) \\
&= \Big|\tfrac{\log|x|}{\mathsf{p}}\Big| + c_{\mathsf{p}} + \Big|\tfrac{\log|x|}{\mathsf{p}} + \tfrac{\log|y|}{\mathsf{q}}\Big| + c_{\mathsf{q}} + \Big|\tfrac{\log|y|}{\mathsf{q}} + z\Big|
\end{aligned}$$

Exchanging the roles of x and y, as well as of p and q, the inequality follows. □

We shall also need the following rough estimate.

(2.9) Lemma. *There is $c > 0$ such that for all $\mathfrak{z} = (x, y, z) \in \mathsf{Sol}(\mathsf{p}, \mathsf{q})$,*

$$\mathsf{d}_{\mathsf{Sol}}(\mathfrak{o}, \mathfrak{z}) \le c \cdot \Big(1 + \log(1 + |x|) + \log(1 + |y|) + |z|\Big)\,.$$

Proof. In the logarithmic model,

$$\begin{aligned}\mathsf{d}_{\mathbb{H}(1)}\big((x,z),(0,0)\big) &\le \mathsf{d}_{\mathbb{H}(1)}\big((x,z),(0,z)\big)+|z| = \log\frac{\sqrt{x^2+4e^{2z}}+|x|}{\sqrt{x^2+4e^{2z}}-|x|}+|z|\\ &\le \log\big(1+|x|e^{-z}+x^2e^{-2z}\big)+|z| \le 2\log(1+|x|)+2\log(1+e^{-z})+|z|\\ &\le 2\log(1+|x|)+2\log 2+3|z|\,.\end{aligned}$$

Combining this with (2.7) and Proposition 2.8(iii), the inequality follows. □

3. Central limit theorem and rate of escape

Let $\mathfrak{Z}_t = (X_t, Y_t, Z_t)$, $t \ge 0$, be the continuous diffusion on $\mathsf{Sol}(\mathsf{p},\mathsf{q}) \equiv \mathbb{R}^3$ whose infinitesimal generator is $\mathfrak{L}_\mathsf{a}$. If the starting point is $\mathfrak{o} = (0,0,0)$, then $\mathfrak{Z}_t$ is given by the stochastic integrals

$$\left\{\begin{aligned} Z_t &= \mathsf{a}\,t + W_t\,,\\ X_t &= \int_0^t e^{\mathsf{p}Z_s}\,dW_s^{(1)}\\ Y_t &= \int_0^t e^{-\mathsf{q}Z_s}\,dW_s^{(2)}\,, \end{aligned}\right. \tag{3.1}$$

where $(W_t, W_t^{(1)}, W_t^{(2)})_{t\ge 0}$ are three independent standard Brownian motions. (We do not attach a superscript to the one that defines the coordinate Z_t, because this is the most important one that determines the behaviour of all three.) See for instance Revuz and Yor [30] or Protter [28], and compare, in particular, with Baldi, Casadio Tarabusi, Figà-Talamanca and Yor [2].

For the following central limit theorem, let

$$\mathcal{N} = W_1\,,\quad \underline{\mathcal{M}} = \min\{W_t : 0 \le t \le 1\}\quad\text{and}\quad \overline{\mathcal{M}} = \max\{W_t : 0\le t\le 1\}\,,$$

so that $\mathcal{N}$ has standard normal distribution.

(3.2) Theorem. (i) *If* $\mathsf{a} > 0$*, then as* $t \to \infty$

$$\frac{1}{\sqrt{t}}\Big(\log|X_t| - \mathsf{pa}\,t\,,\ \log|Y_t|\,,\ Z_t - \mathsf{a}\,t\Big) \to (\mathsf{p}\mathcal{N}\,,\,0\,,\,\mathcal{N})\quad \textit{in law.}$$

(ii) *If* $\mathsf{a} < 0$*, then as* $t\to\infty$

$$\frac{1}{\sqrt{t}}\Big(\log|X_t|\,,\ \log|Y_t| + \mathsf{qa}\,t\,,\ Z_t - \mathsf{a}\,t\Big) \to (0\,,\,\mathsf{q}\mathcal{N}\,,\,\mathcal{N})\quad \textit{in law.}$$

(iii) *If* $\mathsf{a} = 0$ *then*

$$\frac{1}{\sqrt{t}}\Big(\log|X_t|\,,\ \log|Y_t|\,,\ Z_t\Big) \to (\mathsf{p}\overline{\mathcal{M}}\,,\,-\mathsf{q}\,\underline{\mathcal{M}}\,,\,\mathcal{N})\quad \textit{in law.}$$

Proof. For $\alpha\in\mathbb{R}$, set

$$V_t(\alpha) = \int_0^t e^{2\alpha Z_s}\,ds,$$

so that the quadratic variations of X_t and Y_t are $V_t(\mathsf{p})$ and $V_t(-\mathsf{q})$, respectively. Then by a theorem of Dambis, Dubin and Schwartz [10], [12], see also [30, p. 173], there exist two standard Brownian motions $(B_t^{(1)})_{t\ge 0}$ and $(B_t^{(2)})_{t\ge 0}$ such that

$$X_t = B^{(1)}_{V_t(\mathsf{p})} \quad \text{and} \quad Y_t = B^{(2)}_{V_t(-\mathsf{q})}. \tag{3.3}$$

By a theorem of Knight [24], see [30, p. 175], the processes $(B_t^{(1)})_{t\ge 0}$ and $(B_t^{(2)})_{t\ge 0}$ are independent in our case.

By the scaling property of Brownian motion, for $i = 1, 2$ and $\alpha = \mathsf{p}$, resp. $\alpha = -\mathsf{q}$,

$$\frac{\log \big|\, B^{(i)}_{V_t(\alpha)} / \sqrt{V_t(\alpha)}\, \big|}{\sqrt{t}} = \frac{\log |B_1|}{\sqrt{t}} \to 0 \quad \text{in law.}$$

In the following computations we use frequently the following simple fact.

$$\text{If } A_t \to A \text{ and } C_t \to 0 \text{ in law then } (A_t, C_t) \to (A, 0) \text{ in law, as } t \to \infty\,. \tag{3.4}$$

Thus

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{\sqrt{t}} \Big(\log|X_t| - \mathsf{p}\,\mathsf{a}\,t\,,\, \log|Y_t|\,,\, Z_t - \mathsf{a}\,t \Big) \\ &\quad = \lim_{t\to\infty} \frac{1}{\sqrt{t}} \Big(\log\sqrt{V_t(\mathsf{p})} - \mathsf{p}\,\mathsf{a}\,t\,,\, \log\sqrt{V_t(-\mathsf{q})}\,,\, W_t \Big) \quad \text{in law.} \end{aligned} \tag{3.5}$$

Case (i): $\mathsf{a} > 0$. First observe that for all $\alpha > 0$ and $\beta \in \mathbb{R}$,

$$\lim_{t\to\infty} \int_0^t e^{-\alpha s + \beta W_s}\, ds = \int_0^\infty e^{-\alpha s + \beta W_s}\, ds \in (0, +\infty) \quad \text{almost surely,}$$

since $\lim_{s\to\infty} \log(e^{-\alpha s + \beta W_s})/s = -\alpha < 0$. Therefore

$$\frac{1}{\sqrt{t}} \log\sqrt{V_t(-\alpha)} \to 0 \quad \text{in law, when } \alpha > 0. \tag{3.6}$$

Using (3.4), we get (in law) that

$$\begin{aligned} \lim_{t\to\infty} &\frac{1}{\sqrt{t}} \Big(\log\sqrt{V_t(\mathsf{p})} - \mathsf{p}\,\mathsf{a}\,t\,,\, \log\sqrt{V_t(-\mathsf{q})}\,,\, W_t \Big) \\ &= \lim_{t\to\infty} \frac{1}{\sqrt{t}} \Big(\log\sqrt{V_t(\mathsf{p})} - \mathsf{p}\,\mathsf{a}\,t\,,\, 0\,,\, W_t \Big) \\ &= \lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\frac{1}{2} \log\Big(\int_0^t e^{2\mathsf{p}(\mathsf{a}(s-t) + W_s)}\, ds \Big)\,,\, 0\,,\, W_t \right) \\ &= \lim_{t\to\infty} \frac{1}{\sqrt{t}} \left(\frac{1}{2} \log\Big(e^{2\mathsf{p} W_t} \int_0^t e^{2\mathsf{p}(\mathsf{a}(s-t) + W_s - W_t)}\, ds \Big)\,,\, 0\,,\, W_t \right) \end{aligned}$$

Since $(W_{t-s})_{s\le t} = (W_t - W_s)_{s \le t}$ in law,

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{\sqrt{t}} \log\Big(\int_0^t e^{2\mathsf{p}(\mathsf{a}(s-t) + W_s - W_t)}\, ds \Big) &= \lim_{t\to\infty} \frac{1}{\sqrt{t}} \log\Big(\int_0^t e^{-2\mathsf{p}(\mathsf{a}(t-s) - W_{t-s})}\, ds \Big) \\ = \lim_{t\to\infty} \frac{1}{\sqrt{t}} \log\Big(\int_0^t e^{-2\mathsf{p}(\mathsf{a}(s) - W_s)}\, ds \Big) = 0 & \end{aligned}$$

and using (3.4) once more, we get that

$$\lim_{t\to\infty}\frac{1}{\sqrt{t}}\Big(\log|X_t|-\mathsf{p}\,\mathsf{a}\,t\,,\,\log|Y_t|\,,\,Z_t-\mathsf{a}\,t\Big)=\lim_{t\to\infty}\frac{1}{\sqrt{t}}\Big(\mathsf{p}\,W_t\,,\,0\,,\,W_t\Big)=(\mathsf{p}\mathcal{N}\,,\,0\,,\,\mathcal{N})$$

in law.

Case (ii): $\mathsf{a}<0$. This is obtained from Case (i) by exchanging the roles of the x- and y-coordinates.

Case (iii): $\mathsf{a}=0$. We take up (3.5) and continue to compute, with all identities holding in law

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{\sqrt{t}}\Big(\log\sqrt{V_t(\mathsf{p})}\,,\,\log\sqrt{V_t(-\mathsf{q})}\,,\,W_t\Big)\\
&\quad=\lim_{t\to\infty}\frac{1}{2\sqrt{t}}\left(\log\Big(\int_0^t e^{2\mathsf{p}W_s}\,ds\Big)\,,\,\log\Big(\int_0^t e^{-2\mathsf{q}W_s}\,ds\Big)\,,\,2W_t\right)\\
&\quad=\lim_{t\to\infty}\frac{1}{2\sqrt{t}}\left(\log\Big(t\int_0^1 e^{2\mathsf{p}W_{st}}\,ds\Big)\,,\,\log\Big(t\int_0^1 e^{-2\mathsf{q}W_{st}}\,ds\Big)\,,\,2W_t\right)\\
&\quad=\lim_{t\to\infty}\frac{1}{2\sqrt{t}}\left(\log\Big(t\int_0^1 e^{2\mathsf{p}\sqrt{t}\,W_s}\,ds\Big)\,,\,\log\Big(t\int_0^1 e^{-2\mathsf{q}\sqrt{t}\,W_s}\,ds\Big)\,,\,2\sqrt{t}\,W_1\right)\\
&[\text{setting } \tau=2\sqrt{t}\,]\\
&\quad=\lim_{\tau\to\infty}\left(\log\Big(\int_0^1 e^{\tau\mathsf{p}W_s}\,ds\Big)^{1/\tau}\,,\,\log\Big(\int_0^1 e^{-\tau\mathsf{q}W_s}\,ds\Big)^{1/\tau}\,,\,W_1\right)
\end{aligned}$$

since $(W_{st})_{0\le s\le 1}=(\sqrt{t}\,W_s)_{0\le s\le 1}$ in law. Now recall that the L^τ-norm on $C([0\,,\,1])$ converges to the L^∞-norm as $\tau\to\infty$. We apply this to the functions $s\mapsto e^{\mathsf{p}W_s}$ and $s\mapsto e^{-\mathsf{q}W_s}$, respectively, and then take logarithms. Thus, almost surely

$$\begin{aligned}
&\log\Big(\int_0^1 e^{\tau\mathsf{p}W_s}\,ds\Big)^{1/\tau}\to\mathsf{p}\,\max\{W_s:0\le s\le 1\}\quad\text{and}\\
&\log\Big(\int_0^1 e^{-\tau\mathsf{q}W_s}\,ds\Big)^{1/\tau}\to-\mathsf{q}\,\min\{W_s:0\le s\le 1\}.
\end{aligned}$$

This leads to statement (iii). □

Next, with $\mathcal{N}$, $\underline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ as above, we deduce the following central limit theorem for the distance of Brownian motion to the origin.

(3.7) Theorem. *If* $\mathsf{a}\neq 0$ *then*

$$\frac{\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_t\,,\,\mathfrak{o})-|\mathsf{a}|\,t}{\sqrt{t}}\to\mathcal{N}\quad\textit{in law.}$$

If $\mathsf{a}=0$ *then*

$$\frac{\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_t\,,\,\mathfrak{o})}{\sqrt{t}}\to 2(\overline{\mathcal{M}}-\underline{\mathcal{M}})-|\mathcal{N}|\quad\textit{in law,}$$

as $t\to\infty$.

Proof. We start with $\mathsf{a} > 0$. Combining Theorem 3.2(i) with Proposition 2.8(iv), we obtain in law

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_t\,,\,\mathfrak{o}) - \mathsf{a}\,t}{\sqrt{t}} &\le \lim_{t\to\infty} \frac{\left|\frac{\log|X_t|}{\mathsf{p}} + \frac{\log|Y_t|}{\mathsf{q}}\right| + \left|\frac{\log|Y_t|}{\mathsf{q}}\right| + \left|\frac{\log|X_t|}{\mathsf{p}} - Z_t\right| - \mathsf{a}t}{\sqrt{t}} \\ &= \lim_{t\to\infty} \frac{\left|\frac{\log|X_t|}{\mathsf{p}}\right| - \mathsf{a}t + \left|\left(\frac{\log|X_t|}{\mathsf{p}} - \mathsf{a}t\right) - \left(Z_t - \mathsf{a}t\right)\right|}{\sqrt{t}} \\ &= \lim_{t\to\infty} \frac{\left|\frac{\log|X_t|}{\mathsf{p}}\right| - \mathsf{a}t}{\sqrt{t}} \qquad \text{(since } \mathsf{Pr}[\log|X_t| < 0] \to 0) \\ &= \lim_{t\to\infty} \frac{\frac{\log|X_t|}{\mathsf{p}} - \mathsf{a}t}{\sqrt{t}} = \mathcal{N}\,.\end{aligned}$$

On the other hand

$$\lim_{t\to\infty} \frac{\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_t\,,\,\mathfrak{o}) - \mathsf{a}\,t}{\sqrt{t}} \ge \lim_{t\to\infty} \frac{|Z_t| - \mathsf{a}t}{\sqrt{t}} = \mathcal{N} \quad \text{in law.}$$

When $\mathsf{a} < 0$, the result follows once more by exchanging the roles of the x- and y-coordinates.

Now consider the case when $\mathsf{a} = 0$. Combining Theorem 3.2(iii) with Proposition 2.8(iv), we obtain in law

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_t\,,\,\mathfrak{o})}{\sqrt{t}} &\le |\overline{\mathcal{M}} - \underline{\mathcal{M}}| + \min\Big\{|\overline{\mathcal{M}}| + |\mathcal{N} - \underline{\mathcal{M}}|\,,\, |\underline{\mathcal{M}}| + |\overline{\mathcal{M}} - \mathcal{N}|\Big\} \\ &= 2\big(\overline{\mathcal{M}} - \underline{\mathcal{M}}\big) - |\mathcal{N}|.\end{aligned}$$

This upper bound together with the fact that $\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_t\,,\,\mathfrak{o}) \to \infty$ almost surely yields that

$$\frac{\log \mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_t\,,\,\mathfrak{o})}{\sqrt{t}} \to 0$$

in law (and in fact almost surely). We can combine this with Theorem 3.2(iii) and Proposition 2.8(i), and get the required lower bound in the case $\mathsf{a} = 0$. □

Compare this with the analogous result of [6] for simple random walk with drift on Diestel-Leader graphs. We next observe the following.

(3.8) Lemma. *Let* $U_n = \max\big\{\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_n\,,\,\mathfrak{Z}_{n+t}) : 0 \le t \le 1\big\}$. *Then* $\mathsf{E}(U_n) < \infty$, *and*

$$\lim_{n\to\infty} \frac{1}{\log n} \max\big\{\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_n\,,\,\mathfrak{Z}_{n+t}) : 0 \le t \le 1\big\} = 0 \quad \textit{almost surely.}$$

Proof. The random variables U_n, $n \ge 0$, are i.i.d. Let

$$X_* = \max\{|X_t| : 0 \le t \le 1\}\,, \quad Y_* = \max\{|Y_t| : 0 \le t \le 1\}\,, \quad Z_* = \max\{|Z_t| : 0 \le t \le 1\}\,.$$

Then by Lemma 2.9

$$U_0 \le c \cdot \Big(1 + \log(1 + |X_*|) + \log(1 + |Y_*|) + |Z_*|\Big)\,.$$

Observe that by the Burkholder–Davis–Gundy inequality [30, pag. 161], we have for every $r > 2$ that

$$\mathsf{E}(X_*^r) \le c_1 \mathsf{E}\left(\left(\int_0^1 e^{2\mathsf{p}Z_s}\,ds\right)^{r/2}\right) \le c_1 \int_0^1 \mathsf{E}\big(e^{\mathsf{p}rZ_s}\big)\,ds < \infty\,,$$

where E denotes expectation and $c_1 > 0$. The same holds for Y_*. For Z_*, observe that by duality

$$\begin{aligned}\mathsf{Pr}[Z_* > z] &= \mathsf{Pr}[\max\{W_t - \mathsf{a}t : 0 \le t \le 1\} > z \ \text{ or } \ \max\{-W_t + \mathsf{a}t : 0 \le t \le 1\} > z]\\ &\le 4\mathsf{Pr}[W_1 > z - |\mathsf{a}|\,]\,.\end{aligned}$$

Thus

$$\mathsf{E}\big(e^{rZ_*}\big) = \int_0^\infty \mathsf{Pr}[e^{rZ_*} > u]\,du \le 4\int_0^\infty \mathsf{Pr}[e^{r(W_1+|\mathsf{a}|)} > u]\,du = \mathsf{E}\big(e^{r(W_1+|\mathsf{a}|)}\big) < \infty\,.$$

We find that for all $r > 0$,

$$\mathsf{E}\big(e^{rU_n}\big) = \mathsf{E}\big(e^{rU_0}\big) \le e^{cr}\mathsf{E}\Big((1+X_*)^{cr}\,(1+Y_*)^{cr}\,e^{crZ_*}\Big) < \infty\,.$$

By the law of large numbers, $e^{rU_n}/n \to 0$ almost surely, whence $\limsup_{n\to\infty} U_n/\log n < 1/r$ almost surely, for all $r > 0$. □

Given the (left) action on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ on itself by isometries and the group-invariance of our Laplacian (Lemma 2.5), we get that along any time interval $[s\,,\,t]$, the increment of our Brownian motion $\mathfrak{Z}_t = (X_t, Y_t, Z_t)$ of (3.1) satisfies

$$\mathfrak{Z}_s^{-1}\mathfrak{Z}_t = \mathfrak{Z}_{t-s} \quad \text{in law,} \tag{3.9}$$

and for an arbirary number of time intervals which do not overlap (i.e., they meet at most at the endpoints), the associated increments are independent. We now also get the rate of escape for our Brownian motion with drift.

(3.10) Corollary. *For any value of* a*,*

$$\lim_{t\to\infty} \frac{\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_t\,,\,\mathfrak{o})}{t} = |\mathsf{a}| \quad \textit{almost surely.}$$

Proof. In view of Lemma 3.8 and the spatial homogeneity (3.9), the subadditive ergodic theorem of Kingman [23] implies that $\mathsf{d}_{\mathsf{Sol}}(\mathfrak{Z}_t\,,\,\mathfrak{o})/t$ converges almost surely to a constant. (Compare with Derriennic [11] for the case of discrete time.) Theorem 3.7 implies that the limit is a in probability, whence also with probability 1. □

4. Convergence to the boundary at infinity, and the deviation from the limit geodesic

The natural geometric compactification of the hyperbolic plane, in the unit disk model, is just the closed (Euclidean) disk. In the upper half plane model $\mathbb{H}(\mathsf{p})$, the boundary at infinity $\vartheta\mathbb{H}(\mathsf{p})$ of the compactification $\widehat{\mathbb{H}}(\mathsf{p})$ is obtained by adding the bottom line $\vartheta^*\mathbb{H}(\mathsf{p}) = \mathbb{R}$ and the point at infinity, denoted here by ϖ_{p}. In the logarithmic model, convergence to the boundary is as follows: we have that $(x,z) \to \xi \in \vartheta^*\mathbb{H}(\mathsf{p})$ when $z \to -\infty$ and $x \to \xi$, and $(x,z) \to \varpi_{\mathsf{p}}$ if $|x| + e^z \to \infty$.

Now $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ embeds into $\mathbb{H}(\mathsf{p})\times\mathbb{H}(\mathsf{q})$ via (1.5). Therefore the most natural geometric compactification $\widehat{\mathsf{Sol}}(\mathsf{p},\mathsf{q})$ of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ is its closure in the compact bidisk $\widehat{\mathbb{H}}(\mathsf{p})\times\widehat{\mathbb{H}}(\mathsf{q})$. ("Bidisk" because when we use the unit disk model of hyperbolic plane, this is just the direct product of two closed unit disks.) We assemble a brief description of convergence to the boundary in the next lemma; no proof is required. We underline once more the analogy with Diestel-Leader graphs [32] and treebolic space [5]. As pointed out in the Introduction, the boundary at infinity is topologically a filled number "8", that is, two closed disks glued together at a single point. This sheds some light on the observations made by Lyons and Sullivan [27].

(4.1) Lemma. *The boundary at infinity* $\vartheta\mathsf{Sol}(\mathsf{p},\mathsf{q})$ *of* $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ *is*

$$\Big(\underbrace{\vartheta^*\mathbb{H}(\mathsf{p})}_{\mathbb{R}}\times\{\varpi_{\mathsf{q}}\}\Big)\cup\Big(\{\varpi_{\mathsf{p}}\}\times\underbrace{\vartheta^*\mathbb{H}(\mathsf{q})}_{\mathbb{R}}\Big)\cup\Big(\mathbb{H}(\mathsf{p})\times\{\varpi_{\mathsf{q}}\}\Big)\cup\Big(\{\varpi_{\mathsf{p}}\}\times\mathbb{H}(\mathsf{q})\Big)\cup\Big\{(\varpi_{\mathsf{p}},\varpi_{\mathsf{q}})\Big\}.$$

Convergence to the boundary is as follows. In general,

$$\mathfrak{x}=(x,y,z)\to(\xi,\eta)\in\vartheta\mathsf{Sol}(\mathsf{p},\mathsf{q})\,,\quad \text{if}\quad (x,z)\to\xi\ \text{ in }\ \widehat{\mathbb{H}}(\mathsf{p})\ \text{ and }\ (y,-z)\to\eta\ \text{ in }\ \widehat{\mathbb{H}}(\mathsf{q}).$$

This means that

$$\begin{aligned}
&\mathfrak{x}=(x,y,z)\to(\xi,\varpi_{\mathsf{q}})\in\vartheta^*\mathbb{H}(\mathsf{p})\times\{\varpi_{\mathsf{q}}\}\,,\quad \text{if}\quad x\to\xi\ \text{ and }\ z\to-\infty\,,\\
&\mathfrak{x}=(x,y,z)\to(\varpi_{\mathsf{p}},\eta)\in\{\varpi_{\mathsf{p}}\}\times\vartheta^*\mathbb{H}(\mathsf{q})\,,\quad \text{if}\quad y\to\eta\ \text{ and }\ z\to\infty\,,\\
&\mathfrak{x}=(x,y,z)\to\big((x_0,z_0),\varpi_{\mathsf{q}}\big)\in\mathbb{H}(\mathsf{p})\times\{\varpi_{\mathsf{q}}\}\,,\quad \text{if}\quad x\to x_0\,,\ z\to z_0\ \text{ and }\ |y|\to\infty\,,\\
&\mathfrak{x}=(x,y,z)\to\big(\varpi_{\mathsf{p}},(y_0,-z_0)\big)\in\{\varpi_{\mathsf{p}}\}\times\mathbb{H}(\mathsf{q})\,,\quad \text{if}\quad y\to y_0\,,\ z\to -z_0\ \text{ and }\ |x|\to\infty\,,\\
&\mathfrak{x}=(x,y,z)\to(\varpi_{\mathsf{p}},\varpi_{\mathsf{q}})\,,\quad \text{if}\quad |x|+e^z\to\infty\ \text{ and }\ |y|+e^{-z}\to\infty\ \text{ in }\ \mathbb{R}\,.
\end{aligned}$$

A *geodesic ray* is a continuous mapping $\boldsymbol{\gamma}:[0\,,\,\infty)\to\mathsf{Sol}$ (or to any of our other spaces) such that $\mathsf{d}\big(\boldsymbol{\gamma}(t)\,,\,\boldsymbol{\gamma}(s)\big)=|t-s|$ for all s,t. Its starting point is $\boldsymbol{\gamma}(0)$. For any $(x_0,z_0)\in\mathbb{H}(\mathsf{p})$ and $\xi\in\vartheta\mathbb{H}(\mathsf{p})$, there is a unique geodesic ray $\big(x(t),z(t)\big)$ that starts at (x_0,z_0) and converges to ξ. In the case when $\xi=\varpi_{\mathsf{p}}$ then this is the upwards going vertical half-line $t\mapsto(x_0,z_0+t)$ in $\mathbb{H}(\mathsf{p})$.

For $\mathfrak{x}=(x_0,y_0,z_0)\in\mathsf{Sol}(\mathsf{p},\mathsf{q})$ and a boundary point $(\varpi_{\mathsf{p}},\eta)\in\{\varpi_{\mathsf{p}}\}\times\vartheta^*\mathbb{H}(\mathsf{q})$ of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$, we can consider the (unique) upwards geodesic ray starting at $\mathfrak{x}$ given by $\boldsymbol{\gamma}_{\mathfrak{x}}^{\eta}(t)=\big(x_0,y(t),z(t)\big)$, where $\big(y(t),-z(t)\big)_{t\ge0}$ is the geodesic ray from $(y_0,-z_0)$ to η in $\mathbb{H}(\mathsf{q})$.

Analogously, for a boundary point $(\xi,\varpi_{\mathsf{q}})\in\vartheta^*\mathbb{H}(\mathsf{p})\times\{\varpi_{\mathsf{q}}\}$, we have the (unique) downwards geodesic ray starting at $\mathfrak{x}$ given by $\boldsymbol{\gamma}_{\mathfrak{x}}^{\xi}(t)=\big(x(t),y_0,z(t)\big)$, where $\big(x(t),z(t)\big)_{t\ge0}$ is the geodesic ray from (x_0,z_0) to ξ in $\mathbb{H}(\mathsf{p})$. All those geodesics converge to their defining boundary points, as $t\to\infty$, and any two geodescis that converge to the same boundary point are at bounded Hausdorff distance. This is true because it holds in the hyperbolic plane.

In the first of the above two cases, it will be most convenient to use the initial point $\mathfrak{x}=(0,\eta,0)$, and omit the index $\mathfrak{x}$ in that case. Thus, we can parametrise by $z\ge0$ and get $\boldsymbol{\gamma}^{\eta}(z)=(0,\eta,z)$. Analogously, in the second case, we use the standard initial point $\mathfrak{x}=(\xi,0,0)$ and get the corresponding geodesic ray $\boldsymbol{\gamma}^{\xi}(z)=(\xi,0,-z)$, again parametrised by $z \ge 0$. We call these the (upwards, resp. downwards) *vertical* geodesic rays. We remark that there is no geodesic ray in Sol from any starting point that converges to $(\varpi_{\mathsf{p}}, \varpi_{\mathsf{q}})$.

Compare with [13], [14] for further details on the geometry.

Let us return to our Brownian motion $\mathfrak{Z}_t = (X_t, Y_t, Z_t)$ of (3.1).

(4.2) Proposition. (i) *If* $\mathsf{a} > 0$ *then*

$$\lim_{t\to\infty} Y_t = Y_\infty = \int_0^\infty e^{-\mathsf{q}Z_s}\, dW_s^{(2)} \quad \textit{almost surely.}$$

That is, $\mathfrak{Z}_t \to (\varpi_{\mathsf{p}}\,, Y_\infty) \in \vartheta\mathsf{Sol}(\mathsf{p},\mathsf{q})$ *almost surely in the topology of* $\widehat{\mathsf{Sol}}(\mathsf{p},\mathsf{q})$.

(ii) *If* $\mathsf{a} < 0$ *then*

$$\lim_{t\to\infty} X_t = X_\infty = \int_0^\infty e^{\mathsf{p}Z_s}\, dW_s^{(1)} \quad \textit{almost surely.}$$

That is, $\mathfrak{Z}_t \to (X_\infty\,, \varpi_{\mathsf{q}})$ *almost surely in the topology of* $\widehat{\mathsf{Sol}}(\mathsf{p},\mathsf{q})$.

In both cases (i) *and* (ii)*, the respective limiting random variable is a.s. finite.*

(iii*) If* $\mathsf{a} = 0$ *then*

$$|X_t| + e^{Z_t} \to \infty \quad \textit{and} \quad |Y_t| + e^{-Z_t} \to \infty \quad \textit{almost surely.}$$

That is, $\mathfrak{Z}_t \to (\varpi_{\mathsf{p}}\,, \varpi_{\mathsf{q}})$ *almost surely in the topology of* $\widehat{\mathsf{Sol}}(\mathsf{p},\mathsf{q})$.

Proof. (i) and (ii) are immediate from the representation (3.1) via [30, Prop. 1.26].

For (iii), consider $\mathfrak{X}_t = (X_t, Z_t)$ as a process on the affine group $\mathsf{Aff}(\mathsf{p})$ of (2.2).

It also satisfies (3.9) and (3.8). We consider our process at discrete times:

$$\mathfrak{X}_n = (X_n, Z_n) = \mathfrak{X}_1 \cdot (\mathfrak{X}_1^{-1}\mathfrak{X}_2) \cdots (\mathfrak{X}_{n-1}^{-1}\mathfrak{X}_n) \tag{4.3}$$

is a right random walk on $\mathsf{Aff}(\mathsf{p})$. We can apply a result of Brofferio [7]. In the notation of [7], $A_1 = e^{\mathsf{p}Z_1}$ and $B_1 = X_1$. Since the expectation of $\log A_1$ is 0, and all moment conditions of [7, Thm. 1] are satisfied, $\mathfrak{X}_n \to \varpi_{\mathsf{p}}$ almost surely in $\mathbb{H}(\mathsf{p})$, as $n \to \infty$ in $\mathbb{Z}$. By Lemma (3.8), also $\mathfrak{X}_t \to \varpi_{\mathsf{p}}$ almost surely, as $t \to \infty$ in $\mathbb{R}$.

In the same way, $(Y_t, -Z_t) \to \varpi_{\mathsf{q}}$ almost surely, as $t \to \infty$ in $\mathbb{R}$. Statement (iii) follows. □

Thus, when $\mathsf{a} > 0$, we have the vertical *limit geodesic* $\boldsymbol{\gamma}^{Y_\infty}$ to whose limit point our Brownian motion converges, and when $\mathsf{a} < 0$ we have to replace this by $\boldsymbol{\gamma}^{X_\infty}$. In order to simplify notation, we just write $\boldsymbol{\gamma}^\infty$ for the respective limit geodesic in each of those cases.

We now prove that when $\mathsf{a} \ne 0$, the convergence of $\mathfrak{Z}_t$ to its boundary limit is very straight, in the sense that its deviation from $\boldsymbol{\gamma}^\infty$ is of the order of $\log t$.

(4.4) Theorem. *If* $\mathsf{a} \ne 0$ *then Brownian motion on* $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ *satisfies*

$$\limsup_{t\to\infty} \frac{1}{\log t}\, \mathsf{d}\big(\mathfrak{Z}_t\,, \boldsymbol{\gamma}^\infty\big) \le \frac{\mathsf{p}+\mathsf{q}}{2|\mathsf{a}|} \quad \textit{almost surely,}$$

where $\mathsf{d}\big(\mathfrak{Z}_t\,, \boldsymbol{\gamma}^\infty\big) = \inf\big\{\mathsf{d}\big(\mathfrak{Z}_t\,, \boldsymbol{\gamma}^\infty(u)\big) : u \ge 0\big\}$.

Proof. Once more, it is sufficient to consider only the case $\mathsf{a} > 0$.

For each $t \geq 0$, the point $(0, Y_\infty, Z_t)$ lies on the geodesic $\boldsymbol{\gamma}^\infty = \boldsymbol{\gamma}^{Y_\infty}$. We shall show that for integer n,

$$\limsup_{n\to\infty} \frac{1}{\log n}\, \mathsf{d}\big(\mathfrak{Z}_n\,,\,(0, Y_\infty, Z_n)\big) \leq 2|\mathsf{a}|\left(\frac{1}{\mathsf{p}} + \frac{1}{\mathsf{q}}\right) \quad \text{almost surely,} \tag{4.5}$$

Together with Lemma 3.8, this will yield the result.

The metric $\mathsf{d}_{\mathsf{Sol}}$ is invariant under the left action of the group $\mathsf{Sol}(\mathsf{p},\mathsf{q})$. Using the product formula (1.2) and subsequently Lemma 2.9, we find

$$\begin{aligned}\mathsf{d}_{\mathsf{Sol}}\big(\mathfrak{Z}_t, (0, Y_\infty, Z_t)\big) &= \mathsf{d}_{\mathsf{Sol}}\big((e^{-\mathsf{p}Z_t}X_t\,, e^{\mathsf{q}Z_t}(Y_t - Y_\infty), 0), \mathfrak{o}\big)\\ &\leq c\cdot\Big(1 + \log\big(1 + e^{-\mathsf{p}Z_t}|X_t|\big) + \log\big(1 + e^{\mathsf{q}Z_t}|Y_t - Y_\infty|\big)\Big)\,.\end{aligned}$$

We have

$$\begin{aligned}e^{\mathsf{q}Z_t}(Y_\infty - Y_t) &= e^{\mathsf{q}W_t + at}\int_t^\infty e^{-\mathsf{q}(W_s + \mathsf{a}s)}\,dW_s^{(2)} = \int_t^\infty e^{-\mathsf{q}((W_s - W_t) + \mathsf{a}(s-t))}\,dW_s^{(2)}\\ &\overset{\text{in law}}{=} \int_0^\infty e^{-\mathsf{q}(W_s + \mathsf{a}s)}\,dW_s^{(2)} = Y_\infty\,.\end{aligned}$$

Recall from the proof of Proposition 4.2 that $(Y_n, -Z_n)$ can be interpreted as the right random walk $\begin{pmatrix} e^{-\mathsf{q}Z_n} & Y_n \\ 0 & 1\end{pmatrix}$ on the affine group. We can apply a theorem of Kesten [22, Thm. B] to the sequence (Y_n) and its limit Y_∞. Namely, if we set $\kappa(\mathsf{q}) = 2\mathsf{a}/\mathsf{q}$ then $\mathsf{E}\big((e^{-\mathsf{q}Z_1})^{\kappa(\mathsf{q})}\big) = 1$, whence

$$\mathsf{Pr}[|Y_\infty| > y] \asymp y^{-\kappa(\mathsf{q})} \quad \text{as } \ y \to \infty\,.$$

Now take $\delta > 1/\kappa$. Then

$$\begin{aligned}\sum_{n=2}^\infty \mathsf{Pr}\Big[\log\big(1 + e^{\mathsf{q}Z_n}|Y_n - Y_\infty|\big) > \delta\log n\Big] &= \sum_{n=2}^\infty \mathsf{Pr}\Big[\log\big(1 + |Y_\infty|\big) > \delta\log n\Big]\\ &= \sum_{n=2}^\infty \mathsf{Pr}\big[|Y_\infty| > n^\delta - 1\big] \asymp \sum_{n=2}^\infty (n^\delta - 1)^{-\kappa(\mathsf{q})} < +\infty\end{aligned}$$

Thus by the Borel–Cantelli Lemma

$$\limsup_{n\to\infty}\frac{1}{\log n}\log\big(1 + e^{\mathsf{q}Z_n}|Y_n - Y_\infty|\big) \leq \frac{\mathsf{q}}{2\mathsf{a}} \quad \text{almost surely.}$$

We now consider the first coordinate. For fixed t observe that

$$\begin{aligned}e^{-\mathsf{p}Z_t}X_t &= e^{-\mathsf{p}(W_t + \mathsf{a}t)}\int_0^t e^{\mathsf{p}(W_s + \mathsf{a}s)}\,dW_s^{(1)}\\ &= \int_0^t e^{-\mathsf{p}((W_t - W_s) + \mathsf{a}(t-s))}\,dW_s^{(1)}\\ &\overset{\text{in law}}{=} \int_0^t e^{-\mathsf{p}(W_s + \mathsf{a}s)}\,dW_s^{(1)} =: \widetilde{X}_t\,,\end{aligned}$$

and $\widetilde{X}_\infty = \lim_{t\to\infty} \widetilde{X}_t$ exists almost surely. As above, one finds that

$$e^{\mathsf{p}Z_t}(\widetilde{X}_\infty - \widetilde{X}_t) = \int_t^\infty e^{-\mathsf{p}((W_s - W_t)+\mathsf{a}(s-t))}\, dW_s^{(1)} =: \overline{X}_{t,\infty}\,,$$

where $\overline{X}_{t,\infty}$ is independent from $(\mathfrak{Z}_s)_{0\le s\le t}$ and has the same law as $\widetilde{X}_\infty$. Thus, for some constant $C > 0$ and for any $x > 0$,

$$\begin{aligned}
\mathsf{Pr}\big[e^{-\mathsf{p}Z_t}|X_t| > x\big] &= \mathsf{Pr}\big[|\widetilde{X}_n| > x\,,\ |\widetilde{X}_\infty - \widetilde{X}_n| \le \tfrac{x}{2}\big] + \mathsf{Pr}\big[|\widetilde{X}_n| > x\,,\ |\widetilde{X}_\infty - \widetilde{X}_n| > \tfrac{x}{2}\big]\\
&\le \mathsf{Pr}\big[|\widetilde{X}_\infty| > \tfrac{x}{2}\big] + \mathsf{Pr}\big[e^{-\mathsf{p}Z_n}|\overline{X}_{n,\infty}| > \tfrac{x}{2}\big]\\
&\le C\,\big(\tfrac{x}{2}\big)^{\kappa(\mathsf{p})} + \mathsf{E}\Big(\mathsf{Pr}\big[|\overline{X}_{n,\infty}| > \tfrac{x}{2}e^{\mathsf{p}Z_n} \,\big|\, Z_n\big]\Big)\\
&\le C\,\big(\tfrac{x}{2}\big)^{-\kappa(\mathsf{p})} + C\,\mathsf{E}\Big(\big(\tfrac{x}{2}e^{\mathsf{p}Z_n}\big)^{-\kappa(\mathsf{p})}\Big)\\
&\le C\,\big(\tfrac{x}{2}\big)^{-\kappa(\mathsf{p})} + C\big(\tfrac{x}{2}\big)^{-\kappa(\mathsf{p})}\underbrace{\mathsf{E}\big(e^{-\kappa(\mathsf{p})\mathsf{p}Z_1}\big)^n}_{=\,1} = 2C\big(\tfrac{x}{2}\big)^{-\kappa(\mathsf{p})}
\end{aligned}$$

Proceeding as above, the Borel–Cantelli Lemma implies that

$$\limsup_{n\to\infty} \frac{1}{\log n}\log\big(1 + e^{-\mathsf{p}Z_n}|X_n|\big) \le \frac{\mathsf{p}}{2\mathsf{a}}$$

almost surely. □

5. Elements of potential theory

If $\mathfrak{L}$ is any of our different Laplacians on Sol, $\mathbb{H}$, or $\mathbb{R}$, and $\lambda \in \mathbb{R}$, then we denote by $\mathcal{H}(\mathfrak{L},\lambda)$ the space of all functions h on our space which satisfy $\mathfrak{L}f = \lambda \cdot f$. The positive cone $\mathcal{H}^+(\mathfrak{L},\lambda)$ contains non-zero functions if and only if $\lambda \ge \lambda_{\min}(\mathfrak{L})$, the *bottom of the positive spectrum.* Below we shall clarify what the values of $\lambda_{\min}$ are in each of our cases. In any case, $\lambda_{\min} \le 0$, since the space of *harmonic functions* $\mathcal{H}(\mathfrak{L}) = \mathcal{H}(\mathfrak{L},0)$ contains all constant functions.

By the *minimum principle,* every non-zero function in $\mathcal{H}^+(\mathfrak{L},\lambda)$ must be strictly positive in each point.

A function h in $\mathcal{H}^+(\mathfrak{L},\lambda)$ is called *minimal* if $h(0) = 1$ and whenever $h \ge f \in \mathcal{H}^+(\mathfrak{L},\lambda)$ then f/h is constant. A basic fact in classical potential theory of Riemannian manifolds says that every function in $\mathcal{H}^+(\mathfrak{L},\lambda)$ can be expressed uniquely as an integral over the minimal harmonic functions with respect to a finite Borel measure on the latter set.

We shall specify this in more detail in our cases below.

Let us now recall what happens in the case of the standard Laplacian

$$\mathfrak{L}^{\mathbb{H}}_{-1/2} = \frac{1}{2}\Big(e^{2z}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\Big)$$

on standard hyperbolic plane $\mathbb{H} = \mathbb{H}(1) = \{x + \mathfrak{i}\, e^z : x, z \in \mathbb{R}\}$ in the logarithmic model.

The minimal harmonic functions are the *Poisson kernels,* which are parametrised by the (hyperbolic) boundary $\vartheta\mathbb{H} = \mathbb{R} \cup \{\varpi\}$. (Recall that $\varpi = \varpi_1$ is the point at infinity.) In the logarithmic model the kernels are

$$P\big((x,z),\varpi\big) = e^z \quad\text{and}\quad P\big((x,z),\xi\big) = \frac{(\xi^2+1)e^z}{(\xi - x)^2 + e^{2z}}\,, \quad \xi \in \mathbb{R}\,. \tag{5.1}$$

We have $\lambda_{\min}(\mathfrak{L}^{\mathbb{H}}_{-1/2}) = -1/8$, and the minimal elements in $\mathcal{H}^+(\mathfrak{L}^{\mathbb{H}}_{-1/2},\lambda)$ are the functions

$$P(\cdot,\xi)^{\alpha(\lambda)}\,, \quad \text{where} \quad \xi\in\vartheta\mathbb{H} \quad \text{and} \quad \alpha(\lambda)=\frac{1+\sqrt{1+8\lambda}}{2}\,. \tag{5.2}$$

Next, let us turn our attention to $\mathcal{H}^+(\mathfrak{L}^{\mathbb{H}(\mathsf{p})}_{\mathsf{a}},\lambda)$.

(5.3) Lemma. *A function f on $\mathbb{H}(\mathsf{p})$ is in $\mathcal{H}\bigl(\mathfrak{L}^{\mathbb{H}(\mathsf{p})}_{\mathsf{a}},\lambda\bigr)$ if and only if the function on $\mathbb{H}(1)$ given by $\bigl(e^{(\mathsf{a}+\mathsf{p}/2)z}f\bigr)\circ\theta$ is in $\mathcal{H}\bigl(\mathfrak{L}^{\mathbb{H}(1)}_{-1/2},\frac{8\lambda+4\mathsf{a}^2-\mathsf{p}^2}{8\mathsf{p}^2}\bigr)$, where $\theta(x,z)=(x/\mathsf{p},z/\mathsf{p})$. In particular,*

$$\lambda_{\min}(\mathfrak{L}^{\mathbb{H}(\mathsf{p})}_{\mathsf{a}}) = -\mathsf{a}^2/2\,,$$

and for $\lambda\ge -\mathsf{a}^2/2$, the minimal elements in $\mathcal{H}^+\bigl(\mathfrak{L}^{\mathbb{H}(\mathsf{p})}_{\mathsf{a}},\lambda\bigr)$ are the functions

$$P_{\mathsf{p},\mathsf{a},\lambda}\bigl((x,z),\varpi_{\mathsf{p}}\bigr)=e^{\alpha z} \quad \textit{and} \quad P_{\mathsf{p},\mathsf{a},\lambda}\bigl((x,z),\xi\bigr)=e^{\alpha z}\left(\frac{(\xi^2+1)}{(\xi-\mathsf{p}\,x)^2+e^{2\mathsf{p}z}}\right)^{\beta}, \quad \xi\in\vartheta^*\mathbb{H}(\mathsf{p})\equiv\mathbb{R}\,,$$

where $\alpha=\alpha(\lambda,\mathsf{a})=\sqrt{\mathsf{a}^2+2\lambda}-\mathsf{a}$ and $\beta=\beta(\lambda,\mathsf{a},\mathsf{p})=\frac{1}{2}+\frac{\sqrt{\mathsf{a}^2+2\lambda}}{\mathsf{p}}$.

Proof. First of all, it is a straightforward computation that

$$\bigl(\mathfrak{L}^{\mathbb{H}(\mathsf{p})}_{\mathsf{a}}f\bigr)\circ\theta=\mathsf{p}^2\,\mathfrak{L}^{\mathbb{H}(1)}_{\mathsf{a}/\mathsf{p}}(f\circ\theta)\,.$$

Therefore f is in $\mathcal{H}(\mathfrak{L}^{\mathbb{H}(\mathsf{p})}_{\mathsf{a}},\lambda)$ if and only if $\bar f=f\circ\theta$ is in $\mathcal{H}(\mathfrak{L}^{\mathbb{H}(1)}_{\mathsf{a}/\mathsf{p}},\lambda/\mathsf{p}^2)$.

For the moment, set $\bar{\mathsf{a}}=\mathsf{a}/\mathsf{p}$ and $\bar\lambda=\lambda/\mathsf{p}^2$. Then we compute

$$\mathfrak{L}^{\mathbb{H}(1)}_{-1/2}\bigl(e^{(\bar{\mathsf{a}}+1/2)z}\bar f\bigr)=e^{(\bar{\mathsf{a}}+1/2)z}\Bigl(\mathfrak{L}^{\mathbb{H}(1)}_{\bar{\mathsf{a}}}\bar f+\tfrac{4\bar{\mathsf{a}}^2-1}{8}\bar f\Bigr)\,.$$

Therefore $\bar f$ is in $\mathcal{H}(\mathfrak{L}^{\mathbb{H}(1)}_{\bar{\mathsf{a}}},\bar\lambda)$ if and only if $e^{(\bar{\mathsf{a}}+1/2)z}\bar f$ is in $\mathcal{H}(\mathfrak{L}^{\mathbb{H}(1)}_{-1/2},\bar\lambda+\frac{4\bar{\mathsf{a}}^2-1}{8})$.

Combining these computations, the statements follow. □

Thus, every function $h\in\mathcal{H}^+(\mathfrak{L}^{\mathbb{H}(\mathsf{p})}_{\mathsf{a}},\lambda)$ has a unique integral representation

$$h(x,z)=\int_{\vartheta\mathbb{H}}P_{\mathsf{p},\mathsf{a},\lambda}\bigl((x,z),\xi\bigr)\,d\nu(\xi)\,, \tag{5.4}$$

where ν is a (finite, positive) Borel measure on $\vartheta\mathbb{H}$. (This includes $\xi=\varpi_{\mathsf{p}}$.)

(5.5) Remark. When $\lambda=0$, that is, when we consider ordinary harmonic functions, we see that the constant function $\mathbf{1}$ is minimal harmonic if and only if $\mathsf{a}\ge 0$. This can be stated also by saying that $\mathfrak{L}^{\mathbb{H}(\mathsf{p})}_{\mathsf{a}}$ has the (weak) Liouville property, i.e., all bounded harmonic functions are constant, precisely when $\mathsf{a}\ge 0$.

Everything that we have said so far in this section is very well known; see e.g. Helgason [18], or many other sources.

Let us now turn our attention to Sol. The following is immediate.

(5.6) Lemma. *If the function h_1 on $\mathbb{H}(\mathsf{p})$ is such that $h=h_1\circ\pi_1$ is minimal in $\mathcal{H}^+\bigl(\mathfrak{L}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})}_{\mathsf{a}},\lambda\bigr)$, then h_1 is also minimal in $\mathcal{H}^+\bigl(\mathfrak{L}^{\mathbb{H}(\mathsf{p})}_{\mathsf{a}},\lambda\bigr)$.*

In the same way, if the function h_2 on $\mathbb{H}(\mathsf{q})$ is such that $h=h_2\circ\pi_2$ is minimal in $\mathcal{H}^+\bigl(\mathfrak{L}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})}_{\mathsf{a}},\lambda\bigr)$, then h_2 is also minimal in $\mathcal{H}^+\bigl(\mathfrak{L}^{\mathbb{H}(\mathsf{q})}_{-\mathsf{a}},\lambda\bigr)$.

We now need a part of the Martin boundary theory for elliptic operators on manifolds. The reader is referred to ANCONA [1] and TAYLOR [31] for the necessary backgroud material. See also [17, Chapter VI]. In the following propositions, we subsume the necessary material without all proofs.

(5.7) Proposition. *The Markov semigroup* $\mathfrak{H}_t = \mathfrak{H}_t^{\mathsf{a}} = \exp(t\mathfrak{L}_{\mathsf{a}})$, $t > 0$, *admits a symmetric, bounded kernel* $\mathbf{h}_t(\mathfrak{x},\mathfrak{z}) = \mathbf{h}_t^{\mathsf{a}}(\mathfrak{x},\mathfrak{z})$ *with respect to the measure* $\mathbf{m}_{\mathsf{a}}$ *of Lemma 2.4(a), such that*

$$\mathfrak{H}_t f(\mathfrak{x}) = \int_{\mathsf{Sol}} \mathbf{h}_t(\mathfrak{x},\mathfrak{z}) f(\mathfrak{z})\, d\mathbf{m}_{\mathsf{a}}(\mathfrak{z})\,.$$

For each $\mathfrak{z} = (x,y,z) \in \mathsf{Sol}$, *the function* $\mathbf{h}_t(\cdot,\mathfrak{z})$ *is in* $C^2(\mathsf{Sol})$. *Furthermore, its kernel with respect to the volume element* $d\mathfrak{z}$ *of the* Sol*-manifold,*

$$\mathbf{p}_t(\mathfrak{x},\mathfrak{z}) = \mathbf{h}_t(\mathfrak{x},\mathfrak{z})\, e^{(\mathsf{a}+\mathsf{p}-\mathsf{q})z}\,,$$

is stochastic and invariant under the action of the group $\mathsf{Sol}(\mathsf{p},\mathsf{q})$.

(5.8) Proposition. *The associated Green kernel*

$$\mathbf{g}_{\mathsf{a}}(\mathfrak{x},\mathfrak{z}|\lambda) = \mathbf{g}(\mathfrak{x},\mathfrak{z}|\lambda) = \int_0^\infty e^{-\lambda t}\mathbf{p}_t(\mathfrak{x},\mathfrak{z})\, dt \quad (\mathfrak{x},\mathfrak{z} \in \mathsf{Sol}\,,\ \mathfrak{x} \neq \mathfrak{z})$$

is strictly positive and finite for each $\lambda \geq \lambda_{\min}(\mathfrak{L}_{\mathsf{a}})$.

We remark that finiteness at $\lambda = \lambda_{\min}(\mathfrak{L}_{\mathsf{a}})$ follows from the fact that the cone of positive eigenfunctions $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})}, \lambda_{\min}\big)$ does not collapse to a single half-line, as one can see from Lemma 5.3 combined with (5.1) and (5.2).

(5.9) Proposition. *For each* $d > 0$ *and* $\lambda \geq \lambda_{\min}$, *the Green kernel satisfies the Harnack inequality*

$$\frac{\mathbf{g}(\mathfrak{x},\mathfrak{z}'|\lambda)}{\mathbf{g}(\mathfrak{x},\mathfrak{z}|\lambda)} \leq C_d(\lambda) \quad \textit{and} \quad \frac{\mathbf{g}(\mathfrak{z}',\mathfrak{x}|\lambda)}{\mathbf{g}(\mathfrak{z},\mathfrak{x}|\lambda)} \leq C_d(\lambda)\,,$$

whenever $\mathsf{d}(\mathfrak{z},\mathfrak{z}') \leq d$ *and* $\min\{\mathsf{d}(\mathfrak{z},\mathfrak{x}),\mathsf{d}(\mathfrak{z}',\mathfrak{x})\} \geq 10(d+1)$, *where* $C_d(\lambda) > 1$ *is such that* $C_d(\lambda) \to 1$ *when* $d \to 0$.

Furthermore, every function h *in* $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})}, \lambda\big)$ *satisfies*

$$\frac{h(\mathfrak{z}')}{h(\mathfrak{z})} \leq C_d(\lambda) \quad \textit{for all} \quad \mathfrak{z},\mathfrak{z}' \in \mathsf{Sol} \quad \textit{with} \quad \mathsf{d}(\mathfrak{z},\mathfrak{z}') \leq d.$$

Proof (outline). In the case $\mathfrak{L}_{\mathsf{a}}$ is the Laplace-Beltrami operator of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$, one can apply well-known Harnack inequalities of LI AND YAU, see [26], because the Riemannian structure is invariant under a group action and thus the Ricci curvature is bounded below.

For arbitrary values of a, our operator is obtained by adding to the Laplace-Beltrami operator a multiple of $\frac{\partial}{\partial z}$, which leads just to conjugating our functions with an exponential in z, compare with the proof of Lemma 5.3. Thus, the inequalities hold with any drift term a. □

The *Martin kernel* is

$$\mathbf{k}_{\mathsf{a}}(\mathfrak{x},\mathfrak{z}|\lambda) = \mathbf{k}(\mathfrak{x},\mathfrak{z}|\lambda) = \frac{\mathbf{g}_{\mathsf{a}}(\mathfrak{x},\mathfrak{z}|\lambda)}{\mathbf{g}_{\mathsf{a}}(0,\mathfrak{z}|\lambda)}\,, \quad \mathfrak{z} \neq 0,\mathfrak{x}\,. \tag{5.10}$$

The *Martin compactification* is the smallest compactification of the underlying space Sol (i.e., a Hausdorff space into which Sol embeds homeomorphically and densely) such that each function $\mathbf{k}_{\mathsf{a}}(\mathfrak{x},\cdot|\lambda)$ has a continuous extension in the second variable. The *Martin boundary* $\mathcal{M}(\lambda) = \mathcal{M}(\mathfrak{L}_{\mathsf{a}},\lambda)$ is the ideal boundary added to the space in that compactification. The extended kernel is also denoted $\mathbf{k}_{\mathsf{a}}(\cdot,\cdot|\lambda)$

(5.11) Proposition. *Every minimal eigenfunction h in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}},\lambda\big)$, $\lambda \ge \lambda_{\min}$, is of the form*

$$h(\mathfrak{x}) = \mathbf{k}_{\mathsf{a}}(\mathfrak{x},\zeta|\lambda), \quad \textit{where} \;\; \zeta \in \mathcal{M}(\lambda)\,.$$

That is, there is a (suitable) sequence $(\mathfrak{z}_n)$ in Sol with $\mathsf{d}(0,\mathfrak{z}_n) \to \infty$, such that

$$h(\mathfrak{x}) = \lim_{n\to\infty} \mathbf{k}(\mathfrak{x},\mathfrak{z}_n|\lambda)$$

where $(\mathfrak{z}_n)$ is a (suitable) sequence in Sol with $\mathsf{d}(0,\mathfrak{z}_n) \to \infty$.

The *minimal Martin boundary* $\mathcal{M}_{\min}(\lambda) = \mathcal{M}_{\min}(\mathfrak{L}_{\mathsf{a}},\lambda)$ consists of all $\zeta \in \mathcal{M}(\lambda)$ for which $\mathbf{k}_{\mathsf{a}}(\cdot,\zeta|\lambda)$ is minimal. It is a Borel subset of $\mathcal{M}(\lambda)$. The *Poisson-Martin representation theorem* says the following.

(5.12) Proposition. *For every function $h \in \mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}},\lambda\big)$, there is a unique Borel measure ν^h on $\mathcal{M}_{\min}(\lambda)$ such that*

$$h(\mathfrak{x}) = \int_{\mathcal{M}_{\min}(\lambda)} \mathbf{k}(\mathfrak{x},\cdot|\lambda)\,d\nu^h \quad \textit{for every } \mathfrak{x} \in \mathsf{Sol}.$$

All this is of course true for more general manifolds and elliptic operators; see [31].

While we are not able to determine the whole Martin compactification, that is, the directions of convergence of the Martin kernels, we shall determine precisely the minimal positive λ-eigenfunctions for each $\lambda \ge \lambda_{\min}$.

6. Positive harmonic functions on $\mathsf{Sol(p,q)}$

We now show that every positive eigenfunction of our Laplacian on $\mathsf{Sol(p,q)}$ splits as a sum of two eigenfunctions that live on the two respective hyperbolic planes which make up Sol, and we determine precisely all minimal positive eigenfunctions. The first step is the following.

(6.1) Theorem. *Let $h \in \mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}},\lambda\big)$ be minimal, where $\lambda \ge -\mathsf{a}^2/2$.*

Then $h(x,y,z) = h_1(x,z)$, where h_1 is minimal in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}},\lambda\big)$, or $h(x,y,z) = h_2(y,-z)$, where h_2 is minimal in $\mathcal{H}^+\big(\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})},\lambda\big)$.

Proof. Let h be a minimal eigenfunction in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}},\lambda\big)$. Then $h = \lim_{n\to\infty} \mathbf{k}(\cdot,\mathfrak{z}_n|\lambda)$. Write $\mathfrak{z}_n = (x_n,y_n,z_n)$.

Claim. (a) If $\inf_n z_n > -\infty$, then for each $a \in \mathbb{R}$,

$$h(x+a,y,z) = h(x,y,z) \quad \text{for all} \quad (x,y,z) \in \mathsf{Sol}\,.$$

(b) If $\sup_n z_n < +\infty$, then for each $b \in \mathbb{R}$,

$$h(x,y+b,z) = h(x,y,z) \quad \text{for all} \quad (x,y,z) \in \mathsf{Sol}\,.$$

To prove part (a) of this claim, let $a \in \mathbb{R}$ and consider the group element

$$\mathfrak{g}_a = (a,0,0) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{Sol(p,q)}\,.$$

We abbreviate $\tau_a = \tau_{\mathfrak{g}_a}$. Let $\mathfrak{x} = (x,y,z) \in \mathsf{Sol}$. Then, by (2.5), $\tau_a h(\mathfrak{x}) = h(\mathfrak{g}_a\mathfrak{x}) = h(x+a,y,z)$, and Lemma 2.6 tells us that $\tau_a h$ is in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}},\lambda\big)$. Now by (1.4)

$$\mathsf{d}(\mathfrak{g}_a\mathfrak{z}_n\,,\mathfrak{z}_n) = \mathsf{d}_{\mathsf{Sol}}\big((x_n+a,y_n,z_n),(x_n,y_n,z_n) = \mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((x_n+a,z_n),(x_n,z_n)\big)\,.$$

Elementary properties of the hyperbolic metric imply that

$$\mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((x_n+a,z_n),(x_n,z_n)\big) = \mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((a,z_n),(0,z_n)\big) \le \mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((a,c),(0,c)\big) = d_a\,,$$

where $c = \inf_n z_n$. Let $C_{d_a}(\lambda)$ be the corresponding Harnack constant in Lemma 5.9. Then, using that $\mathfrak{g}(\cdot,\cdot|\lambda)$ is $\mathsf{Sol(p,q)}$-invariant,

$$\mathbf{k}(\mathfrak{g}_a\mathfrak{x}\,,\mathfrak{z}_n|\lambda) = \frac{\mathbf{g}(\mathfrak{g}_a\mathfrak{x}\,,\mathfrak{z}_n|\lambda)}{\mathbf{g}(\mathfrak{g}_a\mathfrak{x}\,,\mathfrak{g}_a\mathfrak{z}_n|\lambda)}\,\frac{\mathbf{g}(\mathfrak{g}_a\mathfrak{x}\,,\mathfrak{g}_a\mathfrak{z}_n|\lambda)}{\mathbf{g}(0\,,\mathfrak{z}_n|\lambda)} \le C_{d_a}\,\mathbf{k}(\mathfrak{x}\,,\mathfrak{z}_n|\lambda)$$

Letting $n\to\infty$, we obtain

$$\tau_a h(\mathfrak{x}) = h(\mathfrak{g}_a\mathfrak{x}) \le C_{d_a}\,h(\mathfrak{x}) \quad \text{for all } \mathfrak{x}\in\mathsf{Sol}\,.$$

Now minimality of h implies that the function $\tau_a h/h$ is constant. For $\mathfrak{x} = (x,y,z) \in \mathsf{Sol}$,

$$\mathsf{d}_{\mathsf{Sol}}(\mathfrak{g}_a\mathfrak{x},\mathfrak{x}) = \mathsf{d}_{\mathbb{H}(\mathsf{p})}\big((x+a,z),(x,a)\big)\to 0\,, \quad \text{if} \quad z\to+\infty\,.$$

Therefore the second statement in Lemma 5.9 implies that $h(\mathfrak{g}_a\mathfrak{x})/h(\mathfrak{x}) \to 1$ as $z\to+\infty$, and we conclude that $\tau_a h/h \equiv 1$. This proves statement (a) of the claim, and statement (b) follows by exchanging the roles of the x- and y-coordinates and changing the sign of z.

Now (z_n) must have a subsequence which converges to a limit in $[-\infty\,,+\infty]$. We may assume without loss of generality that (z_n) itself converges.

Case 1. $z_n \to \infty$. Then we can apply part (a) of the Claim, and conclude that h depends only on (y,z). By Lemma 5.6, there is a function h_2 on $\mathbb{H}(\mathsf{q})$ which is minimal in $\mathcal{H}^+\big(\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})},\lambda\big)$, such that $h(x,y,z) = h_2(y,-z)$ for all $\mathfrak{x} = (x,y,z)\in\mathsf{Sol}$.

Case 2. $z_n\to-\infty$. Then we can apply part (b) of the Claim, and again by Lemma 5.6, there is a function h_1 on $\mathbb{H}(\mathsf{p})$ which is minimal in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})},\lambda\big)$, such that $h(x,y,z) = h_1(x,z)$ for all $\mathfrak{x} = (x,y,z)\in\mathsf{Sol}$.

Case 3. $z_n\to z_0\in\mathbb{R}$. Then we can apply both parts (a) and (b) of the claim, and there is a function $\widetilde{h}$ on $\mathbb{R}$ such that $h(x,y,z) = \widetilde{h}(z)$ for all $\mathfrak{x} = (x,y,z)\in\mathsf{Sol}$. It must be minimal both as a function $(x,z)\mapsto\widetilde{h}(z)$ in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})},\lambda\big)$ and as a function $(y,z)\mapsto\widetilde{h}(-z)$ in $\mathcal{H}^+\big(\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})},\lambda\big)$. Of course, it must also be a minimal element of $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathbb{R}},\lambda\big)$. □

(6.2) Remark. If $h\in\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}},\lambda\big)$ is minimal and depends only on z then it must arise by lifting a minimal element of $h\in\mathcal{H}^+\big(\widetilde{\mathfrak{L}}_{\mathsf{a}},\lambda\big)$ from $\mathbb{R}$ to Sol. That is, we must have $h(x,y,z) = e^{\alpha z}$, where $\alpha = \pm\sqrt{\mathsf{a}^2+2\lambda}-\mathsf{a}$. Furthermore, in this case, the function $(x,z)\mapsto e^{\alpha z}$ must be minimal in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})},\lambda\big)$, so that – by Lemma 5.3 – we can only have the "+" sign, that is, $\alpha = \alpha(\lambda,\mathsf{a})$. We shall see below that the corresponding function can really be a minimal λ-eigenfunction on Sol only when $\lambda = \lambda_{\min}$.

(6.3) Corollary. *If $h \in \mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})}, \lambda\big)$, where $\lambda \ge -\mathsf{a}^2/2$, then there are nonnegative functions $h_1 \in \mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})}, \lambda\big)$ and $h_2 \in \mathcal{H}^+\big(\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})}, \lambda\big)$ such that for all $\mathfrak{x} = (x,y,z) \in \mathsf{Sol}(\mathsf{p},\mathsf{q})$,*

$$h(x,y,z) = h_1(x,z) + h_2(y,-z).$$

Proof. We see from Theorem 6.1 that the set of all minimal λ-eigenfunctions on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ is contained in the union of the sets of minimal λ-eigenfunctions on $\mathbb{H}(\mathsf{p})$ and $\mathbb{H}(\mathsf{q})$, with a change of the sign of z for the latter, according to the above cases. Thus, taking into account Remark 6.2, $\mathcal{M}_{\min}(\lambda)$ can be parametrized by a subset of the disjoint union $\vartheta\mathbb{H}(\mathsf{p}) \cup \vartheta^*\mathbb{H}(\mathsf{q}) \cong \vartheta\mathsf{Sol}(\mathsf{p},\mathsf{q})$, or in other terms, of the "8"-shaped outer part of the geometric boundary of Sol (without the interiors of the two disks).

By Proposition 5.12, for every function $h \in \mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})}, \lambda\big)$, there is a Borel measure $\nu = \nu^h$ on $\mathcal{M}_{\min}(\lambda)$ that yields the integral representation of h. Now let ν_1 be the restriction of ν to $\vartheta\mathbb{H}(\mathsf{p})$ and ν_2 the restriction to $\vartheta^*\mathbb{H}(\mathsf{q})$. Then we get for every $\mathfrak{x} = (x,y,z) \in \mathsf{Sol}$

$$h(\mathfrak{x}) = \int_{\vartheta\mathbb{H}(\mathsf{p})} P_{\mathsf{p},\mathsf{a},\lambda}\big((x,z),\xi\big)\,d\nu_1(\xi) + \int_{\vartheta^*\mathbb{H}(\mathsf{q})} P_{\mathsf{q},-\mathsf{a},\lambda}\big((y,-z),\eta\big)\,d\nu_2(\eta) = h_1(x,z) + h_2(y,-z)\,,$$

as proposed. □

(6.4) Corollary. *The Laplacian $\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})}$ has the (weak) Liouville property, i.e., all bounded harmonic functions on Sol are constant, if and only if the rate of escape a vanishes.*

Proof. If $\mathsf{a} = 0$, then all bounded harmonic functions are constant by Corollary 6.3 and Remark 5.5. Conversely, if $\mathsf{a} \ne 0$, then again by Remark 5.5, one of $\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})}$ and $\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})}$ has non-constant bounded harmonic functions, and they lift to harmonic functions on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$. □

The last corollary, which was obtained in a very concrete, case-specific way, should be compared with the theorem of Karlsson and Ledrappier [21], which says that (under very general conditions) the weak Liouville property holds if and only if the rate of escape of Brownian motion is 0.

When $\mathsf{a} \ne 0$, we have the following.

(6.5) Corollary. (i) *If $\mathsf{a} > 0$ then every bounded harmonic function for $\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})}$ has the form $(x,y,z) \mapsto h_2(y,-z)$, where h_2 is a bounded harmonic function for $\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})}$.*

(ii) *If $\mathsf{a} < 0$ then every bounded harmonic function for $\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})}$ has the form $(x,y,z) \mapsto h_1(x,z)$, where h_1 is a bounded harmonic function for $\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})}$.*

Proof. Let h be a bounded harmonic function on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$. We may assume without loss of generality that it is non-negative. We decompose $h(x,y,z) = h_1(x,z) + h_2(y,-z)$ according to Corollary 6.3. Then both h_1 and h_2 are bounded harmonic. When $\mathsf{a} > 0$, Remark 5.5 tells us that h_1 must be constant, so that we can "incorporate" it into h_2. Analogously, when $\mathsf{a} < 0$, the function h_2 must be constant. □

(6.6) Theorem. *The minimal eigenfunctions in* $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}}, \lambda\big)$, $\lambda \ge \lambda_{\min}$, *are precisely the functions*

$$(x,y,z) \mapsto P_{\mathsf{p},\mathsf{a},\lambda}\big((x,z),\xi\big) \quad \textit{and} \quad (x,y,z) \mapsto P_{\mathsf{q},-\mathsf{a},\lambda}\big((y,-z),\eta\big)\,, \quad \xi,\eta \in \mathbb{R}\,,$$

and in addition, when $\lambda = \lambda_{\min}$, *the function*

$$(x,y,z) \mapsto e^{-\mathsf{a}\,z}\,.$$

Proof. Combining Theorem 6.1 with Lemma 5.3, we see that each minimal λ-eigenfunction on Sol must be of the form

$$(x,y,z) \mapsto P_{\mathsf{p},\mathsf{a},\lambda}\big((x,z),\xi\big)\,, \quad \text{where} \quad \xi \in \vartheta\mathbb{H}(\mathsf{p})\,, \quad \text{or}$$
$$(x,y,z) \mapsto P_{\mathsf{q},-\mathsf{a},\lambda}\big((y,-z),\eta\big)\,, \quad \text{where} \quad \eta \in \vartheta\mathbb{H}(\mathsf{q})\,.$$

We have to show that for $\xi \ne \varpi_{\mathsf{p}}$ and for $\eta \ne \varpi_{\mathsf{q}}$, the respective functions are indeed all minimal. Furthermore, we have to show that for $\xi = \varpi_{\mathsf{p}}$ and for $\eta = \varpi_{\mathsf{q}}$, the two resulting functions are *not* minimal λ-eigenfunctions on Sol, unless $\lambda = \lambda_{\min}$. In this last case both coincide and are equal to $e^{-\mathsf{a}\,z}$.

So first we show minimality of $(x,y,z) \mapsto P_{\mathsf{p},\mathsf{a},\lambda}\big((x,z),\xi\big)$ with $\xi \in \vartheta^*\mathbb{H}(\mathsf{p})$. Suppose that $P_{\mathsf{p},\mathsf{a},\lambda}\big((x,z),\xi\big) \ge h(x,y,z)$ for all $\mathfrak{z} = (x,y,z)$, where $h \in \mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}}, \lambda\big)$.

We decompose $h(x,y,z) = h_1(x,z) + h_2(y,-z)$ according to Corollary 6.3. By minimality of $P_{\mathsf{p},\mathsf{a},\lambda}\big((\cdot,\cdot),\xi\big)$ in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})}, \lambda\big)$ (Lemma 5.3), we must have $h_1 = c \cdot P_{\mathsf{p},\mathsf{a},\lambda}\big((\cdot,\cdot),\xi\big)$, where $0 \le c \le 1$. If $c = 1$, we are done. If $c < 1$ then we get

$$P_{\mathsf{p},\mathsf{a},\lambda}\big((x,z),\xi\big) \ge \frac{1}{1-c} h_2(y,-z) = \int_{\vartheta\mathbb{H}(\mathsf{q})} P_{\mathsf{q},-\mathsf{a},\lambda}\big((x,-z),\eta\big)\, d\nu(\eta) \quad \text{for all } (x,y,z) \in \mathsf{Sol}\,,$$

where ν is a Borel measure on $\vartheta\mathbb{H}(\mathsf{q})$. Setting $y = z = 0$, we get

$$P_{\mathsf{p},\mathsf{a},\lambda}\big((x,0),\xi\big) \ge \nu\big(\vartheta\mathbb{H}(\mathsf{q})\big) \quad \text{for all } x \in \mathbb{R}\,.$$

If $x \to \infty$, then we see from the formula for $P_{\mathsf{q},-\mathsf{a},\lambda}$ of Lemma 5.3 that the left hand side in the last inequality tends to 0. Therefore $\nu\big(\vartheta\mathbb{H}(\mathsf{q})\big) = 0$, whence $h_2 \equiv 0$, contradicting the assumption that $c < 1$.

The proof of minimality of $(x,y,z) \mapsto P_{\mathsf{q},-\mathsf{a},\lambda}\big((x,-z),\eta\big)$, where $\eta \in \vartheta^*\mathbb{H}(\mathsf{q})$, follows as usual by exchanging the roles of the x- and y-variables.

Next, let $\xi = \varpi_{\mathsf{p}}$ and $\lambda > \lambda_{min}$, so that we are considering the function

$$(x,y,z) \mapsto P_{\mathsf{p},\mathsf{a},\lambda}\big((x,z),\varpi_{\mathsf{p}}\big) = e^{\alpha(\lambda,\mathsf{a})z}\,.$$

If it were minimal in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}}, \lambda\big)$, then by Lemma 5.6, also the function $(y,z) \mapsto e^{-\alpha(\lambda,\mathsf{a})z}$ would have to be minimal in $\mathcal{H}^+\big(\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})}, \lambda\big)$, which is not the case by Lemma 5.3.

Analogously, when $\lambda > \lambda_{min}$, the function

$$(x,y,z) \mapsto P_{\mathsf{q},-\mathsf{a},\lambda}\big((y,-z),\varpi_{\mathsf{q}}\big) e^{-\alpha(\lambda,-\mathsf{a})z}$$

cannot be minimal in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}}, \lambda\big)$.

Finally, consider the case $\lambda = \lambda_{min}$ and the function $(x,y,z) \mapsto e^{-\mathsf{a}z}$ in $\mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}}, \lambda\big)$. We use a well-known trick, conjugating our operator with this exponential: suppose that $e^{-\mathsf{a}z} \ge h(x,y,z)$, where $h \in \mathcal{H}^+\big(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}}, \lambda\big)$. Then a straightforward computation shows that the function $\tilde h(x,y,z) = e^{\mathsf{a}z}h(x,y,z)$ is in $\mathcal{H}^+\bigl(\mathfrak{L}_0^{\mathsf{Sol(p,q)}},0\bigr)$, that is, it is bounded harmonic, and the new rate of escape is 0. By Corollary 6.4, $\tilde h$ is constant. This proves minimality of $(x,y,z)\mapsto e^{-\mathsf{a}z}$ in $\mathcal{H}^+\bigl(\mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol(p,q)}},\lambda\bigr)$. □

Our results tell us that the *Poisson boundary* of Brownian motion with drift on Sol is the "outer" boundary

$$\Bigl(\vartheta^*\mathbb{H}(\mathsf{p})\times\{\varpi_{\mathsf{q}}\}\Bigr)\cup\Bigl(\{\varpi_{\mathsf{p}}\}\times\vartheta^*\mathbb{H}(\mathsf{q})\Bigr)\cup\Bigl\{(\varpi_{\mathsf{p}}\,,\varpi_{\mathsf{q}})\Bigr\}$$

together with the limit distribution provided by Proposition 4.2. Indeed, for $\mathsf{a}<0$, it is just the first of these three pieces, because the limit distributition is supported by that piece. For $\mathsf{a}>0$, it is just the second piece, and for $\mathsf{a}=0$, it is trivial, i.e., the singleton of the third piece. Here, we do not go into details regarding the construction of the Poisson boundary. (In short, it is the largest probability space that gives rise to an integral representation of all bounded harmonic functions and at the same time provides a model for the limit behavior of the process at infinity.) The reader is referred to the body of work of Kaimanovich, e.g. [20].

Regarding the Martin boundary (which is a metric space, while the Poisson boundary is a measure space), our results underline the evidence that $\mathcal{M}(\lambda_{\min})$ is the boundary in the geometric compactification that we have described in §5, while for $\lambda>\lambda_{\min}$ it should be bigger: one first should consider the *horocyclic compactification* of $\mathbb{H}(\mathsf{p})$, which can be built from the usual one as follows. Replace the boundary point ϖ_{p} by the set $\{\varpi_{\mathsf{p}}^{\zeta}:\zeta\in[-\infty\,,\infty]\}$, which carries the topology of the extended real line. Furthermore, modify the topology by saying that in the new compactification, $(x,z)\to\varpi_{\mathsf{p}}^{\zeta}$ if $|x|\to\infty$ and $z\to\zeta$. Then we expect that the Martin compactification of $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ for $\lambda>\lambda_{\min}$ is the closure of Sol in the direct product of the horocyclic compactifications of the two hyperbolic planes. This evidence comes from the strong analogy with the DL-graphs (the horocyclic product of two homogeneous trees), see [8]; the rigorous proof still has to be carried out.

References

[1] Ancona, A.: *Théorie du potentiel sur les graphes et les variétés.* In:*École d'Été de Probabilités de Saint-Flour XVIII—1988*, pp. 1–112, Lecture Notes in Math. **1427**, Springer, Berlin, 1990.

[2] Baldi, P., Casadio Tarabusi, E., Figà-Talamanca, A., and Yor, M.: *Non-symmetric hitting distributions on the hyperbolic half-plane and subordinated perpetuities.* Rev. Mat. Iberoamericana **17** (2001), 587–605.

[3] Bartholdi, L., and Woess, W.: *Spectral computations on lamplighter groups and Diestel-Leader graphs.* J. Fourier Anal. Appl. **11** (2005) 175 - 202.

[4] Bendikov, A., Saloff-Coste, L., Salvatori, M., and Woess, W. *The heat semigroup and Brownian motionon strip complexes.* Advances in Math. **226** (2011) 992–1055.

[5] Bendikov, A., Saloff-Coste, L., Salvatori, M., and Woess, W. *Positive harmonic functions and Brownian motion on treebolic spaces.* In preparation.

[6] Bertacchi, D.: *Random walks on Diestel-Leader graphs.* Abh. Math. Sem. Univ. Hamburg **71** (2001) 205–224.

[7] Brofferio, S.: *How a centred rndom walk on the affine group goes to infinity.* Ann. Inst. H. Poincaré Probab. Statist. **39** (2003), 371–384.

[8] Brofferio, S., and Woess, W.: *Green kernel estimates and the full Martin boundary for random walks on lamplighter groups and Diestel-Leader graphs.* Annales Inst. H. Poincaré (Prob. & Stat.) **41** (2005) 1101–1123.

[9] Brofferio, S., and Woess, W.: *Positive harmonic functions for semi-isotropic random walks on trees, lamplighter groups, and DL-graphs.* Potential Analysis **24** (2006) 245–265.

[10] Dambis, K. E.: *On decomposition of continuous submartingales.* (Russian) Teor. Verojatnost. i Primenen. **10** (1965) 438–448.

[11] Derriennic, Y.: *Quelques applications du théorème ergodique sous-additif.* Astérisque **74** (1980) 183–201.

[12] Dubins, L. E., and Schwarz, G.: *On continuous martingales.* Proc. Nat. Acad. Sci. U.S.A. **53** (1965) 913–916.

[13] Eskin, A., Fisher, D., and Whyte, K.: *Quasi-isometries and rigidity of solvable groups.* Pure Appl. Math. Q. **3** (2007) 927–947.

[14] Eskin, A., Fisher, D., and Whyte, K.: *Coarse differentiation of quasi-isometries I: spaces non quasi-isometric to Cayley graphs.* preprint, Univ. Chicago (2006).

[15] Farb, B., and Mosher, L.: *A rigidity theorem for the solvable Baumslag-Solitar groups. With an appendix by Daryl Cooper.* Invent. Math. **131** (1998) 419–451.

[16] Grincevičjus, A. K.: *A central limit theorem for the group of linear transformations of the real axis.* Soviet Math. Doklady **15** (1974) 1512–1515.

[17] Guivarc'h, Y., Ji, L., and Taylor, J. C.: *Compactifications of Symmetric Spaces.* Progress in Mathematics **156**. Birkhäuser Boston, Inc., Boston, MA, 1998.

[18] Helgason, S.: *Groups and Geometric Analysis.* Pure and applied mathematics **113**. Academic Press, Orlando, 1984.

[19] Kaimanovich, V. A.: *Boundaries of invariant Markov operators: the identification problem.* In: *Ergodic Theory of Z_d Actions (Warwick, 1993–1994)*, pp. 127–176, London Math. Soc. Lecture Note Ser. **228**, Cambridge Univ. Press, Cambridge, 1996.

[20] Kaimanovich, V. A.: *The Poisson formula for groups with hyperbolic properties.* Ann. of Math. **152** (2000) 659–692.

[21] Karlsson, A., and Ledrappier, F.: *Propriétde Liouville et vitesse de fuite du mouvement Brownien.* C. R. Math. Acad. Sci. Paris **344** (2007) 685–690.

[22] Kesten, H.: *Random difference equations and renewal theory for products of random matrices.* Acta Math. **131** (1973) 207–248.

[23] Kingman, J. F. C.: *Subadditive ergodic theory.* Ann. Probab. **1** (1973), 883–899.

[24] Knight, F. B.: *A reduction of continuous square-integrable martingales to Brownian motion.* In: *Martingales* (Oberwolfach, 1970), pp. 19–31. Lecture Notes in Math. **190**, Springer, Berlin, 1971.

[25] Ledrappier, F.: *Some asymptotic properties of random walks on free groups.* In *Topics in probability and Lie groups: boundary theory, 117–152*, CRM Proc. Lecture Notes **28**, Amer. Math. Soc., Providence, RI, 2001.

[26] Li, P., and Yau, S.-T.: *On the parabolic kernel of the Schrödinger operator.* Acta Math. **156** (1986) 153–201.

[27] Lyons, T., and Sullivan, D.: *Function theory, random paths and covering spaces.* J. Differential Geom. **19** (1984) 299–323.

[28] Protter, Ph. E.: *Stochastic Integration and Differential Equations.* Second edition. Applications of Mathematics, **21**. Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin, 2004.

[29] Raugi, A.: *Fonctions harmoniques positives sur certains groupes de Lie résolubles connexes.* Bull. Soc. Math. France **124** (1996), 649–684.

[30] Revuz, D. and Yor, M. : *Continuous Martingales and Brownian Motion.* Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **293**, Springer-Verlag, Berlin, 1994.

[31] Taylor, J. C.: *The Martin compactification associated with a second order strictly elliptic partial differential operator on a manifold M.* In: *Topics in Probability and Lie Groups: Boundary Theory*, pp. 153–202, CRM Proc. Lecture Notes **28**, Amer. Math. Soc., Providence, RI, 2001.

[32] Woess, W.: *Lamplighters, Diestel-Leader graphs, random walks, and harmonic functions.* Combinatorics, Probability & Computing **14** (2005) 415–433.

[33] Woess, W.: *Denumerable Markov chains. Generating functions, boundary theory, random walks on trees.* EMS Textbooks in Mathematics. European Mathematical Society, Zürich, 2009.