

STOCHASTIC DYNAMICAL SYSTEMS WITH WEAK CONTRACTIVITY PROPERTIES

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WITH A CHAPTER FEATURING RESULTS OF MARTIN BENDA

ABSTRACT. Consider a proper metric space X and a sequence $(F_n)_{n \geq 0}$ of i.i.d. random continuous mappings $X \rightarrow X$. It induces the stochastic dynamical system (SDS) $X_n^x = F_n \circ \dots \circ F_1(x)$ starting at $x \in X$. In this paper, we study existence and uniqueness of invariant measures, as well as recurrence and ergodicity of this process.

In the first part, we elaborate, improve and complete the unpublished work of Martin Benda on local contractivity, which merits publicity and provides an important tool for studying stochastic iterations. We consider the case when the F_n are contractions and, in particular, discuss recurrence criteria and their sharpness for reflected random walk.

In the second part, we consider the case where the F_n are Lipschitz mappings. The main results concern the case when the associated Lipschitz constants are log-centered. Principal tools are the Chacon-Ornstein theorem and a hyperbolic extension of the space X as well as the process (X_n^x) .

The results are applied to the reflected affine stochastic recursion given by $X_0^x = x \geq 0$ and $X_n^x = |A_n X_{n-1}^x - B_n|$, where (A_n, B_n) is a sequence of two-dimensional i.i.d. random variables with values in $\mathbb{R}_*^+ \times \mathbb{R}_*^+$.

CONTENTS

1. Introduction	2
PART I. Strong and local contractivity and examples, including reflected random walk	6
2. Local contractivity and the work of Benda	6
3. Basic example: the affine stochastic recursion	13
4. Iteration of random contractions	14
5. Some remarks on reflected random walk	16
PART II. Stochastic dynamical systems induced by Lipschitz mappings	22
6. The contractive case, and recurrence in the log-centered case	22
7. Hyperbolic extension	25
8. Transient extended SDS	26
9. Conservative extended SDS	29
10. The reflected affine stochastic recursion	35
References	41

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1. INTRODUCTION

We start by reviewing two well known models.

First, let $(B_n)_{n \geq 0}$ be a sequence of i.i.d. real valued random variables. Then *reflected random walk* starting at $x \geq 0$ is the stochastic dynamical system given recursively by $X_0^x = x$ and $X_n^x = |X_{n-1}^x - B_n|$. The absolute value becomes meaningful when B_n assumes positive values with positive probability; otherwise we get an ordinary random walk on \mathbb{R} . Reflected random walk was described and studied by FELLER [19]; apparently, it was first considered by VON SCHELLING [35] in the context of telephone networks. In the case when $B_n \geq 0$, FELLER [19] and KNIGHT [27] have computed an invariant measure for the process when the Y_n are non-lattice random variables, while BOUDIBA [8], [9] has provided such a measure when the Y_n are lattice variables. LEGUESDRON [28], BOUDIBA [9] and BENDA [4] have also studied its uniqueness (up to constant factors). When that invariant measure has finite total mass – which holds if and only if $\mathbb{E}(B_1) < \infty$ – the process is (topologically) recurrent: with probability 1, it returns infinitely often to each open set that is charged by the invariant measure. Indeed, it is positive recurrent in the sense that the mean return time is finite. More general recurrence criteria were provided by SMIRNOV [36] and RABEHERIMANANA [33], and also in our unpublished paper [32]: basically, recurrence holds when $\mathbb{E}(\sqrt{B_1})$ or quantities of more or less the same order are finite. In the present paper, we shall briefly touch the situation when the B_n are not necessarily positive.

Second, let $(A_n, B_n)_{n \geq 0}$ be a sequence of i.i.d. random variables in $\mathbb{R}_*^+ \times \mathbb{R}$. (We shall always write $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}_*^+ = (0, \infty)$, the latter usually seen as a multiplicative group.) The associated *affine stochastic recursion* on \mathbb{R} is given by $Y_0^x = x \in \mathbb{R}$ and $Y_n^x = A_n Y_{n-1}^x + B_n$. There is an ample literature on this process, which can be interpreted in terms of a random walk on the affine group. That is, one applies products of affine matrices:

$$\begin{pmatrix} Y_n^x \\ 1 \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{n-1} & B_{n-1} \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} A_1 & B_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Products of affine transformations were one of the first examples of random walks on non-commutative groups, see GRENANDER [22]. Among the large body of further work, we mention KESTEN [26], GRINCEVIČJUS [23], [24], ELIE [16], [17], [18], and in particular the papers by BABILLOT, BOUGEROL AND ELIE [3] and BROFFERIO [10]. See also the more recent work of BURACZEWSKI [11] and BURACZEWSKI, DAMEK, GUIVARC'H, HULANICKI AND URBAN [12].

As an application of the results of the present paper, we shall study the synthesis of the above two processes. This is the variant of the affine recursion which is forced to stay non-negative: whenever it reaches the negative half-axis, its sign is changed. Thus, we have i.i.d. random variables $(A_n, B_n)_{n \geq 0}$ in $\mathbb{R}_*^+ \times \mathbb{R}$, and our process is

$$(1.1) \quad X_0^x = x \geq 0 \quad \text{and} \quad X_n^x = |A_n X_{n-1}^x - B_n|.$$

We choose the minus sign in the recursion in order to underline the analogy with reflected random walk. Here, we shall only consider the most typical situation, where $B_n > 0$. When $A_n \equiv 1$ then we are back at reflected random walk.

In all those introductory examples, the hardest and most interesting case is the one when A_n is *log-centered*, that is, $\mathbf{E}(\log A_n) = 0$, and the development of tools for handling this case is the main focus of the present work. The easier and well-understood case is the *contractive* one, where $\mathbf{E}(\log A_n) < 0$.

In this paper, stochastic dynamical systems are considered in the following general setting. Let (\mathbf{X}, d) be a proper metric space (i.e., closed balls are compact), and let \mathfrak{G} be the monoid of all continuous mappings $\mathbf{X} \rightarrow \mathbf{X}$. It carries the topology of uniform convergence on compact sets. Now let $\tilde{\mu}$ be a regular probability measure on \mathfrak{G} , and let $(F_n)_{n \geq 1}$ be a sequence of i.i.d. \mathfrak{G} -valued random variables (functions) with common distribution $\tilde{\mu}$, defined on a suitable probability space $(\Omega, \mathfrak{A}, \Pr)$. The measure $\tilde{\mu}$ gives rise to the *stochastic dynamical system* (SDS) $\omega \mapsto X_n^x(\omega)$ defined by

$$(1.2) \quad X_0^x = x \in \mathbf{X}, \quad \text{and} \quad X_n^x = F_n(X_{n-1}^x), \quad n \geq 1.$$

There is an ample literature on processes of this type, see e.g. ARNOLD [2] or BHATTACHARYA AND MAJUMDAR [7]. In the setting of our reflected affine recursion (1.1), we have $\mathbf{X} = \mathbb{R}^+$ with the standard distance, and $F_n(x) = |A_n x - B_n|$, so that the measure $\tilde{\mu}$ is the image of the distribution μ of the two-dimensional i.i.d. random variables (A_n, B_n) under the mapping $\mathbb{R} \times \mathbb{R}_*^+ \rightarrow \mathfrak{G}$, $(a, b) \mapsto f_{a,b}$, where $f_{a,b}(x) = |ax - b|$. Any SDS (1.2) is a Markov chain. The transition kernel is

$$P(x, U) = \Pr[X_1^x \in U] = \tilde{\mu}(\{f \in \mathfrak{G} : f(x) \in U\}),$$

where U is a Borel set in \mathbf{X} . The associated transition operator is given by

$$P\varphi(x) = \int_{\mathbf{X}} \varphi(y) P(x, dy) = \mathbf{E}(\varphi(X_1^x)),$$

where $\varphi : \mathbf{X} \rightarrow \mathbb{R}$ is a measurable function for which this integral exists. The operator is Fellerian, that is, $P\varphi$ is continuous when φ is bounded and continuous. We shall write $\mathcal{C}_c(\mathbf{X})$ for the space of compactly supported continuous functions $\mathbf{X} \rightarrow \mathbb{R}$.

The SDS is called *transient*, if every compact set is visited only finitely often, that is,

$$\Pr[d(X_n^x, x) \rightarrow \infty] = 1 \quad \text{for every } x \in \mathbf{X}.$$

We call it (*topologically*) *recurrent*, if there is a non-empty, closed set $\mathbf{L} \subset \mathbf{X}$ such that for every open set U that intersects \mathbf{L} ,

$$\Pr[X_n^x \in U \text{ infinitely often}] = 1 \quad \text{for every } x \in \mathbf{L}.$$

In our situation, we shall even have this for every starting point $x \in \mathbf{X}$, so that \mathbf{L} is an *attractor* for the SDS. As an intermediate notion, we call the SDS *conservative*, if

$$\Pr[\liminf_n d(X_n^x, x) < \infty] = 1 \quad \text{for every } x \in \mathbf{X}.$$

Besides the question whether the SDS is recurrent, we shall mainly be interested in the question of existence and uniqueness (up to constant factors) of an *invariant measure*. This is a Radon measure ν on \mathbf{X} such that for any Borel set $U \subset \mathbf{X}$,

$$\nu(U) = \int_{\mathbf{X}} \Pr[X_1^x \in U] d\nu(x).$$

We can construct the *trajectory space* of the SDS starting at x . This is

$$(\mathbf{X}^{\mathbb{N}_0}, \mathfrak{B}(\mathbf{X}^{\mathbb{N}_0}), \Pr_x),$$

where $\mathfrak{B}(\mathbf{X}^{\mathbb{N}_0})$ is the product Borel σ -algebra on $\mathbf{X}^{\mathbb{N}_0}$, and \Pr_x is the image of the measure \Pr under the mapping

$$\Omega \rightarrow \mathbf{X}^{\mathbb{N}_0}, \quad \omega \mapsto (X_n^x(\omega))_{n \geq 0}.$$

If we have an invariant Radon measure, then we can construct the measure

$$\Pr_\nu = \int_{\mathbf{L}} \Pr_x d\nu(x)$$

on the trajectory space. It is a probability measure only when ν is a probability measure on \mathbf{X} . In general, it is σ -finite and invariant with respect to the time shift $T : \mathbf{X}^{\mathbb{N}_0} \rightarrow \mathbf{X}^{\mathbb{N}_0}$. Conservativity of the SDS will be used to get conservativity of the shift. We shall study ergodicity of T , which in turn will imply uniqueness of ν (up to multiplication with constants).

As often in this field, ideas that were first developped by FURSTENBERG, e.g. [21], play an important role at least in the background.

(1.3) Proposition. [Furstenberg’s contraction principle.] *Let $(F_n)_{n \geq 1}$ be i.i.d. continuous random mappings $\mathbf{X} \rightarrow \mathbf{X}$, and define the right process*

$$R_n^x = F_1 \circ \cdots \circ F_n(x).$$

If there is an \mathbf{X} -valued random variable Z such that

$$\lim_{n \rightarrow \infty} R_n^x = Z \quad \text{almost surely for every } x \in \mathbf{X},$$

then the distribution ν of the limit Z is the unique invariant probability measure for the SDS $X_n^x = F_n \circ \cdots \circ F_1(x)$.

A proof can be found, e.g., in LETAC [29] in a slightly more general setting.

While being ideally applicable to the contractive case, this contraction principle is not the right tool for handling the log-centered case mentioned above. In the context of the affine stochastic recursion, BABILLOT, BOUGEROL AND ELIE [3] introduced the notion of *local contractivity*, see Definition 2.1 below. This was then exploited systematically by BENDA in interesting and useful work in his PhD thesis [4] (in German) and the two subsequent preprints [5], [6] which were accepted for publication, circulated (not very widely) in preprint version but have remained unpublished. In personal communication, BENDA also gives credit to unpublished work of his late PhD advisor KELLERER, compare with the posthumous publication [25].

We think that this material deserves to be documented in a publication, whence we include – with the consent of M. Benda whom we managed to contact – the next section on weak contractivity (§2). The proofs that we give are “streamlined”, and new aspects and results are added, such as, in particular, ergodicity of the shift on the trajectory space with respect to \Pr_ν (Theorem 2.13). Ergodicity yields uniqueness of the invariant measure. Before that, we explain the alternative between recurrence and transience and the limit set (attractor) \mathbf{L} , which is the support of the invariant measure ν .

We display briefly the classical results regarding the stochastic affine recursion in §3. Then, in §4, we consider the situation when the F_n are contractions with Lipschitz constants $A_n = \mathfrak{l}(F_n) \leq 1$ (not necessarily assuming that $\mathbf{E}(\log A_n) < 0$). We provide a tool for getting strong contractivity in the recurrent case (Theorem 4.2). A typical example is reflected random walk. In §5, we discuss some of its properties, in particular sharpness of recurrence criteria.

This concludes Part I of the paper. In Part II, we examine in detail the iteration of general Lipschitz mappings. That is, the Lipschitz constants $A_n = \mathfrak{l}(F_n)$ of the F_n are positive, finite, i.i.d. random variables. The emphasis is on the case when the A_n are log-centered. We impose natural non-degeneracy assumptions and suitable moment conditions on A_n as well as $B_n = d(F_n(o), o)$, where $o \in \mathbf{X}$ is a reference point. We first prove existence of a non-empty limit set \mathbf{L} on which the SDS is recurrent (§6, Theorem 6.7).

Then (§7) we introduce a *hyperbolic extension* of the space \mathbf{X} as well as of the SDS. The extended SDS turns out to be generated by Lipschitz mappings with Lipschitz constants $= 1$ (Lemma 7.5). The hyperbolic extension appears to be interesting in its own right, and we intend to come back to it in future work. It yields that the extended SDS is either transient or conservative, although in general typically not locally contractive.

First, in §8, we consider the case when the extended SDS is transient. In this case, we can show (8.4) that the original SDS is locally contractive, so that all results of §2 apply. In particular, we get uniqueness of the invariant Radon measure ν (up to constant factors) and ergodicity of the shift on the associated trajectory space. It is worth while to mention that the “classical” instance of this situation is the affine stochastic recursion. Its hyperbolic extension is a random walk on the affine group, which is well known to be transient.

The hardest case turns out to be the one when the extended SDS is conservative (§9). In this case, we are able to obtain a result only under an additional assumption (9.7) on the original SDS that resembles the criterion used in §4 for SDS of contractions. But then we even get ergodicity and uniqueness of the invariant Radon measure for the extended SDS (Theorem 9.14).

In the final section (§10), we explain how to apply all those results to the reflected affine stochastic recursion.

Since we want to present a sufficiently comprehensive picture, we have included – mostly without proof – a few known results, in particular on cases where one has strong contractivity.

PART I. Strong and local contractivity and examples, including reflected random walk

2. LOCAL CONTRACTIVITY AND THE WORK OF BENDA

(2.1) Definition. (i) The SDS is called *strongly contractive*, if for every $x \in \mathbf{X}$,

$$\Pr[d(X_n^x, X_n^y) \rightarrow 0 \text{ for all } y \in \mathbf{X}] = 1.$$

(ii) The SDS is called *locally contractive*, if for every $x \in \mathbf{X}$ and every compact $K \subset \mathbf{X}$,

$$\Pr[d(X_n^x, X_n^y) \cdot \mathbf{1}_K(X_n^y) \rightarrow 0 \text{ for all } y \in \mathbf{X}] = 1.$$

Let $\mathbf{B}(r)$ and $\bar{\mathbf{B}}(r)$, $r \in \mathbb{N}$, be the open and closed balls in \mathbf{X} with radius r and fixed center $o \in \mathbf{X}$, respectively. $\bar{\mathbf{B}}(r)$ is compact by properness of \mathbf{X} .

Using Kolmogorov's 0-1 law, one gets the following alternative.

(2.2) Lemma. *For a locally contractive SDS,*

$$\begin{aligned} & \text{either } \Pr[d(X_n^x, x) \rightarrow \infty] = 0 \text{ for all } x \in \mathbf{X}, \\ & \text{or } \Pr[d(X_n^x, x) \rightarrow \infty] = 1 \text{ for all } x \in \mathbf{X}. \end{aligned}$$

Proof. Consider

$$(2.3) \quad X_{m,m}^x = x \text{ and } X_{m,n}^x = F_n \circ F_{n-1} \circ \dots \circ F_{m+1}(x) \text{ for } n > m,$$

so that $X_n^x = X_{0,n}^x$. Then local contractivity implies that for each $x \in \mathbf{X}$, we have $\Pr(\Omega_0) = 1$ for the event Ω_0 consisting of all $\omega \in \Omega$ with

$$(2.4) \quad \lim_{n \rightarrow \infty} \mathbf{1}_{\mathbf{B}(r)}(X_{m,n}^x(\omega)) \cdot d(X_{m,n}^x(\omega), X_{m,n}^y(\omega)) = 0 \text{ for each } r \in \mathbb{N}, m \in \mathbb{N}_0, y \in \mathbf{X}.$$

Clearly, Ω_0 is invariant with respect to the shift of the sequence (F_n) .

Let $\omega \in \Omega_0$ be such that the sequence $(X_n^x(\omega))_{n \geq 0}$ accumulates at some $z \in \mathbf{X}$. Fix m and set $v = X_m^x(\omega)$. Then also $(X_{m,n}^v(\omega))_{n \geq m}$ accumulates at z . Now let $y \in \mathbf{X}$ be arbitrary. Then there is r such that $v, y, z \in \mathbf{B}(r)$. Therefore also $(X_{m,n}^y(\omega))_{n \geq m}$ accumulates at z . In particular, the fact that $(X_n^x(\omega))_{n \geq 0}$ accumulates at some point does not depend on the initial trajectory, i.e., on the specific realization of F_1, \dots, F_m . We infer that the set

$$\{\omega \in \Omega_0 : (X_n^x(\omega))_{n \geq 0} \text{ accumulates in } \mathbf{X}\}$$

is a tail event of $(F_n)_{n \geq 1}$. On its complement in Ω_0 , we have $d(X_n^x, x) \rightarrow \infty$. \square

If $d(X_n^x, x) \rightarrow \infty$ almost surely, then we call the SDS *transient*.

For $\omega \in \Omega$, let $\mathbf{L}^x(\omega)$ be the set of accumulation points of $(X_n^x(\omega))$ in \mathbf{X} . The following proof is much simpler than the one in [5].

(2.5) Lemma. *For any conservative, locally contractive SDS, there is a set $\mathbf{L} \subset \mathbf{X}$ – the attractor or limit set – such that*

$$\Pr[\mathbf{L}^x(\cdot) = \mathbf{L} \text{ for all } x \in \mathbf{X}] = 1,$$

Proof. The argument of the proof of Lemma 2.2 also shows the following. For every open $U \subset X$,

$$\Pr[X_n^x \text{ accumulates in } U \text{ for all } x \in X] \in \{0, 1\}.$$

X being proper, we can find a countable basis $\{U_k : k \in \mathbb{N}\}$ of the topology of X , where each U_k is an open ball. Let $\mathbb{K} \subset \mathbb{N}$ be the (deterministic) set of all k such that the above probability is 1 for $U = U_k$. Then there is $\Omega_0 \subset \Omega$ such that $\Pr(\Omega_0) = 1$, and for every $\omega \in \Omega_0$, the sequence $(X_n^x(\omega))_{n \geq 0}$ accumulates in U_k for some and equivalently all x precisely when $k \in \mathbb{K}$. Now, if $\omega \in \Omega_0$, then $y \in L^x(\omega)$ if and only if when $k \in \mathbb{K}$ for every k with $U_k \ni y$. We see that $L^x(\omega)$ is the same set for every $\omega \in \Omega_0$. \square

Thus, (X_n^x) is (*topologically*) *recurrent* on L when $\Pr[d(X_n^x, x) \rightarrow \infty] = 0$, that is, every open set that intersects L is visited infinitely often with probability 1.

For a Radon measure ν on X , its transform under P is written as νP , that is, for any Borel set $U \subset X$,

$$\nu P(U) = \int_X P(x, U) d\nu(x).$$

Recall that ν is called *excessive*, when $\nu P \leq \nu$, and *invariant*, when $\nu P = \nu$.

For two transition kernels P, Q , their product is defined as

$$PQ(x, U) = \int_X Q(y, U) P(x, dy).$$

In particular, P^k is the k -fold iterate. The first part of the following is well-known; we outline the proof because it is needed in the second part, regarding $\text{supp}(\nu)$.

(2.6) Lemma. *If the locally contractive SDS is recurrent, then every excessive measure ν is invariant. Furthermore, $\text{supp}(\nu) = L$.*

Proof. For any pair of Borel sets $U, V \subset X$, define the transition kernel $P_{U,V}$ and the measure ν_U by

$$P_{U,V}(x, B) = \mathbf{1}_U(x) P(x, B \cap V) \quad \text{and} \quad \nu_U(B) = \nu(U \cap B),$$

where $B \subset X$ is a Borel set. We abbreviate $P_{U,U} = P_U$. Also, consider the stopping time $\tau_x^U = \inf\{n \geq 1 : X_n^x \in U\}$, and for $x \in U$ let

$$P^U(x, B) = \Pr[\tau_x^U < \infty, X_{\tau_x^U}^x \in B]$$

be the probability that the first return of X_n^x to the set U occurs in a point of $B \subset X$. Then we have

$$\nu_U \geq \nu_U P_U + \nu_{U^c} P_{U^c, U},$$

and by a typical inductive (“balayage”) argument,

$$\nu_U \geq \nu_U \left(P_U + \sum_{k=0}^{n-1} P_{U, U^c} P_{U^c}^k P_{U^c, U} \right) + \nu_{U^c} P_{U^c}^n P_{U^c, U}.$$

In the limit,

$$\nu_U \geq \nu_U \left(P_U + \sum_{k=0}^{\infty} P_{U, U^c} P_{U^c}^k P_{U^c, U} \right) = \nu_U P^U.$$

Now suppose that U is open and relatively compact, and $U \cap \mathbf{L} \neq \emptyset$. Then, by recurrence, for any $x \in U$, we have $\tau_x^U < \infty$ almost surely. This means that P^U is stochastic, that is, $P^U(x, U) = 1$. But then $\nu_U P^U(U) = \nu_U(U) = \nu(U) < \infty$. Therefore $\nu_U = \nu_U P^U$. We now can set $U = \mathbf{B}(r)$ and let $r \rightarrow \infty$. Then monotone convergence implies $\nu = \nu P$, and P is invariant.

Let us next show that $\text{supp}(\nu) \subset \mathbf{L}$.

Take an open, relatively compact set V such that $V \cap \mathbf{L} = \emptyset$.

Now choose r large enough such that $U = \mathbf{B}(r)$ contains V and intersects \mathbf{L} . Let $Q = P^U$. We know from the above that $\nu_U = \nu_U Q = \nu_U Q^n$. We get

$$\nu(V) = \nu_U(V) = \int_U Q^n(x, V) d\nu_U(x).$$

Now $Q^n(x, V)$ is the probability that the SDS starting at x visits V at the instant when it returns to U for the n -th time. As

$$\Pr[X_n^x \in V \text{ for infinitely many } n] = 0,$$

it is an easy exercise to show that $Q^n(x, V) \rightarrow 0$. Since the measure ν_U has finite total mass, we can use dominated convergence to see that $\int_U Q^n(x, V) d\nu_U(x) \rightarrow 0$ as $n \rightarrow \infty$.

We conclude that $\nu(V) = 0$, and $\text{supp}(\nu) \subset \mathbf{L}$.

Since $\nu P = \nu$, we have $f(\text{supp}(\nu)) \subset \text{supp}(\nu)$ for every $f \in \text{supp}(\tilde{\mu})$, where (recall) $\tilde{\mu}$ is the distribution of the random functions F_n in \mathfrak{G} . But then almost surely $X_n^x \in \text{supp}(\nu)$ for all $x \in \text{supp}(\nu)$ and all n , that is, $\mathbf{L}^x(\omega) \subset \text{supp}(\nu)$ for \Pr -almost every ω . Lemma 2.5 yields that $\mathbf{L} \subset \text{supp}(\nu)$. \square

The following holds in more generality than just for recurrent locally contractive SDS.

(2.7) Proposition. *If the locally contractive SDS is recurrent, then it possesses an invariant measure ν .*

Proof. Fix $\psi \in \mathcal{C}_c^+(\mathbf{X})$ such that its support intersects \mathbf{L} . Recurrence implies that

$$\sum_{k=1}^{\infty} P^k \psi(x) = \infty \quad \text{for every } x \in \mathbf{X}.$$

The statement now follows from a result of LIN [30, Thm. 5.1]. \square

Thus we have an invariant Radon measure ν with $\nu P = \nu$ and $\text{supp}(\nu) = \mathbf{L}$. It is now easy to see that the attractor depends only on $\text{supp}(\tilde{\mu}) \subset \mathfrak{G}$.

(2.8) Corollary. *In the recurrent case, \mathbf{L} is the smallest non-empty closed subset of \mathbf{X} with the property that $f(\mathbf{L}) \subset \mathbf{L}$ for every $f \in \text{supp}(\tilde{\mu})$.*

Proof. The reasoning at the end of the proof of Lemma 2.6 shows that \mathbf{L} is indeed a closed set with that property. On the other hand, if $C \subset \mathbf{X}$ is closed, non-empty and such that $f(C) \subset C$ for all $f \in \text{supp}(\tilde{\mu})$ then $(X_n^x(\omega))$ evolves almost surely within C when the starting point x is in C . But then $\mathbf{L}^x(\omega) \subset C$ almost surely, and on the other hand $\mathbf{L}^x(\omega) = \mathbf{L}$ almost surely. \square

(2.9) Remark. Suppose that the SDS induced by the probability measure $\tilde{\mu}$ on \mathfrak{G} is not necessarily locally contractive, resp. recurrent, but that there is another probability measure $\tilde{\mu}'$ on \mathfrak{G} which does induce a weakly contractive, recurrent SDS and which satisfies $\text{supp}(\tilde{\mu}) = \text{supp}(\tilde{\mu}')$. Let L be the limit set of this second SDS. Since it depends only on $\text{supp}(\tilde{\mu}')$, the results that we have so far yield that also for the SDS (X_n^x) associated with $\tilde{\mu}$, L is the unique “essential class” in the following sense: it is the unique minimal non-empty closed subset of X such that

- (i) for every open set $U \subset X$ that intersects L and every starting point $x \in X$, the sequence (X_n^x) visits U with positive probability, and
- (ii) if $x \in L$ then $X_n^x \in L$ for all n . □

For $\ell \geq 2$, we can lift each $f \in \mathfrak{G}$ to a continuous mapping

$$f^{(\ell)} : X^\ell \rightarrow X^\ell, \quad f^{(\ell)}(x_1, \dots, x_\ell) = (x_2, \dots, x_\ell, f(x_\ell)).$$

In this way, the random mappings F_n induce the SDS $(F_n^{(\ell)} \circ \dots \circ F_1^{(\ell)}(x_1, \dots, x_\ell))_{n \geq 0}$ on X^ℓ . For $n \geq \ell - 1$ this is just $(X_{n-\ell+1}^{x_\ell}, \dots, X_n^{x_\ell})$.

(2.10) Lemma. Let $x \in X$, and let $U_0, \dots, U_{\ell-1} \subset X$ be Borel sets such that

$$\begin{aligned} \Pr[X_n^x \in U_0 \text{ for infinitely many } n] &= 1 \quad \text{and} \\ \Pr[X_1^y \in U_j] &\geq \alpha > 0 \quad \text{for every } y \in U_{j-1}, j = 1, \dots, \ell - 1. \end{aligned}$$

Then also

$$\Pr[X_n^x \in U_0, X_{n+1}^x \in U_1, \dots, X_{n+\ell-1}^x \in U_{\ell-1} \text{ for infinitely many } n] = 1.$$

Proof. This is quite standard and true for general Markov chains and not just SDS. Let $\tau(n)$, $n \geq 1$, be the stopping times of the successive visits of (X_n^x) in U . They are all a.s. finite by assumption. We consider the events

$$\Lambda_n = [X_{\tau(\ell n)+1}^x \in U_1, \dots, X_{\tau(\ell n)+\ell-1}^x \in U_{\ell-1}] \quad \text{and} \quad \Lambda_{k,m} = \bigcup_{n=k+1}^{m-1} \Lambda_n,$$

where $k < m$. We need to show that $\Pr(\limsup_n \Lambda_n) = 1$. By the strong Markov property, we have

$$\Pr(\Lambda_n \mid X_{\tau(\ell n)}^x = y) \geq \alpha^\ell \quad \text{for every } y \in U_0.$$

Let $k, m \in \mathbb{N}$ with $k < m$. Just for the purpose of the next lines of the proof, consider the measure on X defined by

$$\sigma(B) = \Pr([X_{\tau(\ell m)}^x \in B] \cap \Lambda_{k,m-1}^c).$$

It is concentrated on U , and using the Markov property,

$$\begin{aligned} \Pr(\Lambda_{k,m}^c) &= \int_U \Pr(\Lambda_m^c \mid X_{\tau(\ell m)}^x = y) d\sigma(y) \\ &\leq (1 - \alpha^\ell) \sigma(U) = (1 - \alpha^\ell) \Pr(\Lambda_{k,m-1}^c) \leq \dots \leq (1 - \alpha^\ell)^{m-k}. \end{aligned}$$

Letting $m \rightarrow \infty$, we see that $\Pr(\bigcap_{n>k} \Lambda_n^c) = 0$ for every k , so that

$$\Pr(\bigcap_k \bigcup_{n>k} \Lambda_n) = 1,$$

as required. □

(2.11) Proposition. *If the SDS is locally contractive and recurrent on \mathbf{X} , then so is the lifted process on \mathbf{X}^ℓ . The limit set of the latter is*

$$\mathbf{L}^{(\ell)} = \left\{ (x, f_1(x), f_2 \circ f_1(x), \dots, f_{\ell-1} \circ \dots \circ f_1(x)) : x \in \mathbf{L}, f_i \in \text{supp}(\tilde{\mu}) \right\}^-,$$

and if the Radon measure ν is invariant for the original SDS on \mathbf{X} , then the measure $\nu^{(\ell)}$ is invariant for the lifted SDS on \mathbf{X}^ℓ , where

$$\int_{\mathbf{X}^\ell} f d\nu^{(\ell)} = \int_{\mathbf{X}} \dots \int_{\mathbf{X}} f(x_1, \dots, x_\ell) P(x_{\ell-1}, dx_\ell) P(x_{\ell-2}, dx_{\ell-1}) \dots P(x_1, dx_2) d\nu(x_1).$$

Proof. It is a straightforward exercise to verify that the lifted SDS is locally contractive and has $\nu^{(\ell)}$ as an invariant measure. We have to prove that it is recurrent. For this purpose, we just have to show that there is some relatively compact subset of \mathbf{X}^ℓ that is visited infinitely often with positive probability. We can find relatively compact open subsets $U_0, \dots, U_{\ell-1}$ of \mathbf{X} that intersect \mathbf{L} such that

$$\Pr[F_1(U_{j-1}) \subset U_j] \geq \alpha > 0 \quad \text{for } j = 1, \dots, \ell - 1.$$

We know that for arbitrary starting point $x \in \mathbf{X}$, with probability 1, the SDS (X_n^x) visits U_0 infinitely often. Lemma 2.10 implies that the lifted SDS on \mathbf{X}^ℓ visits $U_0 \times \dots \times U_{\ell-1}$ infinitely often with probability 1.

By Lemma 2.2, the lifted SDS on \mathbf{X}^ℓ is recurrent. Now that we know this, it is clear from Corollary 2.8 that its attractor is the set \mathbf{L}^ℓ , as stated. \square

As outlined in the introduction, we can equip the trajectory space $X^{\mathbb{N}_0}$ of our SDS with the infinite product σ -algebra and the measure \Pr_ν , which is in general σ -finite.

(2.12) Lemma. *If the SDS is locally contractive and recurrent, then T is conservative on $(X^{\mathbb{N}_0}, \mathfrak{B}(X^{\mathbb{N}_0}), \Pr_\nu)$.*

Proof. Let $\varphi = \mathbf{1}_U$, where $U \subset \mathbf{X}$ is open, relatively compact, and intersects \mathbf{L} . We can extend it to a strictly positive function in $L^1(X^{\mathbb{N}_0}, \Pr_\nu)$ by setting $\varphi(\mathbf{x}) = \varphi(x_0)$ for $\mathbf{x} = (x_n)_{n \geq 0}$. We know from recurrence that

$$\sum_n \varphi(X_n^x) = \infty \quad \text{Pr-almost surely, for every } x \in \mathbf{X}.$$

This translates into

$$\sum_n \varphi(T^n \mathbf{x}) = \infty \quad \text{Pr}_\nu\text{-almost surely, for every } \mathbf{x} \in X^{\mathbb{N}_0}.$$

Conservativity follows; see e.g. [34, Thm. 5.3]. \square

The uniqueness part of the following theorem is contained in [4] and [5]; see also BROFFERIO [10, Thm. 3], who considers SDS of affine mappings. We modify and extend the proof in order to be able to conclude that our SDS is ergodic with respect to T . (This, as well as Proposition 2.11, is new with respect to Benda's work.)

(2.13) Theorem. *For a recurrent locally contractive SDS, let ν be the measure of Proposition 2.7. Then the shift T on $X^{\mathbb{N}_0}$ is ergodic with respect to \Pr_ν .*

In particular, ν is the unique invariant Radon measure for the SDS up to multiplication with constants.

Proof. Let \mathfrak{I} be the σ -algebra of the T -invariant sets in $\mathfrak{B}(\mathbf{X}^{\mathbb{N}_0})$. For $\varphi \in L^1(\mathbf{X}^{\mathbb{N}_0}, \mathbf{Pr}_\nu)$, we write $\mathbf{E}_\nu(\varphi) = \int \varphi d\mathbf{Pr}_\nu$ and $\mathbf{E}_\nu(\varphi | \mathfrak{I})$ for the conditional “expectation” of φ with respect to \mathfrak{I} . The quotation marks refer to the fact that it does not have the meaning of an expectation when ν is not a probability measure. As a matter of fact, what is well defined in the latter case are quotients $\mathbf{E}_\nu(\varphi | \mathfrak{I}) / \mathbf{E}_\nu(\psi | \mathfrak{I})$ for suitable $\psi \geq 0$; compare with the explanations in REVUZ [34, pp. 133–134].

In view of Lemma 2.12, we can apply the ergodic theorem of CHACON AND ORNSTEIN [13], see also [34, Thm.3.3]. Choosing an arbitrary function $\psi \in L^1(\mathbf{X}^{\mathbb{N}_0}, \mathbf{Pr}_\nu)$ with

$$(2.14) \quad \mathbf{Pr}_\nu \left(\left\{ \mathbf{x} \in \mathbf{X}^{\mathbb{N}_0} : \sum_{n=0}^{\infty} \psi(T^n \mathbf{x}) < \infty \right\} \right) = 0,$$

one has for every $\varphi \in L^1(\mathbf{X}^{\mathbb{N}_0}, \mathbf{Pr}_\nu)$

$$(2.15) \quad \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \varphi(T^k \mathbf{x})}{\sum_{k=0}^n \psi(T^k \mathbf{x})} = \frac{\mathbf{E}_\nu(\varphi | \mathfrak{I})}{\mathbf{E}_\nu(\psi | \mathfrak{I})} \quad \text{for } \mathbf{Pr}_\nu\text{-almost every } \mathbf{x} \in \mathbf{X}^{\mathbb{N}_0}.$$

In order to show ergodicity of T , we need to show that the right hand side is just

$$\frac{\mathbf{E}_\nu(\varphi)}{\mathbf{E}_\nu(\psi)}.$$

It is sufficient to show this for non-negative functions that depend only on finitely many coordinates. For a function φ on $\mathbf{X}^{\mathbb{N}_0}$, we also write φ for its extension to $\mathbf{X}^{\mathbb{N}_0}$, given by $\varphi(\mathbf{x}) = \varphi(x_0, \dots, x_{\ell-1})$.

That is, we need to show that for every $\ell \geq 1$ and non-negative Borel functions φ, ψ on \mathbf{X}^ℓ , with ψ satisfying (2.14),

$$(2.16) \quad \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \varphi(X_k^x(\omega), \dots, X_{k+\ell-1}^x(\omega))}{\sum_{k=0}^n \psi(X_k^x(\omega), \dots, X_{k+\ell-1}^x(\omega))} = \frac{\int_{\mathbf{L}} \mathbf{E}(\varphi(X_0^y, \dots, X_{\ell-1}^y)) d\nu(y)}{\int_{\mathbf{L}} \mathbf{E}(\psi(X_0^y, \dots, X_{\ell-1}^y)) d\nu(y)}$$

for ν -almost every $x \in \mathbf{X}$ and \mathbf{Pr} -almost every $\omega \in \Omega$,

when the integrals appearing in the right hand term are finite.

At this point, we observe that we need to prove (2.16) only for $\ell = 1$. Indeed, once we have the proof for this case, we can reconsider our SDS on \mathbf{X}^ℓ , and using Proposition 2.11, our proof for $\ell = 1$ applies to the new SDS as well.

So now let $\ell = 1$. By regularity of ν , we may assume that φ and ψ are non-negative, compactly supported, continuous functions on \mathbf{L} that both are non-zero.

We consider the random variables $S_n^x \varphi(\omega) = \sum_{k=0}^n \varphi(X_k^x(\omega))$ and $S_n^x \psi(\omega)$. Since the SDS is recurrent, both functions satisfy (2.14), i.e., we have almost surely that $S_n^x \varphi$ and $S_n^x \psi > 0$ for all but finitely many n and all x . We shall show that

$$(2.17) \quad \lim_{n \rightarrow \infty} \frac{S_n^x \varphi}{S_n^x \psi} = \frac{\int_{\mathbf{L}} \varphi d\nu}{\int_{\mathbf{L}} \psi d\nu} \quad \mathbf{Pr}\text{-almost surely and for every } x \in \mathbf{L},$$

which is more than what we need (namely that it just holds for ν -almost every x). We know from (2.15) that the limit exists in terms of conditional expectations for ν -almost

every x , so that we only have to show that that it is $\Pr \otimes \nu$ -almost everywhere constant.

Step 1. Independence of x . Let $K_0 \subset \mathbf{L}$ be compact such that the support of φ is contained in K_0 . Define $K = \{x \in \mathbf{L} : d(x, K_0) \leq 1\}$. Given $\varepsilon > 0$, let $0 < \delta \leq 1$ be such that $|\varphi(x) - \varphi(y)| < \varepsilon$ whenever $d(x, y) < \delta$.

By (2.15), there is x such that the limits $\lim_n S_n^x \mathbf{1}_K / S_n^x \varphi$ and $Z_{\varphi, \psi} = \lim_n S_n^x \varphi / S_n^x \psi$ exist and are finite \Pr -almost surely.

Local contractivity implies that for this specific x and each $y \in \mathbf{X}$, we have the following. \Pr -almost surely, there is a random $N \in \mathbb{N}$ such that

$$|\varphi(X_k^x) - \varphi(X_k^y)| \leq \varepsilon \cdot \mathbf{1}_K(X_k^x) \quad \text{for all } k \geq N.$$

Therefore, for every $\varepsilon > 0$ and $y \in \mathbf{X}$

$$\limsup_{n \rightarrow \infty} \frac{|S_n^x \varphi - S_n^y \varphi|}{S_n^x \varphi} \leq \varepsilon \cdot \lim_{n \rightarrow \infty} \frac{S_n^x \mathbf{1}_K}{S_n^x \varphi} \quad \Pr\text{-almost surely.}$$

This yields that for every $y \in \mathbf{L}$,

$$\lim_{n \rightarrow \infty} \frac{S_n^x \varphi - S_n^y \varphi}{S_n^x \varphi} = 0, \quad \text{that is,} \quad \lim_{n \rightarrow \infty} \frac{S_n^y \varphi}{S_n^x \varphi} = 1 \quad \Pr\text{-almost surely.}$$

The same applies to ψ in the place of φ . We get that for all y ,

$$\frac{S_n^x \varphi}{S_n^x \psi} - \frac{S_n^y \varphi}{S_n^y \psi} = \frac{S_n^y \varphi}{S_n^y \psi} \left(\frac{S_n^x \varphi}{S_n^y \varphi} \frac{S_n^y \psi}{S_n^x \psi} - 1 \right) \rightarrow 0 \quad \Pr\text{-almost surely.}$$

In other terms, for the positive random variable $Z_{\varphi, \psi}$ given above in terms of our x ,

$$\lim_{n \rightarrow \infty} \frac{S_n^y \varphi}{S_n^y \psi} = Z_{\varphi, \psi} \quad \Pr\text{-almost surely, for every } y \in \mathbf{L}.$$

Step 2. $Z_{\varphi, \psi}$ is a.s. constant. Recall the random variables $X_{m,n}^x$ of (2.3) and set $S_{m,n}^x \varphi(\omega) = \sum_{k=m}^n \varphi(X_{m,k}^x(\omega))$, $n > m$. Then Step 1 also yields that for our given x and each m ,

$$(2.18) \quad \lim_{n \rightarrow \infty} \frac{S_{m,n}^y \varphi}{S_{m,n}^y \psi} = \lim_{n \rightarrow \infty} \frac{S_{m,n}^x \varphi}{S_{m,n}^x \psi} \quad \Pr\text{-almost surely, for every } x \in \mathbf{L}.$$

Let $\Omega_0 \subset \Omega$ be the set on which the convergence in (2.18) holds for all m , and both $S_n^x \varphi$ and $S_n^x \psi \rightarrow \infty$ on Ω_0 . We have $\Pr(\Omega_0) = 1$. For fixed $\omega \in \Omega_0$ and $m \in \mathbb{N}$, let $y = X_m^x(\omega)$. Then (because in the ratio limit we can omit the first m terms of the sums)

$$Z_{\varphi, \psi}(\omega) = \lim_{n \rightarrow \infty} \frac{S_n^x \varphi(\omega)}{S_n^x \psi(\omega)} = \lim_{n \rightarrow \infty} \frac{S_{m,n}^y \varphi(\omega)}{S_{m,n}^y \psi(\omega)} = \lim_{n \rightarrow \infty} \frac{S_{m,n}^x \varphi(\omega)}{S_{m,n}^x \psi(\omega)}.$$

Thus, $Z_{\varphi, \psi}$ is independent of F_1, \dots, F_m , whence it is constant by Kolmogorov's 0-1 law. This completes the proof of ergodicity. It is immediate from (2.17) that ν is unique up to multiplication by constants. \square

(2.19) Corollary. *Let the locally contractive SDS (X_n^x) be recurrent with invariant Radon measure ν . For relatively compact, open $U \subset \mathbf{X}$ which intersects \mathbf{L} , consider the probability measure \mathbf{m}_U on \mathbf{X} defined by $\mathbf{m}_U(B) = \nu(B \cap U) / \nu(U)$. Consider the SDS with initial distribution \mathbf{m}_U , and let τ^U be its return time to U .*

(a) If $\nu(\mathbf{L}) < \infty$ then the SDS is positive recurrent, that is,

$$\mathbf{E}(\tau^U) = \nu(\mathbf{L})/\nu(U) < \infty.$$

(b) If $\nu(\mathbf{L}) = \infty$ then the SDS is null recurrent, that is,

$$\mathbf{E}(\tau^U) = \infty.$$

This follows from the well known formula of Kac, see e.g. AARONSON [1, 1.5.5., page 44].

(2.20) Lemma. *In the positive recurrent case, let the invariant measure be normalised such that $\nu(\mathbf{L}) = 1$. Then, for every starting point $x \in X$, the sequence (X_n^x) converges in law to ν .*

Proof. Let $\varphi : X \rightarrow \mathbb{R}$ be continuous and compactly supported. Since φ is uniformly continuous, local contractivity yields for all $x, y \in X$ that $\varphi(X_n^x) - \varphi(X_n^y) \rightarrow 0$ almost surely. By dominated convergence, $\mathbf{E}(\varphi(X_n^x) - \varphi(X_n^y)) \rightarrow 0$. Thus,

$$P^n \varphi(x) - \int \varphi d\nu = \int (P^n \varphi(x) - P^n \varphi(y)) d\nu(y) = \int \mathbf{E}(\varphi(X_n^x) - \varphi(X_n^y)) d\nu(y) \rightarrow 0$$

□

3. BASIC EXAMPLE: THE AFFINE STOCHASTIC RECURSION

Here we briefly review the main known results regarding the SDS on $X = \mathbb{R}$ given by

$$(3.1) \quad Y_0^x = x, \quad Y_{n+1}^x = A_n Y_n^x + B_{n+1},$$

where $(A_n, B_n)_{n \geq 0}$ is a sequence of i.i.d. random variables in $\mathbb{R}_*^+ \times \mathbb{R}$. The following results are known.

(3.2) Proposition. *If $\mathbf{E}(\log^+ A_n) < \infty$ and*

$$-\infty \leq \mathbf{E}(\log A_n) < 0$$

then (Y_n^x) is strongly contractive on \mathbb{R} .

If in addition $\mathbf{E}(\log^+ |B_n|) < \infty$ then the affine SDS has a unique invariant probability measure ν , and is (positive) recurrent on $\mathbf{L} = \text{supp}(\nu)$. Furthermore, the shift on the trajectory space is ergodic with respect to the probability measure Pr_ν .

Proof (outline). This is the classical application of Furstenberg's contraction principle. One verifies that for the associated right process,

$$R_n^x \rightarrow Z = \sum_{n=1}^{\infty} A_1 \cdots A_{n-1} B_n$$

almost surely for every $x \in \mathbb{R}$. The series that defines Z is almost surely absolutely convergent by the assumptions on the two expectations. Recurrence is easily deduced via Lemma 2.2. Indeed, we cannot have $|Y_n^x| \rightarrow \infty$ almost surely, because then by dominated convergence $\nu(U) = \nu P^n(U) \rightarrow 0$ for every relatively compact set U . Ergodicity now follows from strong contractivity. □

(3.3) Proposition. *Suppose that $\Pr[A_n = 1] < 1$ and $\Pr[A_n x + B_n = x] < 1$ for all $x \in \mathbb{R}$ (non-degeneracy). If $\mathbb{E}(|\log A_n|) < \infty$ and $\mathbb{E}(\log^+ B_n) < \infty$, and if*

$$\mathbb{E}(\log A_n) = 0$$

then (Y_n^x) is locally contractive on \mathbb{R} .

If in addition $\mathbb{E}(|\log A_n|^2) < \infty$ and $\mathbb{E}((\log^+ |B_n|)^{2+\varepsilon}) < \infty$ for some $\varepsilon > 0$ then the affine SDS has a unique invariant Radon measure ν with infinite mass, and it is (null) recurrent on $L = \text{supp}(\nu)$.

This goes back to [3], with a small gap that was later filled in [5]. With the moment conditions as stated here, a nice and complete “geometric” proof is given in [10]: it is shown that under the stated hypotheses,

$$A_1 \cdots A_n \cdot \mathbf{1}_K(Y_n) \rightarrow 0 \quad \text{almost surely}$$

for very compact set K . Recurrence was shown earlier in [17, Lemma 5.49].

(3.4) Proposition. *If $\mathbb{E}(|\log A_n|) < \infty$ and $\mathbb{E}(\log^+ B_n) < \infty$, and if*

$$\mathbb{E}(\log A_n) > 0$$

then (Y_n^x) is transient, that is, $|Y_n^x| \rightarrow \infty$ almost surely for every starting point $x \in \mathbb{R}$.

A proof is given, e.g., by ELIE [18].

4. ITERATION OF RANDOM CONTRACTIONS

Let us now consider a more specific class of SDS: within \mathfrak{G} , we consider the closed submonoid \mathfrak{L}_1 of all *contractions* of \mathbf{X} , i.e., mappings $f : \mathbf{X} \rightarrow \mathbf{X}$ with Lipschitz constant $\mathfrak{l}(f) \leq 1$. We suppose that the probability measure $\tilde{\mu}$ that governs the SDS is supported by \mathfrak{L}_1 , that is, each random function F_n of (1.2) satisfies $\mathfrak{l}(F_n) \leq 1$. In this case, one does not need local contractivity in order to obtain Lemma 2.2; this follows directly from properness of \mathbf{X} and the inequality

$$D_n(x, y) \leq d(x, y), \quad \text{where} \quad D_n(x, y) = d(X_n^x, X_n^y).$$

When $\Pr[d(X_n^x, x) \rightarrow \infty] = 0$ for every x , we can in general only speak of conservativity, since we do not yet have an attractor on which the SDS is topologically recurrent. Let $\mathfrak{S}(\tilde{\mu})$ be the closed sub-semigroup of \mathfrak{L}_1 generated by $\text{supp}(\tilde{\mu})$.

(4.1) Remark. For strong contractivity it is sufficient that $\Pr[D_n(x, y) \rightarrow 0] = 1$ pointwise for all $x, y \in \mathbf{X}$.

Indeed, by properness, \mathbf{X} has a dense, countable subset Y . If $K \subset \mathbf{X}$ is compact and $\varepsilon > 0$ then there is a finite $W \subset Y$ such that $d(y, W) < \varepsilon$ for every $y \in K$. Therefore

$$\sup_{y \in K} D_n(x, y) \leq \underbrace{\max_{w \in W} D_n(x, w)}_{\rightarrow 0 \text{ a.s.}} + \varepsilon,$$

since $D_n(x, y) \leq D_n(x, w) + D_n(w, y) \leq D_n(x, w) + d(w, y)$.

The following key result of [4] (whose statement and proof we have slightly strengthened here) is inspired by [27, Thm. 2.2], where reflected random walk is studied; see also [28].

(4.2) Theorem. *If the SDS of contractions is conservative, then it is strongly contractive if and only if $\mathfrak{S}(\tilde{\mu}) \subset \mathfrak{L}_1$ contains a constant function.*

Proof. Keeping Remark 4.1 in mind, first assume that $D_n(x, y) \rightarrow 0$ almost surely for all x, y . We can apply all previous results on (local) contractivity, and the SDS has the non-empty attractor \mathbf{L} . If $x_0 \in \mathbf{L}$, then with probability 1 there is a random subsequence (n_k) such that $X_{n_k}^x \rightarrow x_0$ for every $x \in \mathbf{X}$, and by the above, this convergence is uniform on compact sets. Thus, the constant mapping $x \mapsto x_0$ is in $\mathfrak{S}(\tilde{\mu})$.

Conversely, assume that $\mathfrak{S}(\tilde{\mu})$ contains a constant function. Since $D_{n+1}(x, y) \leq D_n(x, y)$, the limit $D_\infty(x, y) = \lim_n D_n(x, y)$ exists and is between 0 and $d(x, y)$. We set $w(x, y) = \mathbb{E}(D_\infty(x, y))$. First of all, we claim that

$$(4.3) \quad \lim_{m \rightarrow \infty} w(X_m^x, X_m^y) = D_\infty(x, y) \quad \text{almost surely.}$$

To see this, consider $X_{m,n}^x$ as in (2.3). Then $D_{m,\infty}(x, y) = \lim_n d(X_{m,n}^x, X_{m,n}^y)$ has the same distribution as $D_\infty(x, y)$, whence $\mathbb{E}(D_{m,\infty}(x, y)) = w(x, y)$. Therefore, we also have

$$\mathbb{E}(D_{m,\infty}(X_m^x, X_m^y) \mid F_1, \dots, F_m) = w(X_m^x, X_m^y).$$

On the other hand, $D_{m,\infty}(X_m^x, X_m^y) = D_\infty(x, y)$, and the bounded martingale

$$\left(\mathbb{E}(D_\infty(x, y) \mid F_1, \dots, F_m) \right)_{m \geq 1}$$

converges almost surely to $D_\infty(x, y)$. Statement (4.3) follows.

Now let $\varepsilon > 0$ be arbitrary, and fix $x, y \in X$. We have to show that the event $\Lambda = [D_\infty(x, y) \geq \varepsilon]$ has probability 0.

(i) By conservativity,

$$\Pr \left(\bigcup_{r \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} [X_n^x, X_n^y \in \mathbf{B}(r)] \right) = 1.$$

On A , we have $D_n(x, y) \geq \varepsilon$ for all n . Therefore we need to show that $\Pr(\Lambda_r) = 0$ for each $r \in \mathbb{N}$, where

$$\Lambda_r = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} [X_n^x, X_n^y \in \mathbf{B}(r), D_n(x, y) \geq \varepsilon].$$

(ii) By assumption, there is $x_0 \in X$ which can be approximated uniformly on compact sets by functions of the form $f_k \circ \dots \circ f_1$, where $f_j \in \text{supp}(\tilde{\mu})$. Therefore, given r there is $k \in \mathbb{N}$ such that

$$\Pr(\Gamma_{k,r}) > 0, \quad \text{where} \quad \Gamma_{k,r} = \left[\sup_{u \in \mathbf{B}(r)} d(X_k^u, x_0) \leq \varepsilon/4 \right].$$

On $\Gamma_{k,r}$ we have $D_\infty(u, v) \leq D_k(u, v) \leq \varepsilon/2$ for all $u, v \in \mathbf{B}(r)$. Therefore, setting $\delta = \Pr(\Gamma_{k,r}) \cdot (\varepsilon/2)$, we have for all $u, v \in \mathbf{B}(r)$ with $d(u, v) \geq \varepsilon$ that

$$\begin{aligned} w(u, v) &= \mathbb{E}(\mathbf{1}_{\Gamma_{k,r}} D_\infty(u, v)) + \mathbb{E}(\mathbf{1}_{\mathbf{X} \setminus \Gamma_{k,r}} D_\infty(u, v)) \\ &\leq \Pr(\Gamma_{k,r}) \cdot (\varepsilon/2) + (1 - \Pr(\Gamma_{k,r})) \cdot d(u, v) \leq d(u, v) - \delta. \end{aligned}$$

We conclude that on Λ_r , there is a (random) sequence (n_ℓ) such that

$$w(X_{n_\ell}^x, X_{n_\ell}^y) \leq D_{n_\ell}(x, y) - \delta.$$

Passing to the limit on both sides, we see that (4.3) is violated on Λ_r , since $\delta > 0$. Therefore $\Pr(\Lambda_r) = 0$ for each r . \square

(4.4) Corollary. *If the semigroup $\mathfrak{S}(\tilde{\mu}) \subset \mathfrak{L}_1$ contains a constant function, then the SDS is locally contractive.*

Proof. In the transient case, X_n^x can visit any compact K only finitely often, whence $d(X_n^x, X_n^y) \cdot \mathbf{1}_K(X_n^x) = 0$ for all but finitely many n . In the conservative case, we even have strong contractivity by Proposition 4.2. \square

5. SOME REMARKS ON REFLECTED RANDOM WALK

As outlined in the introduction, the reflected random walk on \mathbb{R}^+ induced by a sequence $(B_n)_{n \geq 0}$ of i.i.d. real valued random variables is given by

$$(5.1) \quad X_0^x = x \geq 0, \quad X_{n+1}^x = |X_n^x - B_{n+1}|.$$

Let μ be the distribution of the B_n , a probability measure on \mathbb{R} . The transition probabilities of reflected random walk are

$$P(x, U) = \mu(\{y : |x - y| \in U\}),$$

where $U \subset \mathbb{R}^+$ is a Borel set. When $B_n \leq 0$ almost surely, then (X_n^x) is an ordinary random walk (resulting from a sum of i.i.d. random variables). We shall exclude this, and we shall always assume to be in the *non-lattice* situation. That is,

$$(5.2) \quad \text{supp}(\mu) \cap (0, \infty) \neq \emptyset, \quad \text{and there is no } \kappa > 0 \text{ such that } \text{supp}(\mu) \subset \kappa \cdot \mathbb{Z}.$$

For the lattice case, see [32].

For $b \in \mathbb{R}$, consider $g_b \in \mathfrak{L}_1(\mathbb{R}^+)$ given by $g_b(x) = |x - b|$. Then our reflected random walk is the SDS on \mathbb{R}^+ induced by the random continuous contractions $F_n = g_{B_n}$, $n \geq 1$. The law $\tilde{\mu}$ of the F_n is the image of μ under the mapping $b \mapsto g_b$.

In [28, Prop. 3.2], it is shown that $\mathfrak{S}(\tilde{\mu})$ contains the constant function $x \mapsto 0$. Note that this statement and its proof in [28] are completely deterministic, regarding topological properties of the set $\text{supp}(\mu)$. In view of Theorem 4.2 and Corollary 4.4, we get the following.

(5.3) Proposition. *Under the assumptions (5.2), reflected random walk on \mathbb{R}^+ is locally contractive, and strongly contractive if it is recurrent.*

A. Non-negative B_n .

We first consider the case when $\Pr[B_n \geq 0] = 1$. Let

$$N = \sup \text{supp}(\mu) \quad \text{and} \quad \mathbf{L} = \begin{cases} [0, N], & \text{if } N < \infty, \\ \mathbb{R}^+, & \text{if } N = \infty. \end{cases}$$

The distribution function of μ is

$$F_\mu(x) = \Pr[B_n \leq x] = \mu([0, x]), \quad x \geq 0.$$

We next subsume basic properties that are due to [19], [27] and [28]; they do not depend on recurrence.

(5.4) Lemma. *Suppose that (5.2) is verified and that $\text{supp}(\mu) \subset \mathbb{R}^+$. Then the following holds.*

(a) *The reflected random walk with any starting point is absorbed after finitely many steps by the interval \mathbb{L} .*

(b) *It is topologically irreducible on \mathbb{L} , that is, for every $x \in \mathbb{L}$ and open set $U \subset \mathbb{L}$, there is n such that $P^n(x, U) = \Pr[X_n^x \in U] > 0$.*

(c) *The measure ν on \mathbb{L} given by*

$$\nu(dx) = (1 - F_\mu(x)) dx,$$

where dx is Lebesgue measure, is an invariant measure for the transition kernel P .

At this point Lemma 2.6 implies that in the recurrent case, the above set is indeed the attractor, and ν is the unique invariant measure up to multiplication with constants. We now want to understand when we have recurrence.

(5.5) Theorem. *Suppose that (5.2) is verified and that $\text{supp}(\mu) \subset \mathbb{R}^+$. Then each of the following conditions implies the next one and is sufficient for recurrence of the reflected random walk on \mathbb{L} .*

(i)
$$\mathbb{E}(B_1) < \infty$$

(ii)
$$\mathbb{E}(\sqrt{B_1}) < \infty$$

(iii)
$$\int_{\mathbb{R}^+} (1 - F_\mu(x))^2 dx < \infty$$

(iv)
$$\lim_{y \rightarrow \infty} (1 - F_\mu(y)) \int_0^y (F_\mu(y) - F_\mu(x)) dx = 0$$

In particular, one has positive recurrence precisely when $\mathbb{E}(B_1) < \infty$.

The proof of (i) \implies (ii) \implies (iii) \implies (iv) is a basic exercise. For condition (i), see [27]. The implication (ii) \implies recurrence is due to [36], while the recurrence condition (iii) was proved by ourselves in [32]. However, we had not been aware of [36], as well as of [33], where it is proved that already (iv) implies recurrence on \mathbb{L} . Since ν has finite total mass precisely when $\mathbb{E}(B_1) < \infty$, the statement on positive recurrence follows from Corollary 2.19. In this case, also Lemma 2.20 applies and yields that X_n^x converges in law to $\frac{1}{\nu(\mathbb{L})}\nu$. This was already obtained by [27].

Note that the “margin” between conditions (ii), (iii) and (iv) is quite narrow.

B. General reflected random walk.

We now drop the restriction that the random variables B_n are non-negative. Thus, the “ordinary” random walk $S_n = B_1 + \dots + B_n$ on \mathbb{R} may visit the positive as well as the negative half-axis. Since we assume that μ is non-lattice, the closed group generated by $\text{supp}(\mu)$ is \mathbb{R} .

We start with a simple observation ([6] has a more complicated proof).

(5.6) Lemma. *If μ is symmetric, then reflected random walk is (topologically) recurrent if and only if the random walk (S_n) is recurrent.*

Proof. If μ is symmetric, then also $|S_n|$ is a Markov chain. Indeed, for a Borel set $U \subset \mathbb{R}^+$,

$$\begin{aligned} \Pr[|S_{n+1}| \in U \mid S_n = x] &= \mu(-x + U) + \mu(-x - U) - \mu(-x) \delta_0(U) \\ &= \Pr[|S_{n+1}| \in U \mid S_n = -x], \end{aligned}$$

and we see that $|S_n|$ has the same transition probabilities as the reflected random walk governed by μ . \square

Recall the classical result that when $E(|B_1|) < \infty$ and $E(B_1) = 0$ then (S_n) is recurrent; see CHUNG AND FUCHS [15].

(5.7) Corollary. *If μ is symmetric and has finite first moment then reflected random walk is recurrent.*

Let $B_n^+ = \max\{B_n, 0\}$ and $B_n^- = \max\{-B_n, 0\}$, so that $B_n = B_n^+ - B_n^-$. The following is well-known.

(5.8) Lemma. *If (a) $E(B_1^-) < E(B_1^+) \leq \infty$, or if (b) $0 < E(B_1^-) = E(B_1^+) < \infty$, then $\limsup S_n = \infty$ almost surely, so that there are infinitely many reflections.*

In general, we should exclude that $S_n \rightarrow -\infty$, since in that case there are only finitely many reflections, and reflected random walk tends to $+\infty$ almost surely. In the sequel, we assume that $\limsup S_n = \infty$ almost surely. Then the (non-strictly) ascending *ladder epochs*

$$\mathbf{s}(0) = 0, \quad \mathbf{s}(k+1) = \inf\{n > \mathbf{s}(k) : S_n \geq S_{\mathbf{s}(k)}\}$$

are all almost surely finite, and the random variables $\mathbf{s}(k+1) - \mathbf{s}(k)$ are i.i.d. We can consider the *embedded random walk* $S_{\mathbf{s}(k)}$, $k \geq 0$, which tends to ∞ almost surely. Its increments $\bar{B}_k = S_{\mathbf{s}(k)} - S_{\mathbf{s}(k-1)}$, $k \geq 1$, are i.i.d. non-negative random variables with distribution denoted $\bar{\mu}$. Furthermore, if \bar{X}_k^x denotes the reflected random walk associated with the sequence (\bar{B}_k) , while X_n^x is our original reflected random walk associated with (B_n) , then

$$\bar{X}_k^x = X_{\mathbf{s}(k)}^x,$$

since no reflection can occur between times $\mathbf{s}(k)$ and $\mathbf{s}(k+1)$. When $\Pr[B_n < 0] > 0$, one clearly has $\sup \text{supp}(\bar{\mu}) = +\infty$. Lemma 5.4 implies the following.

(5.9) Corollary. *Suppose that (5.2) is verified, $\Pr[B_n < 0] > 0$ and $\limsup S_n = \infty$. Then*

- (a) *reflected random walk is topologically irreducible on $\mathbb{L} = \mathbb{R}^+$, and*
- (b) *the embedded reflected random walk \bar{X}_k^x is recurrent if and only if the original reflected random walk is recurrent.*

Proof. Statement (a) is clear.

Since both processes are locally contractive, each of the two processes is transient if and only if it tends to $+\infty$ almost surely: If $\lim_n X_n^x = \infty$ then clearly also $\lim_k X_{\mathbf{s}(k)}^x = \infty$ a.s. Conversely, suppose that $\lim_k \bar{X}_k^x \rightarrow \infty$ a.s. If $\mathbf{s}(k) \leq n < \mathbf{s}(k+1)$ then $X_n^x \geq X_{\mathbf{s}(k)}^x$. (Here, k is random, depending on n and $\omega \in \Omega$, and when $n \rightarrow \infty$ then $k \rightarrow \infty$ a.s.) Therefore, also $\lim_n X_n^x = \infty$ a.s., so that (b) is also true. \square

We can now deduce the following.

(5.10) Theorem. *Suppose that (5.2) is verified and that $\Pr[B_1 < 0] > 0$. Then reflected random walk (X_n^x) is (topologically) recurrent on $\mathbb{L} = \mathbb{R}^+$, if*

- (a) $\mathbb{E}(B_1^-) < \mathbb{E}(B_1^+)$ and $\mathbb{E}(\sqrt{B_1^+}) < \infty$, or if
- (b) $0 < \mathbb{E}(B_1^-) = \mathbb{E}(B_1^+)$ and $\mathbb{E}(\sqrt{B_1^+}^3) < \infty$.

Proof. We show that in each case the assumptions imply that $\mathbb{E}(\sqrt{\bar{B}_1}) < \infty$. Then we can apply Theorem 5.5 to deduce recurrence of (\bar{X}_k^x) . This in turn yields recurrence of (X_n^x) by Corollary 5.9.

(a) Under the first set of assumptions,

$$\begin{aligned} \mathbb{E}(\sqrt{\bar{B}_1}) &= \mathbb{E}\left(\sqrt{B_1 + \dots + B_{\mathbf{s}(1)}}\right) \leq \mathbb{E}\left(\sqrt{B_1^+ + \dots + B_{\mathbf{s}(1)}^+}\right) \\ &\leq \mathbb{E}\left(\sqrt{B_1^+} + \dots + \sqrt{B_{\mathbf{s}(1)}^+}\right) = \mathbb{E}(\sqrt{B_1^+}) \cdot \mathbb{E}(\mathbf{s}(1)) \end{aligned}$$

by Wald's identity. Thus, we now are left with proving $\mathbb{E}(\mathbf{s}(1)) < \infty$. If $\mathbb{E}(B_1^+) < \infty$, then $\mathbb{E}(|B_1|) < \infty$ and $\mathbb{E}(B_1) > 0$ by assumption, and in this case it is well known that $\mathbb{E}(\mathbf{s}(1)) < \infty$; see e.g. [19, Thm. 2 in §XII.2, p. 396-397]. If $\mathbb{E}(B_1^+) = \infty$ then there is $M > 0$ such that $B_n^{(M)} = \min\{B_n, M\}$ (which has finite first moment) satisfies $\mathbb{E}(B_n^{(M)}) = \mathbb{E}(B_1^{(M)}) > 0$. The first increasing ladder epoch $\mathbf{s}^{(M)}(1)$ associated with $S_n^{(M)} = B_1^{(M)} + \dots + B_n^{(M)}$ has finite expectation by what we just said, and $\mathbf{s}(1) \leq \mathbf{s}^{(M)}(1)$. Thus, $\mathbf{s}(1)$ is integrable.

(b) If the B_n are centered, non-zero and $\mathbb{E}((B_1^+)^{1+a}) < \infty$, where $a > 0$, then $\mathbb{E}((\bar{B}_1)^a) < \infty$, as was shown by CHOW AND LAI [14]. In our case, $a = 1/2$. \square

We conclude our remarks on reflected random walk by discussing sharpness of the sufficient recurrence conditions $\mathbb{E}(\sqrt{B_1^+}^3) < \infty$ in the centered case, resp. $\mathbb{E}(\sqrt{\bar{B}_1}) < \infty$ in the case when $B_1 \geq 0$.

(5.11) Example. Define a symmetric probability measure μ on \mathbb{R} by

$$\mu(dx) = \frac{dx}{(1 + |x|)^{1+a}},$$

where $a > 0$ and c is the proper normalizing constant (and dx is Lebesgue measure). Then it is well known and quite easy to prove via Fourier analysis that the associated symmetric random walk S_n on \mathbb{R} is recurrent if and only if $a \geq 1$. By Lemma 5.6, the associated reflected random walk is also recurrent, but when $1 \leq a \leq 3/2$ then condition (b) of Theorem 5.10 does not hold.

Nevertheless, we can also show that in general, the sufficient condition $\mathbb{E}(\sqrt{B_1}) < \infty$ for recurrence of reflected random walk with non-negative increments \overline{B}_n is very close to being sharp. (We write \overline{B}_n because we shall represent this as an embedded random walk in the next example.)

(5.12) Proposition. *Let μ_0 be a probability measure on \mathbb{R}^+ which has a density $\phi_0(x)$ with respect to Lebesgue measure that is decreasing and satisfies*

$$\phi(x) \sim c(\log x)^b/x^{3/2}, \quad \text{as } x \rightarrow \infty,$$

where $b > 1/2$ and $c > 0$. Then the associated reflected random walk on \mathbb{R}^+ is transient.

Note that μ_0 has finite moment of order $\frac{1}{2} - \varepsilon$ for every $\varepsilon > 0$, while the moment of order $\frac{1}{2}$ is infinite.

The proof needs some preparation. Let (B_n) be i.i.d. random variables with values in \mathbb{R} that have finite first moment and are non-constant and centered, and let μ be their common distribution.

The first *strictly ascending* and *strictly descending ladder epochs* of the random walk $S_n = B_1 + \dots + B_n$ are

$$\mathbf{t}_+(1) = \inf\{n > 0 : S_n > 0\} \quad \text{and} \quad \mathbf{t}_-(1) = \inf\{n > 0 : S_n < 0\},$$

respectively. They are almost surely finite. Let μ_+ be the distribution of $S_{\mathbf{t}_+(1)}$ and μ_- the distribution of $S_{\mathbf{t}_-(1)}$, and – as above – $\overline{\mu}$ the distribution of $\overline{B}_1 = S_{\mathbf{s}(1)}$. We denote the characteristic function associated with any probability measure σ on \mathbb{R} by $\widehat{\sigma}(t)$, $t \in \mathbb{R}$. Then, following FELLER [19, (3.11) in §XII.3], *Wiener-Hopf-factorization* tells us that

$$\mu = \overline{\mu} + \mu_- - \overline{\mu} * \mu_- \quad \text{and} \quad \overline{\mu} = u \cdot \delta_0 + (1 - u) \cdot \mu_+,$$

$$\text{where } u = \overline{\mu}(0) = \sum_{n=1}^{\infty} \Pr[S_1 < 0, \dots, S_{n-1} < 0, S_n = 0] < 1.$$

Here $*$ is convolution. Note that when μ is absolutely continuous (i.e., absolutely continuous with respect to Lebesgue measure) then $u = 0$, so that

$$(5.13) \quad \overline{\mu} = \mu_+ \quad \text{and} \quad \mu = \mu_+ + \mu_- - \mu_+ * \mu_-.$$

(5.14) Lemma. *Let μ_0 be a probability measure on \mathbb{R}^+ which has a decreasing density $\phi_0(x)$ with respect to Lebesgue measure. Then there is an absolutely continuous symmetric probability measure μ on \mathbb{R} such that the associated first (non-strictly) ascending ladder random variable has distribution μ_0 .*

Proof. If μ_0 is the law of the first strictly ascending ladder random variable associated with some absolutely continuous, symmetric measure μ , then by (5.13) we must have $\mu_+ = \mu_0$ and $\mu_- = \check{\mu}_0$, the reflection of μ_0 at 0, and

$$(5.15) \quad \mu = \mu_0 + \check{\mu}_0 - \mu_0 * \check{\mu}_0.$$

We *define* μ in this way. The monotonicity assumption on μ_0 implies that μ is a probability measure: indeed, by the monotonicity assumption it is straightforward to check that the function $\phi = \phi_0 + \check{\phi}_0 - \phi_0 * \check{\phi}_0$ is non-negative; this is the density of μ .

The measure μ of (5.15) is non-degenerate and symmetric. If it induces a recurrent random walk (S_n) , then the ascending and descending ladder epochs are a.s. finite. If (S_n) is transient, then $|S_n| \rightarrow \infty$ almost surely, but it cannot be $\Pr[S_n \rightarrow \infty] > 0$ since in that case this probability had to be 1 by Kolmogorov's 0-1-law, while symmetry would yield $\Pr[S_n \rightarrow -\infty] = \Pr[S_n \rightarrow \infty] \leq 1/2$. Therefore $\liminf S_n = -\infty$ and $\limsup S_n = +\infty$ almost surely, a well-known fact, see e.g. [19, Thm. 1 in §XII.2, p. 395]. Consequently, the ascending and descending ladder epochs are again a.s. finite. Therefore the probability measures μ_+ and $\mu_- = \check{\mu}_+$ (the laws of $S_{t_{\pm}(1)}$) are well defined. By the uniqueness theorem of Wiener-Hopf-factorization [19, Thm. 1 in §XII.3, p. 401], it follows that $\mu_- = \check{\mu}_0$ and that the distribution of the first (non-strictly) ascending ladder random variable is $\bar{\mu} = \mu_0$. \square

Proof of Proposition 5.12. Let μ be the symmetric measure associated with μ_0 according to (5.15) in Lemma 5.14. Then its characteristic function $\widehat{\mu}(t)$ is non-negative real. A well-known criterion says that the random walk S_n associated with μ is transient if and only if (the real part of) $1/(1 - \widehat{\mu}(t))$ is integrable in a neighbourhood of 0. Returning to $\mu_0 = \mu_+$, it is a standard exercise (see [19, Ex. 12 in Ch. XVII, Section 12]) to show that there is $A \in \mathbb{C}$, $A \neq 0$ such that its characteristic function satisfies

$$\widehat{\mu}_0(t) = 1 + A \sqrt{t} (\log t)^b (1 + o(t)) \quad \text{as } t \rightarrow 0.$$

By (5.13),

$$1 - \widehat{\mu}(t) = (1 - \widehat{\mu}_+(t))(1 - \widehat{\mu}_-(t)).$$

We deduce

$$\widehat{\mu}(t) = 1 - |A|^2 |t| (\log |t|)^{2b} (1 + o(t)) \quad \text{as } t \rightarrow 0.$$

The function $1/(1 - \widehat{\mu}(t))$ is integrable near 0. By Lemma 5.6, the associated reflected random walk is transient. But then also the embedded reflected random walk associated with $S_{s(n)}$ is transient by Corollary 5.9. This is the reflected random walk governed by μ_0 . \square

PART II. Stochastic dynamical systems induced by Lipschitz mappings

6. THE CONTRACTIVE CASE, AND RECURRENCE IN THE LOG-CENTERED CASE

We now consider the situation when the i.i.d. random mappings $F_n : \mathsf{X} \rightarrow \mathsf{X}$ belong to the semigroup $\mathfrak{L} \subset \mathfrak{G}$ of Lipschitz mappings. Recall our notation $\mathfrak{l}(f)$ for the Lipschitz constant of $f \in \mathfrak{L}$. We assume that

$$(6.1) \quad \Pr[\mathfrak{l}(F_n) > 0] = 1 \quad \text{and} \quad \Pr[\mathfrak{l}(F_n) < 1] > 0.$$

In this situation, the real random variables

$$(6.2) \quad A_n = \mathfrak{l}(F_n) \quad \text{and} \quad B_n = d(F_n(o), o)$$

play an important role. Indeed, let (X_n^x) be the SDS starting at $x \in \mathsf{X}$ which is associated with the sequence (F_n) , and for any starting point $y \geq 0$, let (Y_n^y) the affine SDS on \mathbb{R}^+ associated with (A_n, B_n) according to (3.1). Then

$$(6.3) \quad d(X_n^x, o) \leq Y_n^{|x|}, \quad \text{where} \quad |x| = d(x, o).$$

Thus, we can use the results of Section 3. First of all, Propositions 1.3, resp. 3.2 yield the following.

(6.4) Corollary. *Given the random i.i.d. Lipschitz mappings F_n , let A_n and B_n be as in (6.2).*

If $\mathbb{E}(\log^+ A_n) < \infty$ and $-\infty \leq \mathbb{E}(\log A_n) < 0$ then the SDS (X_n^x) generated by the F_n is strongly contractive on X .

If in addition $\mathbb{E}(\log^+ B_n) < \infty$ then the SDS has a unique invariant probability measure ν on X , it is (positive) recurrent on $\mathsf{L} = \text{supp}(\nu)$, and the time shift on the tracetary space $\mathsf{X}^{\mathbb{N}_0}$ is ergodic with respect to the probability measure \Pr_ν .

Proof. Strong contractivity is obvious. When $\mathbb{E}(\log^+ B_n) < \infty$, (6.3) tells us that along with $(Y_n^{|x|})$ also (X_n^x) is positive recurrent. \square

The interesting and much harder case is the one where $\log A_n$ is integrable and centered, that is, $\mathbb{E}(\log A_n) = 0$. The assumptions of Proposition 3.2, applied to A_n and B_n of (6.2), will in general not imply that our SDS is locally contractive.

(6.5) Remarks. (a) In the log-centered case, we can apply Proposition 3.3 to $(Y_n^{|x|})$. Among its hypotheses, also need that

$$(6.6) \quad \Pr[A_n y + B_n = y] < 1 \quad \text{for all } y \in \mathbb{R}.$$

A sufficient condition for this is that

$$\Pr[F_n(x) = x] < 1 \quad \text{for every } x \in \mathsf{X}.$$

Indeed, when $y = 0$, then $\Pr[A_n y + B_n = y] < 1$ is the same as $\Pr(F_n(o) = o) < 1$ from (6.1). If $y \neq 0$ then observe that $A_n - 1$ assumes both positive and negative values with positive probability, so that the requirement is again met.

When the assumptions of Proposition 3.3 hold for the random variables (A_n, B_n) of (6.2), the affine SDS $(Y_n^{|x|})$ on \mathbb{R} is locally contractive and recurrent on its limit set $\mathsf{L}_{\mathbb{R}}$,

which is contained in \mathbb{R}^+ by construction. Note that it depends on the reference point $o \in \mathbf{X}$ through the definition of B_n .

(b) In view of our assumptions (6.1), we can always modify the measure $\tilde{\mu}$ on \mathfrak{L} to obtain a new one, say $\tilde{\mu}'$, which has the same support and satisfies

$$-\infty < \int_{\mathfrak{L}} \log \mathfrak{l}(f) d\tilde{\mu}'(f) < 0.$$

Then $\tilde{\mu}'$ gives rise to a strongly contractive SDS. Let \mathbf{L} be its limit set. Remark 2.9 tells us that also our original SDS governed by $\tilde{\mu}$ is topologically irreducible on \mathbf{L} and that it evolves within \mathbf{L} when started in a point of \mathbf{L} . This set is given by Corollary 2.8. We may assume that the reference point o belongs to \mathbf{L} .

In the sequel, we shall write

$$A_{m,m} = 1 \quad \text{and} \quad A_{m,n} = A_{m+1} \cdots A_{n-1} A_n \quad (n > m).$$

(6.7) Theorem. *If in addition to (6.1) and (6.6), one has*

$$(6.8) \quad \mathbf{E}(\log A_n) = 0, \quad \mathbf{E}(|\log A_n|^2) < \infty, \quad \text{and} \quad \mathbf{E}((\log^+ |B_n|)^{2+\varepsilon}) < \infty$$

for some $\varepsilon > 0$, then the SDS is topologically recurrent on the set \mathbf{L} of Corollary 2.8. Moreover, for every $x \in \mathbf{X}$ (and not just $\in \mathbf{L}$) and every open set $U \subset \mathbf{X}$ that intersects \mathbf{L} ,

$$\Pr[X_n^x \in U \text{ for infinitely many } n] = 1.$$

Proof. The (non-strictly) *descending* ladder epochs are

$$\ell(0) = 0, \quad \ell(k+1) = \inf\{n > \ell(k) : A_{0,n} \leq A_{0,\ell(k)}\}$$

Since $(A_{0,n})$ is a recurrent multiplicative random walk on \mathbb{R}_*^+ , these epochs are stopping times with i.i.d. increments. The induced SDS is $(\bar{X}_k^x)_{k \geq 0}$, where $\bar{X}_k^x = X_{\ell(k)}^x$. It is also generated by random i.i.d. Lipschitz mappings, namely

$$\bar{F}_k = F_{\ell(k)} \circ F_{\ell(k)-1} \circ \cdots \circ F_{\ell(k-1)+1}, \quad k \geq 1.$$

With the same stopping times, we also consider the induced affine recursion given by $\bar{Y}_k^{|x|} = Y_{\ell(k)}^{|x|}$. It is generated by the i.i.d. pairs $(\bar{A}_k, \bar{B}_k)_{k \geq 1}$, where

$$\bar{A}_k = A_{\ell(k-1), \ell(k)} \quad \text{and} \quad \bar{B}_k = \sum_{j=\ell(k-1)+1}^{\ell(k)} |B|_j A_{j, \ell(k)}.$$

It is known [17, Lemma 5.49] that under our assumptions, $\mathbf{E}(\log^+ \bar{A}_k) < \infty$, $\mathbf{E}(\log \bar{A}_k) < 0$ and $\mathbf{E}(\log^+ \bar{B}_k) < \infty$. Returning to (\bar{X}_k^x) , we have $\mathfrak{l}(\bar{F}_k) \leq \bar{A}_k$ and $d(\bar{F}_k(o), o) \leq \bar{B}_k$. Corollary 6.4 applies, and the induced SDS is strongly contractive. It has a unique invariant probability measure $\bar{\nu}$, and it is (positive) recurrent on $\bar{\mathbf{L}} = \text{supp}(\bar{\nu})$. Moreover, for every starting point $x \in \mathbf{X}$ and each open set $U \subset \mathbf{X}$ that intersects $\bar{\mathbf{L}}$, we get that almost surely, (\bar{X}_k^x) visits U infinitely often.

In view of the fact that the original SDS is topologically irreducible on \mathbf{L} , we have $\bar{\mathbf{L}} \subset \mathbf{L}$. We now define a sequence of subsets of \mathbf{L} by

$$\mathbf{L}_0 = \bar{\mathbf{L}} \quad \text{and} \quad \mathbf{L}_m = \bigcup \{f(\mathbf{L}_{m-1}) : f \in \text{supp}(\tilde{\mu})\}.$$

Then the closure of $\bigcup_m \mathsf{L}_m$ is a subset of L that is mapped into itself by every $f \in \text{supp}(\tilde{\mu})$. Corollary 2.8 yields that

$$\mathsf{L} = \left(\bigcup_m \mathsf{L}_m\right)^-.$$

We now show by induction on m that for every starting point $x \in \mathsf{X}$ and every open set U that intersects L_m ,

$$\Pr[X_n^x \in U \text{ for infinitely many } n] = 1,$$

and this will conclude the proof.

For $m = 0$, the statement is true. Suppose it is true for $m - 1$. Given an open set U that intersects L_m , we can find an open, relatively compact set V that intersects L_{m-1} such that $\tilde{\mu}(\{f \in \mathfrak{L} : f(V) \subset U\}) = \alpha > 0$.

By the induction hypothesis, (X_n^x) visits U infinitely often with probability 1. We can now apply Lemma 2.10 with $\ell = 2$, $U_0 = U$ and $U_1 = V$ to conclude that also V is visited infinitely often with probability 1. \square

(6.9) Lemma. (a) *Under the assumptions (6.1), every invariant Radon measure ν satisfies $\mathsf{L} \subset \text{supp}(\nu)$.*

(b) *If in addition to (6.1), one has (6.6) and (6.8), then the SDS possesses an invariant Radon measure ν with $\text{supp}(\nu) = \mathsf{L}$. Furthermore, the transition operator P is a conservative contraction of $L^1(\mathsf{X}, \nu)$ for every invariant measure ν .*

Proof. (a) Let ν be invariant. The argument at the end of the proof of Lemma 2.6 shows that $f(\text{supp}(\nu)) \subset \text{supp}(\nu)$ for all $f \in \text{supp}(\tilde{\mu})$. As explained in Remark 6.5(b), Corollary 2.8 applies here and yields statement (a).

(b) Theorem 6.7 yields conservativity. Indeed, let $\mathsf{B}(r)$ be a ball that intersects L . For every starting point $x \in \mathsf{X}$, the SDS (X_n^x) visits $\mathsf{B}(r)$ infinitely often with probability 1. We can choose $\varphi \in \mathcal{C}_c^+(\mathsf{X})$ such that $\varphi \geq 1$ on $\mathsf{B}(r)$. Then

$$\sum_{k=1}^{\infty} P^k \varphi(x) = \infty \quad \text{for every } x \in \mathsf{X},$$

The existence of an invariant Radon measure follows once more from [30, Thm. 5.1], and conservativity of P on $L^1(\mathsf{X}, \nu)$ follows, see e.g. [34, Thm. 5.3]. If right from the start we consider the whole process only on L with the induced metric, then we obtain an invariant measure ν with $\text{supp}(\nu) = \mathsf{L}$. \square

Note that unless we know that the SDS is locally contractive, we cannot argue right away that every invariant measure must be supported exactly by L . The assumptions (6.1) & (6.8) will in general not imply local contractivity, as we shall see below. Thus, the question of uniqueness of the invariant measure is more subtle. For a sufficient condition that requires a more restrictive (Harris type) notion of irreducibility, see [30, Def. 5.4 & Thm. 5.5].

7. HYPERBOLIC EXTENSION

In order to get closer to answering the uniqueness question in a more “topological” spirit, we also want to control the Lipschitz constants A_n . We shall need to distinguish between two cases.

A. Non-lattice case

If the random variables $\log A_n$ are non-lattice, i.e., there is no $\kappa > 0$ such that $\log A_n \in \kappa \cdot \mathbb{Z}$ almost surely, then we consider the extended SDS

$$(7.1) \quad \hat{X}_n^{x,a} = (X_n^x, A_n A_{n-1} \cdots A_1 a)$$

on the extended space $\hat{\mathbf{X}} = \mathbf{X} \times \mathbb{R}_*^+$, with initial point $(x, a) \in \hat{\mathbf{X}}$. We also extend ν to a Radon measure $\lambda = \lambda_\nu$ on $\hat{\mathbf{X}}$ by

$$(7.2) \quad \int_{\hat{\mathbf{X}}} \varphi(x, a) d\lambda(x, a) = \int_{\mathbf{X}} \int_{\mathbb{R}} \varphi(x, e^u) d\nu(x) du.$$

This is the product of ν with the multiplicative Haar measure on \mathbb{R}_*^+ .

B. Lattice case

Otherwise, there is a maximal $\kappa > 0$ such that $\log A_n \in \kappa \cdot \mathbb{Z}$ almost surely. Then we consider again the extended SDS (7.1), but now the extended space is $\hat{\mathbf{X}} = \mathbf{X} \times \exp(\kappa \cdot \mathbb{Z})$, where of course $\exp(\kappa \cdot \mathbb{Z}) = \{e^{\kappa m} : m \in \mathbb{Z}\}$. The initial point (x, a) now has to be such that also $a \in \exp(\kappa \mathbb{Z})$. In this case, we define λ by

$$(7.3) \quad \int_{\hat{\mathbf{X}}} \varphi(x, a) d\lambda(x, a) = \int_{\mathbf{X}} \sum_{m \in \mathbb{Z}} \varphi(x, e^{\kappa m}) d\nu(x).$$

In both cases, it is straightforward to verify that λ is an invariant Radon measure for the extended SDS on $\hat{\mathbf{X}}$.

Consider the hyperbolic upper half plane $\mathbb{H} \subset \mathbb{C}$ with the Poincaré metric

$$\theta(z, w) = \log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|},$$

where $z, w \in \mathbb{H}$ and \bar{w} is the complex conjugate of w . We use it to define a “hyperbolic” metric on $\hat{\mathbf{X}}$ by

$$(7.4) \quad \begin{aligned} \hat{d}((x, a), (y, b)) &= \theta(\mathbf{i}a, d(x, y) + \mathbf{i}b) \\ &= \log \frac{\sqrt{d(x, y)^2 + (a + b)^2} + \sqrt{d(x, y)^2 + (a - b)^2}}{\sqrt{d(x, y)^2 + (a + b)^2} - \sqrt{d(x, y)^2 + (a - b)^2}}. \end{aligned}$$

It is a good exercise, using the specific properties of θ , to verify that this is indeed a metric. The metric space $(\hat{\mathbf{X}}, \hat{d})$ is again proper, and for any $a > 0$, the embedding $\mathbf{X} \rightarrow \hat{\mathbf{X}}, x \mapsto (x, a)$, is a homeomorphism.

(7.5) Lemma. *Let $f : \mathbf{X} \rightarrow \mathbf{X}$ be a Lipschitz mapping with Lipschitz constant $\mathfrak{l}(f) > 0$. Then the mapping $\hat{f} : \hat{\mathbf{X}} \rightarrow \hat{\mathbf{X}}$, defined by*

$$\hat{f}(x, a) = (f(x), \mathfrak{l}(f)a)$$

is a contraction of $(\hat{\mathbf{X}}, \hat{d})$ with Lipschitz constant 1.

Proof. We have by the dilation invariance of the hyperbolic metric

$$\begin{aligned} \tilde{d}(\hat{f}(x, a), \hat{f}(y, b)) &= \theta(\mathbf{i} \mathfrak{l}(f)a, d(f(x), f(y)) + \mathbf{i} \mathfrak{l}(f)b) \leq \theta(\mathbf{i} \mathfrak{l}(f)a, \mathfrak{l}(f)d(x, y) + \mathbf{i} \mathfrak{l}(f)b) \\ &= \theta(\mathbf{i} a, d(x, y) + \mathbf{i} b) = \hat{d}((x, a), (y, b)). \end{aligned}$$

Thus, $\mathfrak{l}(\hat{f}) \leq 1$. Furthermore, if $\varepsilon > 0$ and $x, y \in \mathbf{X}$ are such that $d(f(x), f(y)) \geq (1 - \varepsilon)\mathfrak{l}(f)d(x, y)$ then we obtain in the same way that

$$\hat{d}(\tilde{f}(x, a), \tilde{f}(y, b)) \geq \theta(\mathbf{i} a, (1 - \varepsilon)d(x, y) + \mathbf{i} b).$$

when $\varepsilon \rightarrow 0$, the right hand side tends to $\hat{d}((x, a), (y, b))$. Hence $\mathfrak{l}(\hat{f}) = 1$. \square

Thus, with the sequence (F_n) , we associate the sequence (\hat{F}_n) of i.i.d. Lipschitz contractions of $\hat{\mathbf{X}}$ with Lipschitz constants 1. The associated SDS on $\hat{\mathbf{X}}$ is $(\hat{X}_n^{x,a})$, as defined in (7.1). From Lemma 2.2, which is true for any SDS of contractions, we get the following, where $o \in \mathbf{X}$ and $\hat{o} = (o, 1)$.

(7.6) Corollary. $\Pr[\hat{d}(\hat{X}_n^{x,a}, \hat{o}) \rightarrow \infty] \in \{0, 1\}$, and the value is the same for all $(x, a) \in \hat{X}$.

8. TRANSIENT EXTENDED SDS

We first consider the situation when $(\hat{X}_n^{x,a})$ is transient, i.e., the probability in Corollary 7.6 is $= 1$. We shall use the comparison (6.3) of (X_n^x) with the affine stochastic recursion $(Y_n^{|x|})$. Recall that $|x| = d(o, x)$ and that $B_n \geq 0$. The hyperbolic extension $(\hat{Y}_n^{|x|,a})$ of $(Y_n^{|x|})$ is a random walk on the hyperbolic upper half plane. It can be also seen as a random walk on the affine group of all mappings $g_{a,b}(z) = az + b$. Under the non-degeneracy assumptions of Proposition 3.3, this random walk is well-known to be transient.

(8.1) Lemma. *Assume that (6.1), (6.6) and (6.8) hold.*

Then for every sufficiently large $r > 0$ and every $s > 1$ there are $\alpha = \alpha_{r,s}$ and $\delta = \delta_{r,s} > 0$ such that, setting $K_{r,s} = [0, r] \times [1/s, s]$ and $Q_{r,\alpha} = [0, r] \times [\alpha, \infty)$, one has for the affine recursion that

$$\Pr[\hat{Y}_n^{y,a} \in K_{r,s} \text{ for some } n \geq 1] \geq \delta \quad \text{for all } (y, a) \in Q_{r,\alpha}.$$

Proof. In this proof only, we write ν for the invariant Radon measure associated with $(Y_n^{|x|})$. Its existence is guaranteed by Proposition 3.3. Let $\lambda = \lambda_\nu$ be its hyperbolic extension according to (7.2), resp. (7.3). We normalize ν , and consequently λ , so that ν is the measure which is denoted $m(f)$ in [3, p. 482].

The random walk $(\hat{Y}_n^{y,a})$ on the affine group (parametrized by $\mathbb{R}_*^+ \times \mathbb{R}$) evolves on $\mathbb{R}_*^+ \times \mathbb{R}^+$, when $y \geq 0$. By [3], its potential kernel

$$\mathcal{U}\varphi(y, a) = \sum_{n=0}^{\infty} \mathbb{E}(\varphi(\hat{Y}_n^{y,a})), \quad \varphi \in \mathcal{C}_c(\mathbb{R}_*^+ \times \mathbb{R}^+),$$

is finite and weakly compact as a family of Radon measures that are parametrized by (y, a) . Furthermore [3, Thm. 2.2],

$$\lim_{a \rightarrow \infty} \mathcal{U}\varphi(y, a) = \int \varphi d\lambda,$$

and convergence is uniform when y remains in a compact set. We fix $r > 1$ large enough so that $\nu([0, r']) > 0$, where $r' = r - 1$, and let $s > 1$ be arbitrary. We set $s' = (s + 1)/2$ and $c_{r,s} = \lambda(K_{r',s'})/2$, which is strictly positive, and choose $\varphi \in \mathcal{C}_c^+(\mathbb{R}_*^+ \times \mathbb{R}^+)$ so that $\mathbf{1}_{K_{r',s'}} \leq \varphi \leq \mathbf{1}_{K_{r,s}}$. By the above, there is $\alpha = \alpha_{r,s} > 0$ such that $\mathcal{U}\varphi(y, a) \geq c_{r,s}$ for all $(y, a) \in Q_{r,\alpha}$. Given any starting point (y, a) , let

$$\tau = \inf\{n \geq 1 : \hat{Y}_n^{y,a} \in K_{r,s}\}.$$

We know that

$$M_{r,s} = \sup \mathcal{U}\mathbf{1}_{K_{r,s}} < \infty.$$

Let $(y, a) \in Q_{r,\alpha}$. Just for the purpose of this proof, we consider the hitting distribution $\sigma_{(y,a)}$ on $K_{r,s}$ defined by $\sigma_{(y,a)}(B) = \Pr[\tau < \infty, \hat{Y}_\tau^{y,a} \in B]$. Then by the Markov property,

$$\begin{aligned} \mathcal{U}\mathbf{1}_{K_{r,s}}(y, a) &= \mathbb{E}\left(\sum_{n=0}^{\infty} \mathbf{1}_{K_{r,s}}(\hat{Y}_n^{y,a})\right) = \mathbb{E}\left(\mathbf{1}_{[\tau < \infty]} \sum_{n=\tau}^{\infty} \mathbf{1}_{K_{r,s}}(\hat{Y}_n^{y,a})\right) \\ &= \int_{K_{r,s}} \mathbb{E}\left(\sum_{n=0}^{+\infty} \mathbf{1}_{K_{r,s}}(\hat{Y}_n^{z,b})\right) d\sigma_{(y,a)}(z, b) \\ &\leq M_{r,s} \sigma_{(y,a)}(K_{r,s}) = M_{r,s} \Pr_{(y,a)}[\tau < \infty], \end{aligned}$$

where the index (y, a) indicates the starting point. Therefore we can set $\delta = M_{r,s}/c_{r,s}$, and $\Pr_{(y,a)}[\tau < \infty] \geq \delta$ for all $(y, a) \in Q_{r,\alpha}$. \square

Let $\bar{B}(r)$ be the closed ball in \mathbf{X} with center 0 and radius r . Set $B_{r,s} = \bar{B}(r) \times [1/s, s]$ and $C_{r,\alpha} = \bar{B}(r) \times [\alpha, \infty)$.

(8.2) Lemma. *Assume that (6.1), (6.6) and (6.8) hold and that $(\hat{X}_n^{x,a})$ is transient. Then for every sufficiently large $r > 0$, there is $\alpha > 0$ such that*

$$\Pr[\hat{X}_n^{x,a} \in C_{r,\alpha} \text{ for infinitely many } n] = 0 \quad \text{for all } (x, a) \in \hat{X}.$$

Proof. Let

$$\Lambda = \Lambda^{x,a} = \{\omega \in \Omega : \hat{X}_n^{x,a}(\omega) \in C_{r,\alpha} \text{ for infinitely many } n\}.$$

Given r sufficiently large so that Lemma 8.1 applies, choose $s > 1$ and let α and $\delta > 0$ be as in that lemma. For each $(c, a) \in Q_{r,\alpha}$ there is an index $N_{c,a} \in \mathbb{N}$ such that

$$(8.3) \quad \Pr[\hat{Y}_n^{y,a} \in K_{r,s} \text{ for some } n \text{ with } 1 \leq n \leq N_{c,a}] \geq \delta/2.$$

If $(c, a) \notin Q_{r,\alpha}$ then we set $N_{c,a} = 0$. Since $B_{r,s}$ is compact, the transience assumption yields that $\Pr(\bigcup_{j=2}^{\infty} \Omega_j) = 1$, where

$$\Omega_j = \Omega_j^{x,a} = \{\omega \in \Omega : \hat{X}_n^{x,a}(\omega) \notin B_{r,s} \text{ for every } n \geq j\}.$$

Thus, we need to show that $\Pr(\Lambda \cap \Omega_j) = 0$ for every $j \geq 2$. We define a sequence of stopping times $\tau_k = \tau_k^{x,a}$ and (when $\tau_k < \infty$) associated pairs $(x_k, a_k) = \widehat{X}_{\tau_k}^{x,a}$ by

$$\begin{aligned} \tau_1 &= \inf\{n > N^{|x|,a} : \widehat{X}_n^{x,a} \in \mathbb{C}_{r,\alpha}\} \quad \text{and} \\ \tau_{k+1} &= \begin{cases} \inf\{n > \tau_k + N^{|x_k|,a_k} : \widehat{X}_n^{x,a} \in \mathbb{C}_{r,\alpha}\}, & \text{if } \tau_k < \infty, \\ \infty, & \text{if } \tau_k = \infty. \end{cases} \end{aligned}$$

Unless explained separately, we always use $\tau_k = \tau_k^{x,a}$. Note that $\omega \in \Lambda$ if and only if $\tau_k(\omega) < \infty$ for all k . Therefore

$$\Lambda \cap \Omega_j = \bigcap_{k \geq j} \Lambda_{j,k}, \quad \text{where} \quad \Lambda_{j,k} = [\tau_k < \infty, \widehat{X}_n^{x,a} \notin \mathbb{B}_{r,s} \text{ for all } n \text{ with } j \leq n \leq \tau_k].$$

We have $\Lambda_{j,k} \subset \Lambda_{j,k-1}$. Next, note that

$$\text{if } \widehat{X}_n^{x,a}(\omega) \notin \mathbb{B}_{r,s} \quad \text{then} \quad \widehat{Y}_n^{|x|,a}(\omega) \notin K_{r,s}.$$

This follows from (6.3).

We have that $\widehat{X}_{\tau_{k-1}}^{x,a} \in \mathbb{C}_{r,\alpha}$ for $k \geq 2$. Just for the purpose of the next lines of the proof, we introduce the measure σ on $\mathbb{C}_{r,\alpha}$ given by $\sigma(\widehat{B}) = \Pr(\Lambda_{j,k-1} \cap [\widehat{X}_{\tau_{k-1}}^{x,a} \in \widehat{B}])$, where $\widehat{B} \subset \mathbb{C}_{r,\alpha}$ is a Borel set. Then, using the strong Markov property and (8.3),

$$\begin{aligned} \Pr(\Lambda_{j,k}) &= \Pr([\tau_k < \infty, \widehat{X}_n^{x,a} \notin \mathbb{B}_{r,s} \text{ for all } n \text{ with } \tau_{k-1} < n \leq \tau_k] \cap \Lambda_{j,k-1}) \\ &= \int_{\mathbb{C}_{r,\alpha}} \Pr[\tau_1^{y,b} < \infty, \widehat{X}_n^{y,b} \notin \mathbb{B}_{r,s} \text{ for all } n \text{ with } 0 < n \leq \tau_1^{y,b}] d\sigma(y, b) \\ &\leq \int_{\mathbb{C}_{r,\alpha}} \Pr[\tau_1^{y,b} < \infty, \widehat{Y}_n^{|y|,b} \notin K_{r,s} \text{ for all } n \text{ with } 0 < n \leq N^{|y|,b}] d\sigma(y, b) \\ &\leq \int_{\mathbb{C}_{r,\alpha}} (1 - \delta/2) d\sigma(y, b) = (1 - \delta/2) \Pr(\Lambda_{j,k-1}). \end{aligned}$$

We continue recursively downwards until we reach $k = 2$ (since $k = 1$ is excluded unless $(x, a) \in \mathbb{C}_{r,\alpha}$). Thus, $\Pr(\Lambda_{j,k}) \leq (1 - \delta/2)^{k-1}$, and as $k \rightarrow \infty$, we get $\Pr(\Lambda \cap \Omega_j) = 0$, as required. \square

(8.4) Theorem. *Given the random i.i.d. Lipschitz mappings F_n , let A_n and B_n be as in (6.2). Suppose that (6.1), (6.6) and (6.8) hold, and that $\Pr[\widehat{d}(\widehat{X}_n^{x,a}, \hat{o}) \rightarrow \infty] = 1$. Then the SDS induced by the F_n on \mathbf{X} is locally contractive.*

In particular, it has an invariant Radon measure ν that is unique up to multiplication with constants.

Also, the shift T on $(\mathbf{X}^{\mathbb{N}_0}, \mathfrak{B}(\mathbf{X}^{\mathbb{N}_0}), \Pr_\nu)$ is ergodic, where \Pr_ν is the measure on $\widehat{\mathbf{X}}^{\mathbb{N}_0}$ associated with ν .

Proof. Fix any starting point (x, a) of the extended SDS. Let r be sufficiently large so that the last two lemmas apply, and such that

$$\Pr[X_n^x \in \overline{\mathbb{B}}(r) \text{ for infinitely many } n] = 1.$$

We claim that

$$(8.5) \quad \lim_{n \rightarrow \infty} A_{0,n} \mathbf{1}_{\overline{\mathbf{B}}(r)}(X_n^x) = 0 \quad \text{almost surely.}$$

We consider α associated with r as in Lemma 8.2. Then we choose an arbitrary $s \geq \alpha$. We know by transience of the extended SDS that

$$\Pr[\widehat{X}_n^{x,a} \in \mathbf{B}_{r,s} \text{ for infinitely many } n] = 0.$$

We combine this with Lemma 8.2 and get

$$\Pr[\widehat{X}_n^{x,a} \in \mathbf{B}_{r,s} \cup \mathbf{C}_{r,\alpha} \text{ for infinitely many } n] = 0.$$

Since $s \geq \alpha$, we have $\mathbf{B}_{r,s} \cup \mathbf{C}_{r,\alpha} = \overline{\mathbf{B}}(r) \times [1/s, \infty)$.

Thus, if $\mathbb{N}(x, r)$ denotes the a.s. infinite random set of all n for which $X_n^x \in \overline{\mathbf{B}}(r)$, then for all but finitely many $n \in \mathbb{N}(x, r)$, we have $A_{0,n} < 1/s$. This holds for every $s > \alpha$, and we have proved (8.5). We conclude that

$$d(X_n^x, X_n^y) \mathbf{1}_{\overline{\mathbf{B}}(r)}(X_n^x) \leq A_{0,n} d(x, y) \mathbf{1}_{\overline{\mathbf{B}}(r)}(X_n^x) \rightarrow 0 \quad \text{almost surely.}$$

Now that we have local contractivity, the remaining statements follow from Theorem 2.13. \square

9. CONSERVATIVE EXTENDED SDS

Now we assume to be in the conservative case, i.e., the probability in Corollary 7.6 is $= 0$. We start with an invariant measure ν for the SDS on \mathbf{X} . If (6.1), (6.6) & (6.8) hold, its existence is guaranteed by Lemma 6.9. Then we extend ν to the measure $\lambda = \lambda_\nu$ on $\widehat{\mathbf{X}}$ of (7.2), resp. (7.3).

We can realize the extended SDS, starting at $(x, a) \in \widehat{\mathbf{X}}$, on the space

$$(\widehat{\mathbf{X}}^{\mathbb{N}_0}, \mathfrak{B}(\widehat{\mathbf{X}}^{\mathbb{N}_0}), \Pr_{x,a}),$$

where $\mathfrak{B}(\widehat{\mathbf{X}}^{\mathbb{N}_0})$ is the product Borel σ -algebra, and $\Pr_{x,a}$ is the image of the measure \Pr under the mapping

$$\Omega \rightarrow \widehat{\mathbf{X}}^{\mathbb{N}_0}, \quad \omega \mapsto (\widehat{X}_n^{x,a}(\omega))_{n \geq 0}.$$

Then we consider the Radon measure on $\widehat{\mathbf{X}}^{\mathbb{N}_0}$ defined by

$$\Pr_\lambda = \int_{\widehat{\mathbf{X}}} \Pr_{x,a} d\lambda(x, a).$$

The integral with respect to \Pr_λ is denoted \mathbf{E}_λ . We write \widehat{T} for the time shift on $\widehat{\mathbf{X}}^{\mathbb{N}_0}$. Since λ is invariant for the extended SDS, \widehat{T} is a contraction of $L^1(\widehat{\mathbf{X}}^{\mathbb{N}_0}, \Pr_\lambda)$. Also, in this section, \mathfrak{I} stands for the σ -algebra of the \widehat{T} -invariant sets in $\mathfrak{B}(\widehat{\mathbf{X}}^{\mathbb{N}_0})$. As before, any function $\varphi : \widehat{\mathbf{X}}^\ell \rightarrow \mathbb{R}$ is extended to $\widehat{\mathbf{X}}^{\mathbb{N}_0}$ by setting $\varphi(\mathbf{x}, \mathbf{a}) = \varphi((x_0, a_0), \dots, (x_{\ell-1}, a_{\ell-1}))$, if $(\mathbf{x}, \mathbf{a}) = ((x_n, a_n))_{n \geq 0}$. In analogy with (2.3), we define

$$\widehat{X}_{m,n}^{x,a} = (X_{m,n}^x, A_{m,n}a) \quad (n \geq m).$$

We now set for $n \geq m$ and $\varphi : \widehat{\mathbf{X}}^{\mathbb{N}_0} \rightarrow \mathbb{R}$

$$S_{m,n}^{x,a} \varphi(\omega) = \sum_{k=m}^n \varphi((\widehat{X}_{m,k}^{x,a}(\omega))_{k \geq m})$$

and in particular $S_n^{x,a}\varphi(\omega) = S_{0,n}^{x,a}\varphi(\omega)$. Consider the sets

$$(9.1) \quad \Omega_r = \{\omega \in \Omega : \liminf \hat{d}(\hat{X}_n^{\hat{o}}(\omega), \hat{o}) \leq r\} \quad (r \in \mathbb{N}) \quad \text{and} \quad \Omega_\infty = \bigcup_r \Omega_r.$$

By our assumption of conservativity, $\Pr(\Omega_\infty) = 1$. For $r \in \mathbb{N}$, write $\widehat{\mathbf{B}}(r)$ for the *closed* ball in $(\widehat{\mathbf{X}}, \hat{d})$ with center \hat{o} and radius r . Then for every $\omega \in \Omega_r$ and $s \in \mathbb{N}_0$, the set $\{n : \hat{X}_n^{x,a}(\omega) \in \widehat{\mathbf{B}}(r+s) \text{ for all } (x,a) \in \widehat{\mathbf{B}}(s)\}$ is infinite. For each r , set $\psi_r(x,a) = \max\{1 - \hat{d}((x,a), \widehat{\mathbf{B}}(r)), 0\}$. Then $\psi_r \in \mathcal{C}_c^+(\widehat{\mathbf{X}})$ satisfies

$$(9.2) \quad \begin{aligned} \mathbf{1}_{\widehat{\mathbf{B}}(r+1)} &\geq \psi_r \geq \mathbf{1}_{\widehat{\mathbf{B}}(r)}, \\ |\psi(x,a) - \psi(y,b)| &\leq \hat{d}((x,a), (y,b)) \text{ on } \widehat{\mathbf{X}}, \quad \text{and} \\ S_n^{x,a}\psi_{r+s}(\omega) &\rightarrow \infty \quad \text{for all } \omega \in \Omega_r, (x,a) \in \widehat{\mathbf{B}}(s). \end{aligned}$$

Then we can find a decreasing sequence of numbers $c_r > 0$ such that $\sum_r c_r \max \psi_{r+2} < \infty$ and the functions

$$(9.3) \quad \Phi = \sum_r c_r \psi_{r+2} \quad \text{and} \quad \Psi = \sum_r c_r \psi_r$$

are in $L^1(\widehat{\mathbf{X}}, \lambda)$ and thus (there extensions to $\mathbf{X}^{\mathbb{N}_0}$) in $L^1(\widehat{\mathbf{X}}^{\mathbb{N}_0}, \Pr_\lambda)$. They will be used below several times. Both are continuous and strictly positive on $\widehat{\mathbf{X}}$, and by construction,

$$\sum_n \Psi(\hat{X}_n^{x,a}(\omega)) = \infty \quad \text{for all } \omega \in \Omega_\infty \text{ and } (x,a) \in \widehat{\mathbf{X}}.$$

We have obtained the following.

(9.4) Lemma. *When the extended SDS is conservative, \widehat{T} is conservative.*

Next, for any $\varphi \in L^1(\widehat{\mathbf{X}}^{\mathbb{N}_0}, \Pr_\lambda)$, consider the function $\mathbf{v}_\varphi = \mathbf{E}_\lambda(\varphi | \mathfrak{I}) / \mathbf{E}_\lambda(\Psi | \mathfrak{I})$ on $\widehat{\mathbf{X}}^{\mathbb{N}_0}$. A priori, the quotient of conditional expectations is defined only \Pr_λ -almost everywhere, and we consider a representative which is always finite. We turn this into the family of finite positive random variables

$$V_\varphi^{x,a}(\omega) = \mathbf{v}_\varphi\left((\hat{X}_n^{x,a}(\omega))_{n \geq 0}\right), \quad (x,a) \in \widehat{\mathbf{X}}.$$

(9.5) Lemma. *In the conservative case, let $\tau : \Omega \rightarrow \mathbb{N}$ be any a.s. finite random time. Then, on the set where $\tau(\omega) < \infty$, for every $\varphi \in L^1(\widehat{\mathbf{X}}^{\mathbb{N}_0}, \Pr_\lambda)$,*

$$\lim_{n \rightarrow \infty} \frac{S_n^{x,a}\varphi - S_\tau^{x,a}\varphi}{S_n^{x,a}\Psi - S_\tau^{x,a}\Psi} = V_\varphi^{x,a} \quad \text{Pr-almost surely, for } \lambda\text{-almost every } (x,a) \in \widehat{\mathbf{X}}.$$

Proof. We know that $S_n^{x,a}\Psi(\omega) \rightarrow \infty$ for all $\omega \in \Omega_\infty$. Once more by the Chacon-Ornstein theorem, $S_n^{x,a}\varphi / S_n^{x,a}\Psi \rightarrow V_\varphi^{x,a}$ almost surely on Ω_∞ , for λ -almost every $(x,a) \in \widehat{\mathbf{X}}$. Furthermore, both $S_\tau^{x,a}\varphi / S_n^{x,a}\Psi$ and $S_\tau^{x,a}\Psi / S_n^{x,a}\Psi$ tend to 0 on Ω_∞ , as $n \rightarrow \infty$. When $n > \tau$,

$$\frac{S_n^{x,a}\varphi}{S_n^{x,a}\Psi} = \underbrace{\frac{S_\tau^{x,a}\varphi}{S_n^{x,a}\Psi}}_{\rightarrow 0 \text{ a.s.}} + \left(1 - \underbrace{\frac{S_\tau^{x,a}\Psi}{S_n^{x,a}\Psi}}_{\rightarrow 0 \text{ a.s.}}\right) \frac{S_n^{x,a}\varphi - S_\tau^{x,a}\varphi}{S_n^{x,a}\Psi - S_\tau^{x,a}\Psi}.$$

The statement follows. □

When the extended SDS is conservative, we do not see how to involve local contractivity, but we can provide a reasonable additional assumption which will yield uniqueness of the invariant Radon measure. We set

$$(9.6) \quad D_n(x, y) = \frac{d(X_n^x, X_n^y)}{A_1 \cdots A_n}.$$

(Compare with the proof of Theorem 4.2, which corresponds to $A_n \equiv 1$.) The assumption is

$$(9.7) \quad \Pr[D_n(x, y) \rightarrow 0] = 1 \quad \text{for all } x, y \in \mathbf{X}.$$

(9.8) Remark. If we set $D_{m,n}(x, y) = d(X_{m,n}^x, X_{m,n}^y)/A_{m,n}$ then (9.7) implies that

$$\Pr \left[\lim_{n \rightarrow \infty} D_{m,n}(x, y) = 0 \text{ for all } x, y \in \mathbf{X}, m \in \mathbb{N} \right] = 1.$$

Indeed, let \mathbf{X}_0 be a countable, dense subset of \mathbf{X} . Then (9.7) implies that

$$\Pr \left[\lim_{n \rightarrow \infty} D_{m,n}(x, y) = 0 \text{ for all } x, y \in \mathbf{X}_0, m \in \mathbb{N} \right] = 1.$$

Let Ω_0 be the subset of Ω_∞ where this holds.

Note that $D_{m,n}(x, y) \leq d(x, y)$. Given arbitrary $x, y \in \mathbf{X}$ and $x_0, y_0 \in \mathbf{X}_0$, we get on Ω_0

$$D_{m,n}(x, y) \leq D_{m,n}(x_0, y_0) + d(x, x_0) + d(y, y_0),$$

and the statement follows. \square

In the next lemma, we give a condition for (9.7). It will be useful, in §10.

(9.9) Lemma. *In the case when the extended SDS is conservative, suppose that for every $\varepsilon > 0$ and $r \in \mathbb{N}$ there is k such that $\Pr[D_k(x, y) < \varepsilon \text{ for all } x, y \in \mathbf{B}(r)] > 0$. Then (9.7) holds.*

Proof. We set $D_\infty(x, y) = \lim_n D_n(x, y)$ and $w(x, y) = \mathbb{E}(D_\infty(x, y))$. A straightforward adaptation of the argument used in the proof of Theorem 4.2 yields that

$$(9.10) \quad \lim_{m \rightarrow \infty} \frac{w(X_m^x, X_m^y)}{A_1 \cdots A_m} = D_\infty(x, y) \quad \text{almost surely.}$$

Again, we claim that $\Pr[D_\infty(x, y) \geq \varepsilon] = 0$. By conservativity, it is sufficient to show that $\Pr(\Lambda_r) = 0$ for every $r \in \mathbb{N}$, where

$$\Lambda_r = \bigcap_{m \geq k} \bigcup_{n \geq m} [\hat{X}_n^x, \hat{X}_n^y \in \mathbf{B}(r) \times [1/r, r], D_n(x, y) \geq \varepsilon].$$

By assumption, there is k such that the event $\Gamma_{k,r} = [D_k(x, y) < \varepsilon/2 \text{ for all } x, y \in \mathbf{B}(r)]$ satisfies $\Pr(\Gamma_{k,r}) > 0$.

We now continue as in the proof of Theorem 4.2, and find that for all $u, v \in \mathbf{B}(r)$ with $d(u, v) \geq \varepsilon$,

$$w(u, v) \leq d(u, v) - \delta, \quad \text{where } \delta = \Pr(\Gamma_{k,r}) \cdot (\varepsilon/2) > 0.$$

This yields that on Λ_r , almost surely we have infinitely many $n \geq k$ for which $w(X_n^x, X_n^y) \leq d(X_n^x, X_n^y) - \delta$ and $A_1 \cdots A_n \leq r$, that is,

$$\frac{w(X_n^x, X_n^y)}{A_1 \cdots A_n} \leq D_n(x, y) - \frac{\delta}{r} \quad \text{infinitely often.}$$

Letting $n \rightarrow \infty$, we get $D_\infty(x, y) < D_\infty(x, y)$ almost surely on Λ_r , so that indeed $\Pr(\Lambda_r) = 0$. \square

We now elaborate the main technical prerequisite for handling the case when the extended SDS is conservative. Some care may be in place to have a clear picture regarding the dependencies of sets on which various “almost everywhere” statements hold. Let $\varphi \in L^1(\widehat{X}^{\mathbb{N}_0}, \Pr_\lambda)$. Let Ω_0 be as in Remark 9.8. For λ -almost every $(x, a) \in \widehat{X}$, there is a set $\Omega_\varphi^{x,a} \subset \Omega_0$ with $\Pr(\Omega_\varphi^{x,a}) = 1$, such that

$$\frac{S_n^{x,a}\varphi(\omega)}{S_n^{x,a}\Psi(\omega)} \rightarrow V_\varphi^{x,a}(\omega)$$

for every $\omega \in \Omega_\varphi^{x,a}$. For the remaining $(x, a) \in \widehat{X}$, we set $\Omega_\varphi^{x,a} = \emptyset$.

(9.11) Proposition. *In the case when the extended SDS is conservative, assume (9.7). Let $\varphi \in \mathcal{C}_c^+(\widehat{X}^\ell)$ with $\ell \geq 1$. Then for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, \varphi) > 0$ with the following property.*

For all $(x, a), (y, b) \in \widehat{X}$ and any a.s. finite random time $\tau : \Omega \rightarrow \mathbb{N}_0$, one has on the set of all $\omega \in \Omega_\Phi^{x,a}$ with $\tau(\omega) < \infty$ and $|\log(A_{0,\tau}(\omega)a/b)| < \delta$ that

$$\limsup_{n \rightarrow \infty} \left| \frac{S_n^{x,a}\varphi}{S_n^{x,a}\Psi} - \frac{S_{\tau,n}^{y,b}\varphi}{S_{\tau,n}^{y,b}\Psi} \right| \leq \varepsilon W^{x,a},$$

where $W^{x,a} = V_\Phi^{x,a} + 1$.

Proof. Recall that Φ , Ψ , φ and ψ_r are also considered as functions on $X^{\mathbb{N}_0}$ via their extensions defined above.

Since Ψ is continuous and > 0 , there is $C = C_\varphi > 0$ such that $\varphi \leq C \cdot \Psi$. Also, there is some $r_0 \in \mathbb{N}$ such that the projection of $\text{supp}(\varphi)$ onto the first coordinate in \widehat{X} (i.e., the one with index 0) is contained in $\widehat{B}(r_0)$. We let $\varepsilon' = \min\{\varepsilon/2, \varepsilon/(2C), c_{r_0+1}\varepsilon/2, 1\}$, where c_{r_0+1} comes from the definition (9.3) of Φ and Ψ . Since φ is uniformly continuous, there is $\delta > 0$ with $2\delta \leq \varepsilon'$ such that

$$\begin{aligned} & |\varphi((x_0, a_0), \dots, (x_{\ell-1}, a_{\ell-1})) - \varphi((y_0, b_0), \dots, (y_{\ell-1}, b_{\ell-1}))| \leq \varepsilon' \\ & \text{whenever } \hat{d}((x_j, a_j), (y_j, b_j)) < 2\delta, \quad j = 0, \dots, \ell-1. \end{aligned}$$

We write

$$\left| \frac{S_n^{x,a}\varphi}{S_n^{x,a}\Psi} - \frac{S_{\tau,n}^{y,b}\varphi}{S_{\tau,n}^{y,b}\Psi} \right| \leq \underbrace{\frac{|S_n^{x,a}\varphi - S_{\tau,n}^{y,b}\varphi|}{S_n^{x,a}\Psi}}_{\text{Term 1}} + \underbrace{\frac{S_{\tau,n}^{y,b}\varphi}{S_{\tau,n}^{y,b}\Psi}}_{\leq C_\varphi} \underbrace{\frac{|S_n^{x,a}\Psi - S_{\tau,n}^{y,b}\Psi|}{S_n^{x,a}\Psi}}_{\text{Term 2}}.$$

We consider the random element $z = X_\tau^x$, so that $X_n^x = X_{\tau,n}^z$. Using the dilation invariance of hyperbolic metric,

$$\begin{aligned} \hat{d}(\widehat{X}_n^{x,a}, \widehat{X}_{\tau,n}^{y,b}) &= \theta(\mathbf{i} A_{0,n}a, d(X_{\tau,n}^z, X_{\tau,n}^y) + \mathbf{i} A_{\tau,n}b) \\ &= \theta(\mathbf{i} A_{0,\tau}a, D_{\tau,n}(z, y) + \mathbf{i} b) \leq |\log(A_{0,\tau}a/b)| + D_{\tau,n}(z, y) + \mathbf{i} b. \end{aligned}$$

By (9.7), for $\omega \in \Omega_{\Phi}^{x,a}$ with $\tau(\omega) < \infty$ there is a finite $\sigma(\omega) \geq \tau(\omega)$ in \mathbb{N} such that $\theta(\mathbf{i}a, D_{\tau,n}(z, y) + \mathbf{i}a) < \delta$ for all $n \geq \sigma(\omega)$. In the sequel, we assume that our $\omega \in \Omega_{\Phi}^{x,a}$ also satisfies $|\log(A_{0,\tau}(\omega)a/b)| < \delta$.

Now, we first bound the limsup of Term 1 by $\varepsilon/2$. If $n \geq \sigma$ and $|A_{0,\tau}(\omega)a/b| < \delta$, then we obtain that

$$|\varphi(\hat{X}_n^{x,a}, \hat{X}_{n+1}^{x,a}, \dots, \hat{X}_{n+\ell-1}^{x,a}) - \varphi(\hat{X}_{\tau,n}^{y,b}, \hat{X}_{\tau,n+1}^{y,b}, \dots, \hat{X}_{\tau,n+\ell-1}^{y,b})| < \varepsilon' \leq \varepsilon/2.$$

Suppose in addition that at least one of the two values $\varphi(\hat{X}_n^{x,a}, \hat{X}_{n+1}^{x,a}, \dots, \hat{X}_{n+\ell-1}^{x,a})$ or $\varphi(\hat{X}_{\tau,n}^{y,b}, \hat{X}_{\tau,n+1}^{y,b}, \dots, \hat{X}_{\tau,n+\ell-1}^{y,b})$ is positive. Then at least one of $\hat{X}_n^{x,a}$ or $\hat{X}_{\tau,n}^{y,b}$ belongs to $\hat{\mathbf{B}}(r_0)$, and by the above (since $\delta < 1$) both belong to $\hat{\mathbf{B}}(r_0 + 1)$. Thus, for $n \geq \sigma$,

$$\begin{aligned} |\varphi(\hat{X}_n^{x,a}, \hat{X}_{n+1}^{x,a}, \dots, \hat{X}_{n+\ell-1}^{x,a}) - \varphi(\hat{X}_{\tau,n}^{y,b}, \hat{X}_{\tau,n+1}^{y,b}, \dots, \hat{X}_{\tau,n+\ell-1}^{y,b})| &\leq \varepsilon' \psi_{r_0+1}(\hat{X}_n^{x,a}) \\ &\leq (\varepsilon/2) \Psi(\hat{X}_n^{x,a}). \end{aligned}$$

We get

$$\frac{|(S_n^{x,a}\varphi - S_{\sigma}^{x,a}\varphi) - (S_{\tau,n}^{y,b}\varphi - S_{\tau,\sigma}^{y,b}\varphi)|}{S_n^{x,a}\Psi - S_{\sigma}^{x,a}\Psi} \leq \varepsilon/2.$$

Since $S_n^{x,a}\Psi \rightarrow \infty$ almost surely, when passing to the limsup, we can omit all terms in the last inequality that contain a σ ; see Lemma 9.5. This yields the bound on the limsup of Term 1.

Next, we bound the limsup of Term 2 by $\varepsilon/2$. We start in the same way as above, replacing φ with an arbitrary one among the functions ψ_r and replacing ℓ with 1. Using the specific properties (9.2) of ψ_r (in particular, Lipschitz continuity with constant 1), and replacing $\hat{\mathbf{B}}(r_0)$ with $\hat{\mathbf{B}}(r+1) = \text{supp}(\psi_r)$, we arrive at the inequality

$$|\psi_r(\hat{X}_n^{x,a}) - \psi_r(\hat{X}_{\tau,n}^{y,b})| \leq \frac{\varepsilon}{2C} \psi_{r+2}(\hat{X}_n^{x,a}).$$

It holds for all $n \geq \sigma$, with probability 1. We deduce

$$|\Psi(\hat{X}_n^{x,a}) - \Psi(\hat{X}_{\tau,n}^{y,b})| \leq \frac{\varepsilon}{2C} \Phi(\hat{X}_n^{x,a})$$

and

$$\frac{|(S_n^{x,a}\Psi - S_{\sigma}^{x,a}\Psi) - (S_{\tau,n}^{y,b}\Psi - S_{\tau,\sigma}^{y,b}\Psi)|}{S_n^{x,a}\Psi - S_{\sigma}^{x,a}\Psi} \leq \frac{\varepsilon}{2C} \frac{S_n^{x,a}\Phi - S_{\sigma}^{x,a}\Phi}{S_n^{x,a}\Psi - S_{\sigma}^{x,a}\Psi}$$

Passing to the limsup as above, and using the Chacon-Ornstein theorem here, we get that the limsup of Term 2 is bounded almost surely by $\frac{\varepsilon}{2C} V_{\Phi}^{x,a}$. \square

In the sequel, when we sloppily say “for almost every $a > 0$ ”, we shall mean “for Lebesgue-almost every $a > 0$ ” in the non-lattice case, resp. “for every $a = e^{-\kappa m}$ ($m \in \mathbb{Z}$)” in the lattice case.

(9.12) Corollary. *Let $\varphi \in \mathcal{C}_c^+(\hat{\mathbf{X}}^{\ell})$ as above. For almost every $a > 0$, there is a set $\Omega_{\varphi}^a \subset \Omega_0$ with $\Pr(\Omega_{\varphi}^a) = 1$ such that for all $x, y \in \mathbf{X}$,*

$$V_{\varphi}^{x,a} = V_{\varphi}^{y,a} =: V_{\varphi}^a.$$

Proof. For almost every a , there is at least one $x_a \in \mathbf{X}$ such that $\Pr(\Omega_{\varphi}^{x_a,a}) = 1$. We can apply Proposition 9.11 with arbitrary $y \in \mathbf{X}$, $b = a$ and $\tau = 0$. Then we are allowed to take any $\varepsilon > 0$ and get that $V_{\varphi}^{x,a} = V_{\varphi}^{y,a}$ on $\Omega_{\varphi}^{x_a,a} \cap \Omega_{\Phi}^{x_a,a}$. \square

(9.13) Proposition. *Suppose that (6.1), (6.6), (6.8) and (9.7) hold, and that the extended SDS is conservative. Let $\varphi \in \mathcal{C}_c^+(\widehat{\mathbf{X}}^\ell)$, as above. Then for almost every $a > 0$, the random variable V_φ^a is almost surely constant (depending on φ and – so far – on a).*

Proof. Let a be such that $\Pr(\Omega_\varphi^a) = 1$, and choose $x = x_a$ as in the proof of Corollary 9.12.

For $s \in \mathbb{N}$, let $\varepsilon_s = 1/s$ and $\delta_s = \delta(\varepsilon_s, \varphi)$ according to Proposition 9.11. By our assumptions, $(A_{0,n})_{n \geq 1}$ is a topologically recurrent random walk on \mathbb{R}_*^+ , starting at 1. Choose $m \in \mathbb{N}$ and let $\tau_{m,s}$ be the m -th return time to the interval $(e^{-\delta_s}, e^{\delta_s})$. For every m and s , this is an almost surely finite stopping time, and we can find $\bar{\Omega}_\varphi^a \subset \Omega_\varphi^a \cap \Omega_\Phi^{x,a}$ with $\Pr(\bar{\Omega}_\varphi^a) = 1$ such that all $\tau_{m,s}$ are finite on that set.

We now apply Proposition 9.11 with $(y, b) = (x, a)$ and $\tau = \tau_{m,s}$. Then

$$\limsup_{n \rightarrow \infty} \left| V^a \varphi - \underbrace{\frac{S_{\tau,n}^{x,a} \varphi}{S_{\tau,n}^{x,a} \Psi}}_{=: U_{n,m,s}} \right| \leq \frac{1}{s} W^{x,a}.$$

Since our stopping time satisfies $\tau_{m,s} \geq m$, the random variable $U_{n,m,s}$ (depending also on φ and (x, a)) is independent of the basic random mappings F_1, \dots, F_m . (Recall that the F_k that appear in $S_{\tau,n}^{x,a}$ are such that $k \geq \tau + 1$.) We get

$$\lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} |V^a \varphi - U_{n,m,s}| = 0$$

on $\bar{\Omega}_\varphi^a$. Therefore also $V^a \varphi$ is independent of F_1, \dots, F_m . This holds for every m . By Kolmogorov's 0-1-law, $V^a \varphi$ is almost surely constant.

Note that in the lattice case, the proof simplifies, because we can just take the first return times of $A_{0,n}$ to 1. \square

(9.14) Theorem. *Given the random i.i.d. Lipschitz mappings F_n , let A_n and B_n be as in (6.2). Suppose that besides (6.1) and (6.6) [non-degeneracy] and (6.8) [moment conditions], also (9.7) holds, and that $\Pr[\hat{d}(\widehat{X}_n^{x,a}, \hat{o}) \rightarrow \infty] = 0$. Then the SDS induced by the F_n on \mathbf{X} has an invariant Radon measure ν that is unique up to multiplication with constants.*

Also, the shift \hat{T} on $(\widehat{\mathbf{X}}^{\mathbb{N}_0}, \mathfrak{B}(\widehat{\mathbf{X}}^{\mathbb{N}_0}), \Pr_\lambda)$ is ergodic, where λ is the extension of ν to $\widehat{\mathbf{X}}$ and \Pr_λ the associated measure on $\widehat{\mathbf{X}}^{\mathbb{N}_0}$.

Proof. Let $\varphi \in \mathcal{C}_c^+(\widehat{\mathbf{X}}^\ell)$. Recall that the function $\mathbf{v}_\varphi = \mathbf{E}_\lambda(\varphi | \mathfrak{I}) / \mathbf{E}_\lambda(\Psi | \mathfrak{I})$ on $\widehat{\mathbf{X}}^{\mathbb{N}_0}$ is \hat{T} -invariant. For the random variables $V_\varphi^{x,a} = V_\varphi^a$, this means that for almost every $a > 0$,

$$V_\varphi^a = V_\varphi^{A_{0,n}a} \quad \text{Pr-almost surely for all } n.$$

By Proposition 9.13, these random variables are constant on a set $\bar{\Omega}_\varphi^a \subset \Omega_\varphi^a$ with $\Pr(\bar{\Omega}_\varphi^a) = 1$. Fix one $a_0 > 0$ for which this holds.

In the lattice case, since we have chosen the maximal κ for which $\log A_n \in \kappa \cdot \mathbb{Z}$ a.s., the associated centered random walk $\log A_{0,n}$ is recurrent on $\kappa \cdot \mathbb{Z}$: for every starting point $a \in \exp(\kappa \cdot \mathbb{Z})$, we have that $(A_{0,n}a)_{n \geq 0}$ visits a_0 almost surely. We infer that $V_\varphi^a = V_\varphi^{a_0}$ Pr-almost surely for every $a \in \exp(\kappa \cdot \mathbb{Z})$.

In the non-lattice case, the multiplicative random walk $(A_{0,n}a)_{n \geq 0}$ starting at any $a > 0$ is topologically recurrent on \mathbb{R}_*^+ . This means that for every $a > 0$, with probability 1 there is a random sequence $(n_k)_{k \geq 0}$ such that $A_{0,n_k}a \rightarrow a_0$ as $k \rightarrow \infty$. Proposition 9.11 yields that $V_\varphi^a = V_\varphi^{a_0}$ on a set $\tilde{\Omega}_\varphi^a \subset \Omega_\varphi^{a_0}$ with probability 1.

Now let $\{a_k : k \in \mathbb{N}\}$ be dense in \mathbb{R}_*^+ and such that $\Pr(\tilde{\Omega}_\varphi^{a_k}) = 1$ for all \mathbb{N} . Using Proposition 9.11 once more, we get that for every $a > 0$, $V_\varphi^a = V_\varphi^{a_k} = V_\varphi^{a_0}$ on $\bigcap_k \tilde{\Omega}_\varphi^{a_k}$.

We conclude that \mathbf{v}_φ is constant \Pr_λ -almost surely.

This is true for any $\varphi \in \mathcal{C}_c^+(\hat{\mathbf{X}}^\ell)$. Therefore \hat{T} is ergodic. It follows that up to multiplication with constants, λ is the unique invariant measure on $\hat{\mathbf{X}}$ for the extended SDS, so that ν is the unique invariant measure on \mathbf{X} for the original SDS. By Lemma 6.9(b), $\text{supp}(\nu) = \mathbf{L}$. \square

We remark that by projecting, also the shift T on $(\mathbf{X}^{\mathbb{N}_0}, \mathfrak{B}(\mathbf{X}^{\mathbb{N}_0}), \Pr_\nu)$ is ergodic.

10. THE REFLECTED AFFINE STOCHASTIC RECURSION

We finally consider in detail the SDS of (1.1). Thus, $F_n(x) = |A_n x - B_n|$, so that $\mathfrak{l}(F_n) = A_n$ and $d(F_n(0), 0) = |B_n|$. We assume (6.1).

In the case when $\mathbb{E}(\log A_n) < 0$, we can once more apply Propositions 1.3, resp. 3.2, and Corollary 6.4.

(10.1) Corollary. *If $\mathbb{E}(\log^+ A_n) < \infty$ and $-\infty \leq \mathbb{E}(\log A_n) < 0$ then the reflected affine stochastic recursion is strongly contractive on \mathbb{R}^+ .*

If in addition $\mathbb{E}(\log^+ |B_n|) < \infty$ then it has a unique invariant probability measure ν on \mathbb{R}^+ , and it is (positive) recurrent on $\mathbf{L} = \text{supp}(\nu)$.

From now on, we shall be interested in the case when $\log A_n$ is centered.

For the time being, we shall only deal with the case when $B_n > 0$. We can use Remark 2.9; compare with the arguments used after Corollary 6.4. Thus, the reflected affine stochastic recursion is topologically irreducible on the set \mathbf{L} given by Corollary 2.8. Here, we shall not investigate the nature of \mathbf{L} in detail. It may be unbounded or compact.

Since we have $\mathbf{X} = \mathbb{R}^+$, the extended space $\hat{\mathbf{X}}$ is just the first quadrant with hyperbolic metric, and if $f(x) = |ax - b|$ then $\hat{f}(x, y) = (|ax - b|, ay)$. We can apply Corollary 7.6 to the extended process.

(10.2) Proposition. *Assume that (6.1) and (6.6) hold, $\mathbb{E}(|\log A_n|) < \infty$, $\mathbb{E}(\log A_n) = 0$, $B_n > 0$ almost surely, and $\mathbb{E}(\log^+ B_n) < \infty$.*

If the extended process $(\hat{X}_n^{x,a})$ is conservative, then the normalized distances $D_n(x, y)$ of (9.6) satisfy (9.7), that is, $\Pr[d(Z_n^x, Z_n^y) \rightarrow 0] = 1$ for all $x, y \in \mathbf{X}$, where $Z_n^x = X_n/A_{0,n}$.

Proof. We have the recursion $Z_0^x = x$ and $Z_n^x = |Z_{n-1}^x - B_n/A_{0,n}|$. We start with a simple exercise whose proof we omit. Let $c_j > 0$ and $f_j(x) = |x - c_j|$, $j = 1, \dots, s$. Then

$$(10.3) \quad f_s \circ \dots \circ f_2 \circ f_1(x) \leq \max\{c_1, \dots, c_s\} \quad \text{for all } x \in [0, c_1 + \dots + c_s].$$

We prove that for every $\varepsilon > 0$ and $M > 0$ there is N such that

$$\Pr(\Gamma_{M,N,\varepsilon}) > 0, \quad \text{where} \quad \Gamma_{M,N,\varepsilon} = [D_N(x, y) < \varepsilon \text{ for all } x, y \text{ with } 0 \leq x, y \leq M].$$

To show this, let μ be the probability measure on $\mathbb{R}_*^+ \times \mathbb{R}_*^+$ governing our SDS, that is, $\Pr[(A_k, B_k) \in U] = \mu(U)$ for any Borel set $U \subset \mathbb{R}_*^+ \times \mathbb{R}_*^+$. By our assumptions, there are $(a_1, b_1), (a_2, b_2) \in \text{supp}(\mu)$, such that $0 < a_1 < 1 < a_2$ and $b_1, b_2 > 0$. We choose $\Delta > 1$ such that $a_1 \Delta < 1 < a_2/\Delta$ and $b_* = \min\{b_1, b_2\}/\Delta > 0$, and we set $b^* = \max\{b_1, b_2\} \Delta$.

Let $r, s \in \mathbb{N}$. For $k = r + 1, \dots, r + s$, we recursively define indices $i(k) \in \{1, 2\}$ by

$$i(r+1) = 1, \quad i(k+1) = \begin{cases} 1, & \text{if } a_{i(r+1)} \cdots a_{i(k)} \geq 1, \\ 2, & \text{if } a_{i(r+1)} \cdots a_{i(k)} < 1. \end{cases}$$

Therefore $a_1 \leq a_{i(r+1)} \cdots a_{i(k)} \leq a_2$ for all $k > r$. We have

$$\begin{aligned} \Pr[a_2/\Delta^{1/r} \leq A_k \leq a_2 \Delta^{1/r} \text{ and } b_* \leq B_k \leq b^*] &> 0, \quad k = 1, \dots, r, \quad \text{and} \\ \Pr[a_{i(k)}/\Delta^{1/s} \leq A_k \leq a_{i(k)} \Delta^{1/s} \text{ and } b_* \leq B_k \leq b^*] &> 0, \quad k = r+1, \dots, r+s. \end{aligned}$$

Since the (A_k, B_k) are i.i.d., we also get that with positive probability,

$$\begin{aligned} \frac{a_2^k}{\Delta} &\leq A_{0,k} \leq a_2^k \Delta \quad \text{for } k = 1, \dots, r, \\ \frac{a_1}{\Delta} &\leq A_{r,r+j} \leq a_2 \Delta \quad \text{for } j = 1, \dots, s, \\ b_* &\leq B_k \leq b^* \quad \text{for } k = 1, \dots, r+s, \end{aligned}$$

and thus, again with positive probability,

$$(10.4) \quad \begin{aligned} \frac{B_k}{A_{0,k}} &\leq \frac{b^* \Delta^2}{a_2} \quad \text{for } k = 1, \dots, r \quad \text{and} \\ \frac{b_*}{a_2^{r+1} \Delta^2} &\leq \underbrace{\frac{B_{r+j}}{A_{0,r+j}}}_{=: c_j} \leq \frac{b^* \Delta^2}{a_1 a_2^r} \quad \text{for } j = 1, \dots, s. \end{aligned}$$

We now set $M' = b^* \Delta^2 / a_2$ and then choose r and s sufficiently large such that

$$\frac{b^* \Delta^2}{a_1 a_2^r} < \varepsilon \quad \text{and} \quad s \frac{b_*}{a_2^{r+1} \Delta^2} \geq M + M'.$$

We set $N = r + s$ and let $\Gamma_{M,N,\varepsilon}$ be the event on which the inequalities (10.4) hold. On $\Gamma_{M,N,\varepsilon}$, we can use (10.3) to get $Z_r^0 \leq M'$. Since $D_n(x, y)$ is decreasing in n , we have for $x \in [0, M]$ that $|Z_r^x - Z_r^0| \leq x \leq M$ and thus $\xi = Z_r^x \in [0, M + M']$. Now we can apply (10.3) with c_j as in (10.4) and obtain $\max_j c_j < \varepsilon$ and $c_1 + \dots + c_s \geq M + M'$. But for the associated mappings f_1, \dots, f_s according to (10.3), we have $Z_n^x = f_s \circ \dots \circ f_1(\xi)$. We see that on the event $\Gamma_{M,N,\varepsilon}$, one has $Z_n^x < \varepsilon$ for all $x \in [0, M]$, whence $D_N(x, y) < \varepsilon$ for all $x, y \in [0, M]$.

We can use Lemma 9.9 to conclude. \square

Combining the last proposition with theorems 8.4 and Theorem 9.14, we obtain the main result of this section.

(10.5) Theorem. *Consider the reflected affine stochastic recursion (1.1) with $A_n, B_n > 0$. Suppose*

(1) *non-degeneracy: $\Pr[A_n = 1] < 1$ and $\Pr[A_n x + B_n = x] < 1$ for all $x \in \mathbb{R}$*

(2) *moment conditions*: $E(|\log A_n|^2) < \infty$ and $E((\log^+ B_n)^{2+\varepsilon}) < \infty$ for some $\varepsilon > 0$

(3) *centered case*: $E(\log A_n) = 0$.

Then the SDS has a unique invariant Radon measure ν on \mathbb{R}^+ , it is topologically recurrent on $\mathbf{L} = \text{supp}(\nu)$. The time shift on the trajectory space $((\mathbb{R}^+)^{\mathbb{N}_0}, \text{Pr}_\nu)$ is ergodic.

We now answer the additional question when there is an invariant *probability* measure, i.e., when $\nu(\mathbf{L}) < \infty$.

(10.6) Theorem. *In the situation of Theorem 10.5, suppose also that $E(|\log A_n|^{2+\varepsilon}) < \infty$ and $\text{Pr}[B_n \geq b] = 1$ for some $b > 0$. Then we have $\nu(\mathbf{L}) < \infty$ if and only if the set \mathbf{L} is bounded.*

The proof will be based on the next proposition, which may be of interest in its own right.

(10.7) Proposition. *For any $x, t \geq 0$, let*

$$\tau_x^{[0,t)} = \inf\{n \geq 1 : X_n^x < t\}$$

be the time of the first visit in the interval $[0, t)$. Under the assumptions of Theorem 10.6, there is $x(t) > 0$ such that for all $x \geq x(t)$, one has

$$E(\tau_x^{[0,t)}) = \infty.$$

Proof. Consider the affine recursion without reflection $Y_n^x = A_n Y_{n-1}^x - B_n$. If $Y_k^x \geq t$ for $k = 1, \dots, n$ then $X_k^x = Y_k^x$ for those k , and then we have $\tau_x^{[0,t)} > n$. That is,

$$\text{Pr}[\tau_x^{[0,t)} > n] \geq \text{Pr}[Y_k^x \geq t, k = 1, \dots, n].$$

We have

$$(10.8) \quad Y_k^x \geq t \iff \underbrace{\sum_{j=1}^k \frac{B_j}{A_{0,j}}}_{\check{R}_k^0} + \frac{t}{A_{0,k}} \leq x.$$

Now consider the affine stochastic recursion generated by the inverses of the affine mappings $F_n(x) = A_n x - B_n$. These are

$$\check{F}_n(y) = \check{A}_n y + \check{B}_n, \quad \text{where} \quad \check{A}_n = 1/A_n \quad \text{and} \quad \check{B}_n = B_n/A_n.$$

They satisfy moment conditions of the same order as A_n , resp. B_n , so that the associated affine recursion (\check{Y}_n^y) is recurrent on the support of its unique invariant measure. Thus, there is $u > 0$ (sufficiently large) such that $\text{Pr}[\check{Y}_n^y \leq u \text{ infinitely often}] = 1$ for any starting point y . The right process induced by the \check{F}_n is $\check{R}_n^y = \check{F}_1 \circ \dots \circ \check{F}_n(y)$. It is not a Markov chain, but \check{R}_n^y has the same distribution as \check{Y}_n^y . In particular, \check{R}_k^0 appears above in (10.8), and

$$\sum_n \text{Pr}[\check{R}_n^0 \leq u] = \sum_n \text{Pr}[\check{Y}_n^0 \leq u] = \infty.$$

Now, if $\check{R}_n^0 \leq u$, then for $k = 1, \dots, n$,

$$\check{R}_k^0 + \frac{t}{A_{0,k}} \leq \check{R}_n^0 + \underbrace{\frac{B_k}{A_{0,k}}}_{\leq \check{R}_n^0} \frac{t}{B_k} \leq u(1 + t/b) =: x(t).$$

If $x \geq x(t)$ then we see that

$$\Pr[\check{R}_n^0 \leq u] \leq \Pr[Y_k^x \geq t, \ k = 1, \dots, n].$$

Therefore

$$\sum_n \Pr[\tau_x^{[0,t)} > n] \geq \sum_n \Pr[\check{R}_n^0 \leq u],$$

and the statement follows. \square

Proof of Theorem 10.6. Suppose that \mathbf{L} is unbounded. We use the distinction between positive and null recurrence as in Corollary 2.19. We fix a suitable $t > 0$ such that the interval $[0, t)$ intersects \mathbf{L} . We consider the probability measure $\nu_t = \frac{1}{\nu([0,t))} \nu|_{[0,t)}$ and the SDS $(X_n^{\nu_t})$ with initial distribution ν_t . We shall show that its return time $\tau^{[0,t)}$ to $[0, t)$ has infinite expectation. Then ν cannot be finite.

We know that there is $u \in \mathbf{L}$ with $u > x(t)$, with $x(t)$ as in Proposition 10.7. We let U be an open interval that contains u and does not intersect $[0, t]$. We apply Theorem 6.7 to a starting point $x_0 \in [0, t) \cap \mathbf{L}$. There is m such that $\Pr[X_m^{x_0} \in U] > 0$. This means that there are $f_1, \dots, f_m \in \text{supp}(\tilde{\mu})$ such that $f_m \circ \dots \circ f_1(x_0) \in U$. (Each f_j is of the form $f_k(x) = |a_j x - b_j|$.) There must be a maximal $k < m$ for which $x_k = f_k \circ \dots \circ f_1(0) \in [0, t]$. Note that $x_j \in \mathbf{L}$ for all j by Corollary 2.8, compare with Remark 6.5(b).

We now may assume without loss of generality that $k = 0$. Therefore we can find neighbourhoods (open intervals) $U_0, U_1, \dots, U_{m-1}, U_m = U$ of the respective x_j such that $U_0 \subset [0, t)$, while $U_j \cap [0, t) = \emptyset$ for $j > 0$, and

$$\tilde{\mu}(\{f : f(U_{j-1}) \subset U_j\}) > 0, \quad j = k+1, \dots, m.$$

This translates into

$$\Pr(\Lambda_x) \geq \alpha > 0 \quad \text{for all } x \in U_0, \quad \text{where } \Lambda_x = [X_j^x \in U_j, \ j = 1, \dots, m].$$

So we can now consider the SDS starting at $x \in U_0$, leaving $(0, t]$ at the first step, and reaching some $y \in U$ in m steps. After that, it takes $\tau_y^{[0,t)}$ steps to return to $(0, t]$. We formalize this, and remember that $U_j \cap \mathbf{L} \neq \emptyset$ for every j . Just for the purpose of the next lines, we consider the measure $\sigma_x(B) = \Pr(\Lambda_x \cap [X_m^x \in B])$, where $x \in U_0$. It is concentrated on U with $\sigma_x(U) \geq \alpha$, and

$$\begin{aligned} \mathbb{E}(\tau^{[0,t)}) &\geq \int_{U_0} \mathbb{E}(\tau_x^{[0,t)} \cdot \mathbf{1}_{\Lambda_x}) d\nu_t(x) \geq \int_{U_0} \underbrace{\left(\int_U (m + \mathbb{E}(\tau_y^{[0,t)})) d\sigma_x(y) \right)}_{=\infty \text{ by Proposition 10.7}} d\nu_t(x) = \infty. \end{aligned}$$

Therefore ν must have infinite mass. \square

We now discuss an example.

(10.9) Example. We let $0 < p < 1$ and

$$A_n = \begin{cases} 2 & \text{with probability } p, \\ 1/2 & \text{with probability } q = 1 - p, \end{cases} \quad B_n = 1 \text{ always.}$$

Thus, we randomly iterate the transformations $f_1(x) = |2x - 1|$ and $f_{-1}(x) = |x/2 - 1|$. In other words, $F_n(x) = |2^{\varepsilon_n}x - 1|$, where $(\varepsilon_n)_{n \geq 1}$ is a sequence of i.i.d. ± 1 -valued random variables with $\Pr[\varepsilon_n = 1] = p$ and $\Pr[\varepsilon_n = -1] = q$.

Keeping in mind Remark 6.5(b), we now determine \mathbf{L} as the smallest non-empty closed set which satisfies $f_{\pm 1}(\mathbf{L}) \subset \mathbf{L}$. First of all, we see that each of the two functions maps the interval $[0, 1]$ into itself. Thus, we must have $\mathbf{L} \subset [0, 1]$.

Let $\alpha = \max \mathbf{L}$. Then $\alpha \geq 2/3$, because $2/3 \in \mathbf{L}$ as the attracting fixed point of f_{-1} . We must have $(1 + \alpha)/2 = f_{-1} \circ f_1 \circ f_{-1}(\alpha) \in \mathbf{L}$, whence it is $\leq \alpha$. Therefore $\alpha = 1$. We get that $1 \in \mathbf{L}$. The set of all iterates of 1 under $f_{\pm 1}$ is

$$\{f_{i_1} \circ \cdots \circ f_{i_n}(1) : n \geq 0, i_j = \pm 1\} = \mathbb{D}, \quad \text{where } \mathbb{D} = \mathbb{Z}[\frac{1}{2}] \cap [0, 1],$$

and $\mathbb{Z}[\frac{1}{2}]$ stands for the dyadic rationals, i.e., rationals whose denominator is a power of 2. Since \mathbb{D} is dense, $\mathbf{L} = [0, 1]$.

Contractive case ($p < 1/2$). We can apply Corollary 10.1 and get a unique invariant probability measure ν , which is supported on $[0, 1]$.

Log-centered case ($p = 1/2$). Since \mathbf{L} is compact, the extended SDS is clearly conservative. In particular, $D_n(x, y) \rightarrow 0$ almost surely for all x, y . We now undertake an additional effort to clarify that the SDS is *not* locally contractive.

For the symmetric random walk $S_n = \varepsilon_1 + \cdots + \varepsilon_n$ on \mathbb{Z} , let $M_n = \max\{0, S_1, \dots, S_n\}$. Now consider our SDS $(X_n^x)_{n \geq 0}$ with $x \in [0, 1]$. It is an instructive exercise to prove the following by induction on n .

(10.10) Lemma. *The map $x \mapsto X_n^x$ is continuous and piecewise affine and continuous on $[0, 1]$, and there are random variables $\delta \in \{-1, 1\}$ and $C_j = C_{j, M_n} \in \mathbb{Z}[\frac{1}{2}]$ such that*

$$X_n^x = (-1)^j \delta 2^{S_n} x + C_j \quad \text{on } I_{j, M_n}, \quad \text{where } I_{j, k} = [(j-1)2^{-k}, j2^{-k}], \quad j = 1, \dots, 2^k.$$

In particular, the images of each of the intervals I_{j, M_n} under $x \mapsto X_n^x$ coincide and have the form

$$[(L_n - 1)/2^{M_n - S_n}, L_n/2^{M_n - S_n}],$$

where L_n is an integer random variable with $1 \leq L_n \leq 2^{M_n - S_n}$.

Recall the *strictly ascending ladder epochs* of the random walk (S_n) ,

$$\mathbf{t}(0) = 0 \quad \text{and} \quad \mathbf{t}(k+1) = \inf\{n > \mathbf{t}(k) : S_n > S_{\mathbf{t}(k)}\}.$$

They are all a.s. finite, and $S_{\mathbf{t}(k)} = M_{\mathbf{t}(k)} = k$. By Lemma 10.10, the image of each interval $I_{j, k}$ is the whole of $[0, 1]$. From this and the specific form that $x \mapsto X_n^x$ has to take, one sees that the only two choices for the mapping $x \mapsto X_{\mathbf{t}(k)}^x$ are

$$X_{\mathbf{t}(k)}^x = f_1^{(k)}(x) \quad \text{or} \quad X_{\mathbf{t}(k)}^x = 1 - f_1^{(k)}(x),$$

where $f^{(k)}$ denotes the k -th iterate of the function f . Therefore, considering the fixed points $x_0 = 1$ and $y_0 = 1/3$ of f_1 , we get

$$|X_{\mathbf{t}(k)}^{x_0} - X_{\mathbf{t}(k)}^{y_0}| = 2/3 \quad \text{for all } k.$$

Thus, we do not have local contractivity.

Expanding case ($p > 1/2$). Since \mathbb{L} is compact, the SDS is conservative for any value of p , so that there are always invariant probability measures. We show that in the expanding case, there are infinitely many mutually singular ones. Fix r , an odd prime or $r = 1$, and define

$$\mathbb{D}_r = \left\{ \frac{k}{r 2^n} : k, n \in \mathbb{N}_0, k \leq r 2^n, \text{lcd}(k, r 2^n) = 1 \right\}.$$

(Note that we must have $0 < k < r 2^n$ when $r > 1$.) Then it is easy to verify that $f_{\pm 1}(\mathbb{D}_r) \subset \mathbb{D}_r$. Thus, when we start at a point $x \in \mathbb{D}_r$, then (X_n^x) can be seen as a Markov chain on the denumerable state space \mathbb{D}_r . Let $p(x, y) = \Pr[X_1^x = y]$ denote its transition matrix. It is not hard to verify that it is irreducible (all states communicate), although we do not really need this. We partition $\mathbb{D}_r = \bigcup_n \mathbb{D}_{r,n}$, where $\mathbb{D}_{r,n}$ consists of all $\frac{k}{r 2^n}$ as above with the specific value of n . If $n \geq 1$, then we see that for each $x \in \mathbb{D}_{r,n}$, we have that

$$p(x, \mathbb{D}_{r,m}) = \sum_{y \in \mathbb{D}_{r,m}} p(x, y) = \begin{cases} p, & \text{if } m = n - 1, \\ q, & \text{if } m = n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

A similar identity for $x \in \mathbb{D}_{r,0}$ does not hold, so that we cannot define the factor chain on \mathbb{N}_0 . Nevertheless, since each $\mathbb{D}_{r,n}$ is finite, we can use comparison with the birth-and-death chain on \mathbb{N}_0 with transition probabilities $\bar{p}(n, n+1) = q$ and $\bar{p}(n, n-1) = p$ for $n \geq 1$. (We do not need to specify the outgoing probabilities at 0.) Thus, our Markov chain on \mathbb{D}_r is positive recurrent when $p > 1/2$, null recurrent when $p = 1/2$, and transient when $p < 1/2$. In particular, when $p > 1/2$, it has a unique invariant probability measure ν_r on the countable set \mathbb{D}_r . Since it is a probability measure, we can lift it to a Borel measure on $[0, 1]$ by setting $\nu_r(B) = \sum_{x \in \mathbb{D}_r \cap B} \nu_r(x)$. Thus, each ν_r is also an invariant probability measure for the (“topological”) SDS on $[0, 1]$, and all the ν_r are pairwise mutually singular.

(10.11) Remark. Regarding the last example, we underline that the respective discrete, denumerable Markov chains on \mathbb{D}_r have precisely the opposite behaviour of the SDS on $[0, 1]$: the Markov chain is transient precisely when the SDS is strongly contractive (and positive recurrent), and it is null recurrent precisely when the SDS is weakly, but not strongly contractive (and null-recurrent). But this fact should not be surprising. Indeed, let us compare this with the affine stochastic recursion $Y_n^x = 2^{L_n}x + B_n$, where (L_n, B_n) are 2-dimensional i.i.d. random variables with $L_n \in \mathbb{Z}$ and $B_n \in \mathbb{Z}[\frac{1}{2}]$. If the starting point x is also a dyadic rational, then we can consider (Y_n^x) as an SDS both on \mathbb{R} with Euclidean distance and on the field \mathbb{Q}_2 of dyadic numbers with the distance induced by the dyadic norm. Under the usual moment conditions, this SDS is transient on \mathbb{R} precisely when it is strongly contractive on \mathbb{Q}_2 , and weakly (but not strongly) contractive on \mathbb{R} precisely when it has the same property on \mathbb{Q}_2 .

In conclusion, we briefly touch another example, considering only the log-centered case.

(10.12) Example. We let $0 < p < 1$ and

$$A_n = \begin{cases} 3 & \text{with probability } 1/2, \\ 1/3 & \text{with probability } 1/2, \end{cases} \quad B_n = 1 \text{ always.}$$

This time, we randomly iterate $g_1(x) = |3x - 1|$ and $g_{-1}(x) = |x/3 - 1|$. A brief discussion shows that the limit set must be unbounded: suppose that $\alpha = \sup \mathbf{L} < \infty$. Then we must have $g_{i_n} \circ \cdots \circ g_{i_1}(\alpha) \in \mathbf{L}$ for any choice of n and $i_j \in \{-1, 1\}$ ($j = 1, \dots, n$). But for any α we can find some choice where $g_{i_n} \circ \cdots \circ g_{i_1}(\alpha) > \alpha$, a contradiction.

Thus, the invariant Radon measure has infinite mass.

A more detailed study of these and similar classes of reflected affine stochastic recursions are planned to be the subject of future work.

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