# PROOF OF THE ASM-DPP CONJECTURE

(PDF + Roger Behrend + Paul Zinn-Justin)

- Physical Combinatorics

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- Physical Combinatorics
- Descending Plane Partitions **DPP**

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- Alternating Sign Matrices

# PROOF OF THE ASM-DPP CONJECTURE

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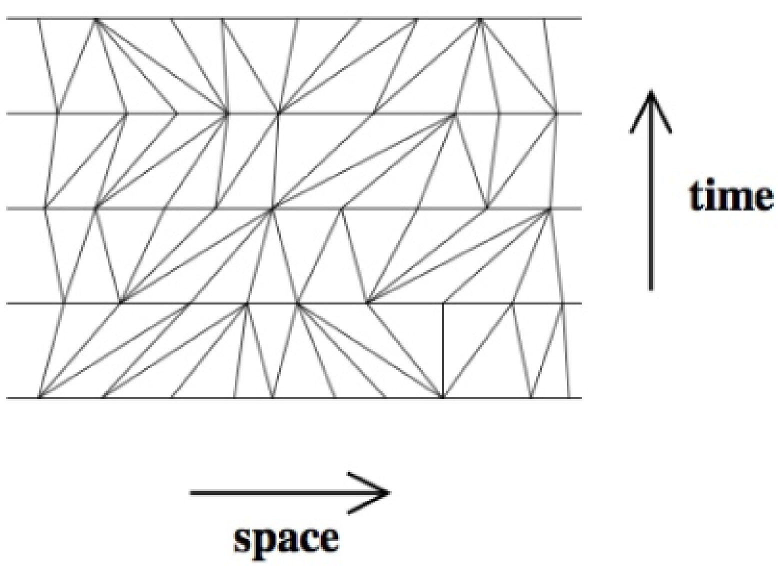
identity between refined enumerations

$$\det(DPP) = \det(ASM)$$

# DIGRESSION: 1+1 Dimensional Lorentzian quantum gravity

(PDF + Emmanuel Guitter + Charlotte Kristjansen '99)



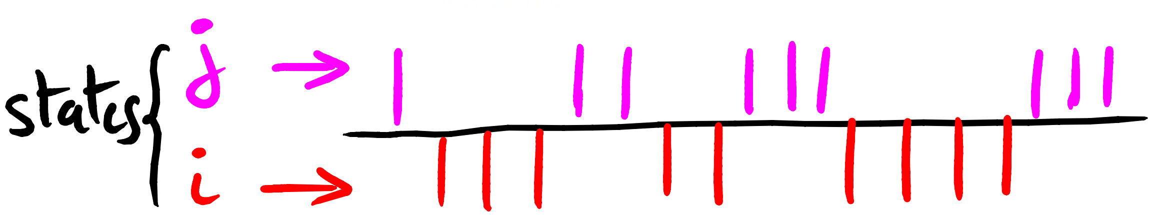

Triangulations that are

1. Random in space direction
2. Regular in time direction

$\Rightarrow$ TRANSFER MATRIX

$$\boxed{T_{ij} = \binom{i+j}{i}}$$

$(i,j \in \mathbb{Z}_+)$



Include $\left\{ \begin{array}{l} \text{curvature weight } a \ / \perp\!\!\perp \text{ or } \Pi \\ \text{area weight } g \ / \perp \text{ or } \top \end{array} \right.$

Then

$$\boxed{T_{i,j}(g,a) = (ag)^{i+j} \sum_{k=0}^{\mathrm{Min}(i,j)} \binom{i}{k}\binom{j}{k} a^{-2k}}$$

Generating Function

$$\boxed{\sum_{i,j\geq 0} z^i w^j T_{i,j}(g,a) = \frac{1}{1-ga(z+w)-g^2(1-a^2)zw}}$$

## Integrability

$$\boxed{\begin{gathered}\left[T(g,a)\,,\,T(g',a')\right]=0\\ \iff \quad \varphi(g,a)=\varphi(g',a')\\ \varphi(g,a)=\frac{1-g^2(1-a^2)}{ag}\qquad (=q+q^{-1})\end{gathered}}$$

END OF DIGRESSION

# DPP = Arrays of positive integers

$$\begin{array}{cccccc}
a_{11} & a_{12} & \cdots & & & a_{1,\mu_1} \\
 & a_{22} & a_{23} & \cdots & & a_{2,\mu_2} \\
 & & \ddots & & & \\
 & & & a_{r,r} & \cdots & a_{r,\mu_r}
\end{array}$$

1. $a \geq b$ $\longrightarrow$

2. $\begin{array}{c} a \\ \vee \\ b \end{array}$ $\downarrow$

3. $\lambda_i = \mu_i + i - 1$ = # parts in row $i$

   $\lambda_i < a_{i,i} \leq \lambda_{i-1}$

Vocabulary

- $a_{ij}$ = part
- $a_{ij} \leq j-i$ = special part
- order $n$ = $a_{ij} \leq n \ \forall ij$.

## OBSERVABLES

# parts $= n$

# special parts

# non-special parts

$n=3$

## 7 DPP's

$\emptyset$ $\quad \boxed{2}$ $\quad \boxed{3}$ $\quad \boxed{3\ 3}$ $\quad \boxed{3\ 2}$ $\quad \boxed{3\ 1}$ $\quad \begin{array}{|cc|} \hline 3 & 3 \\ \hline & \multicolumn{1}{|c|}{2} \\ \cline{2-2} \end{array}$

# DPP = Arrays of positive integers

$$\boxed{\begin{array}{llllll} a_{11} & a_{12} & \cdots & & & a_{1,\mu_1} \\ & a_{22} & a_{23} & \cdots & a_{2,\mu_2} & \\ & & \ddots & & & \\ & & & a_{r,r} & \cdots & a_{r,\mu_r} \end{array}}$$

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## OBSERVABLES

$\#$ parts $= n \longrightarrow 3$

$\#$ special parts $\longrightarrow$

$\#$ non-special parts

$n=3$

## 7 DPP's

$\emptyset$ $\quad \boxed{2}$ $\quad \boxed{3}$ $\quad \boxed{3\ 3}$ $\quad \boxed{3\ 2}$ $\quad \boxed{3\ \ }$ $\quad \begin{array}{|cc|}\hline 3 & 3 \\ \hline & \multicolumn{1}{|c|}{2} \\ \cline{2-2}\end{array}$

# ASM

- $n \times n$ matrices with elements $0, \pm 1$
- $\pm 1$ alternate along raws and columns
- raw and column sums $= 1$

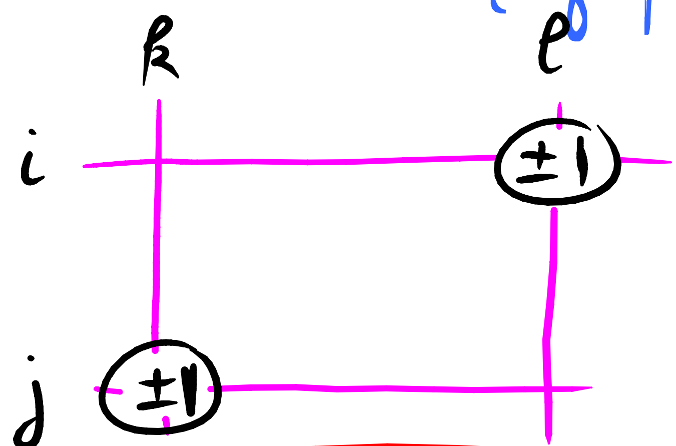
Generalize Permutation matrices ($\lambda$-determinant of Mills-Robbins-Rumsey)

$n=3$: 7 ASM's

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

# OBSERVABLES

- position of the 1 in the first row
- $\#(-1)$ number of $-1$'s
- inversion number (of permutations)




$$\boxed{\mathrm{inv}(A) = \sum_{i<j} \sum_{k<\ell} A_{i\ell} A_{jk}}$$

# THE ASM-DPP CONJECTURE

JOURNAL OF COMBINATORIAL THEORY, Series A 34, 340–359 (1983)

## Alternating Sign Matrices and Descending Plane Partitions

W. H. MILLS, DAVID P. ROBBINS, AND HOWARD RUMSEY, JR.

*Institute for Defense Analyses, Princeton, New Jersey 08540-3699*

*Communicated by the Managing Editors*

Received March 15, 1982

*Conjecture* 3. Suppose that $n, k, m, p$ are nonnegative integers, $1 \leqslant k \leqslant n$. Let $\mathscr{A}(n, k, m, p)$ be the set of alternating sign matrices such that

(i) the size of the matrix is $n \times n$,

(ii) the 1 in the top row occurs in position $k$,

(iii) the number of $-1$'s in the matrix is $m$,

(iv) the number of inversions in the matrix is $p$.

On the other hand, let $\mathscr{D}(n, k, m, p)$ be the set of descending plane partitions such that

(I) no parts exceed $n$,

(II) there are exactly $k-1$ parts equal to $n$,

(III) there are exactly $m$ special parts,

(IV) there are a total of $p$ parts.

Then $\mathscr{A}(n, k, m, p)$ and $\mathscr{D}(n, k, m, p)$ have the same cardinality.

ASM $(A(n))$
- $n$ = size
- position top 1
- # $-1$'s
- # inversions

DPP $(D(n))$
- $n$ = order
- # parts $= n$
- # special parts
- # parts

# Counting

- DPP : Andrews '79 $\rightarrow$ formula $D(n)$

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- ASM : Zeilberger '96 $A(n) = \mathrm{TSSCPP}(n) = D(n)$

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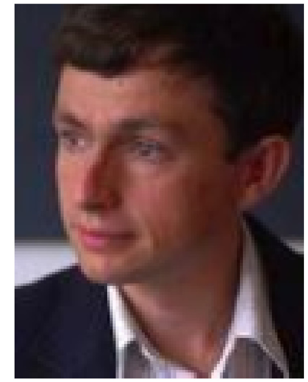


- DPP : Andrews '79 $\to$ formula $D(n)$
- ASM : Zeilberger '96   $A(n) = \mathrm{TSSCPP}(n) = D(n)$
  Kuperberg '96   6 vertex model

$\uparrow$

Izergin–Korepin integrable lattice model

Refinements ?

# Computation of the refined numbers

Strategy: 1. reformulate in terms of known (and manageable objects)

- DPP → lattice paths (lattice fermions)
- ASM → 6 vertex model (integrable lattice model)

2. write refined generating functions as determinants

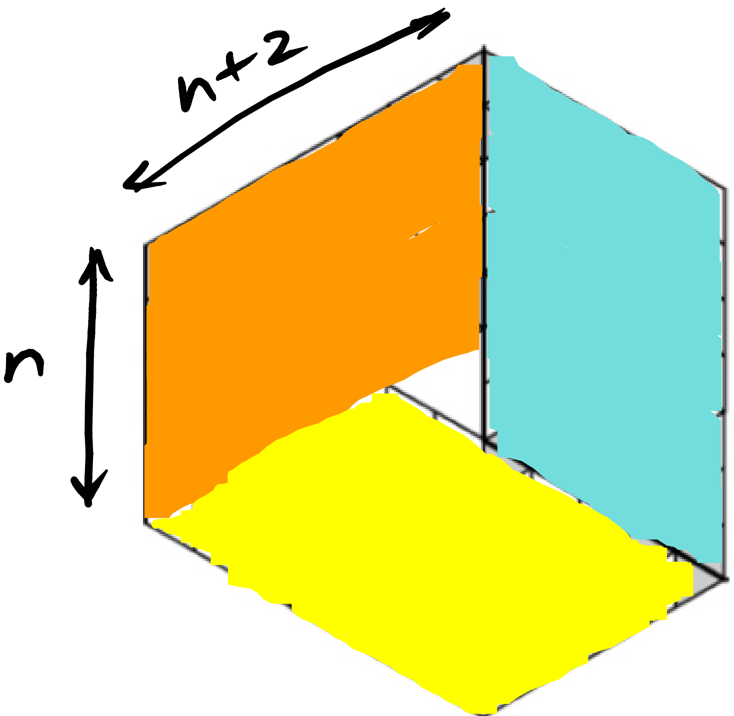
3. Prove identity between determinants

$\boxed{\text{DPP}}$

# From DPP to Lattice Paths

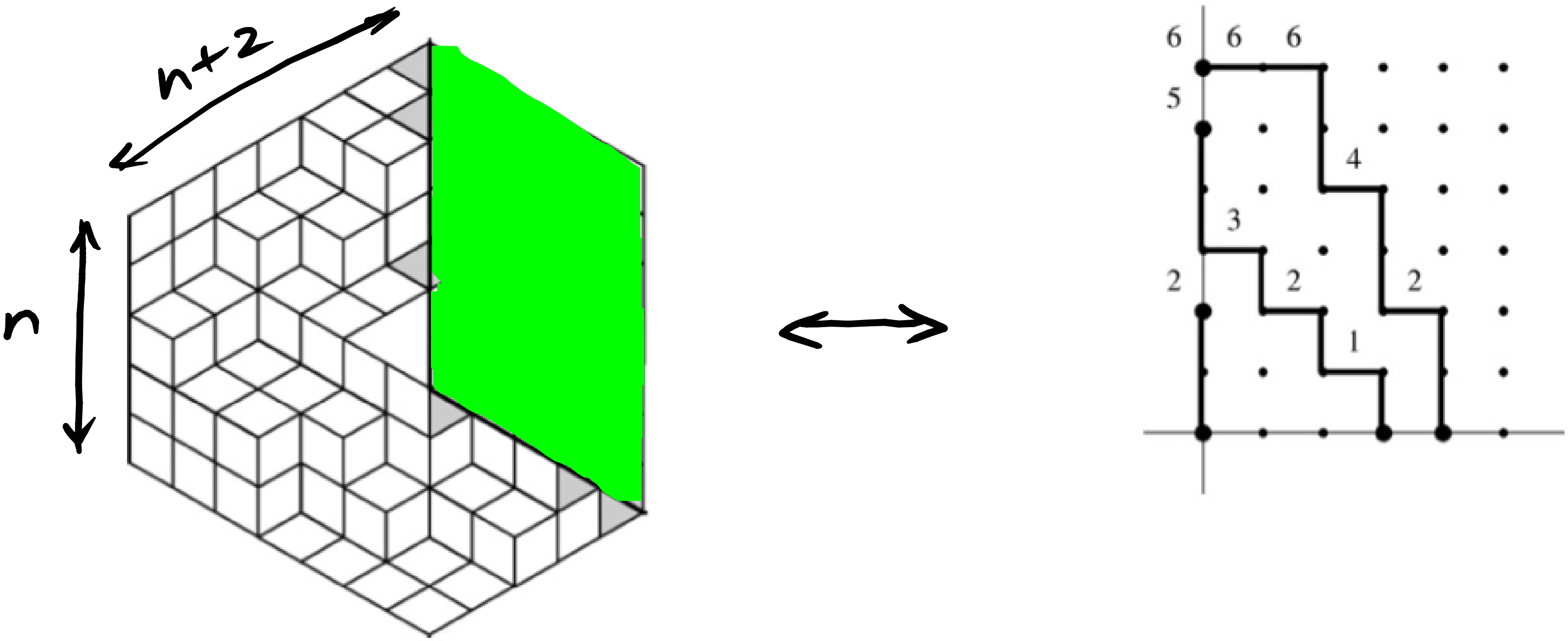
Lalonde '03

(Krattenthaler '06)




Cyclically symmetric Rhombus tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with $\Delta$ hole $\longleftrightarrow$ Lattice Paths

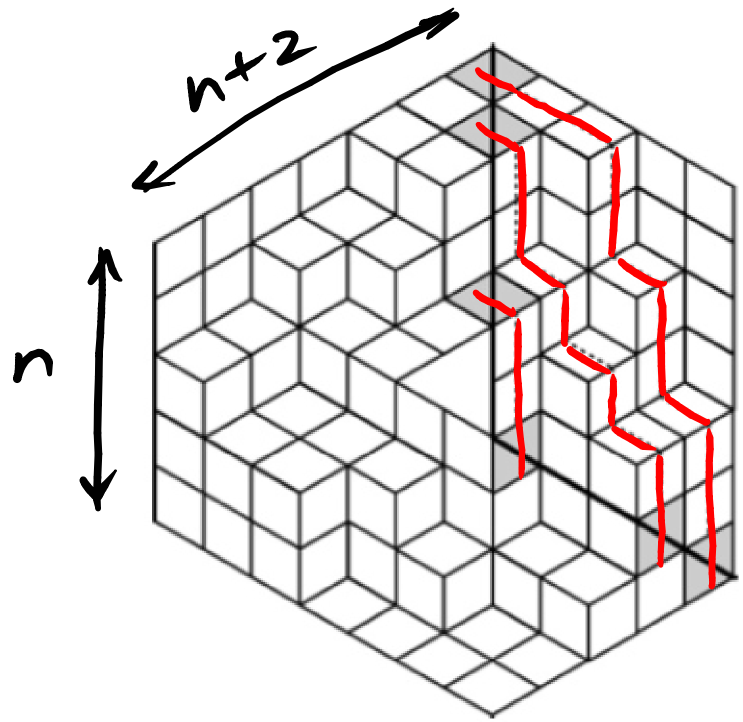
# From DPP to Lattice Paths



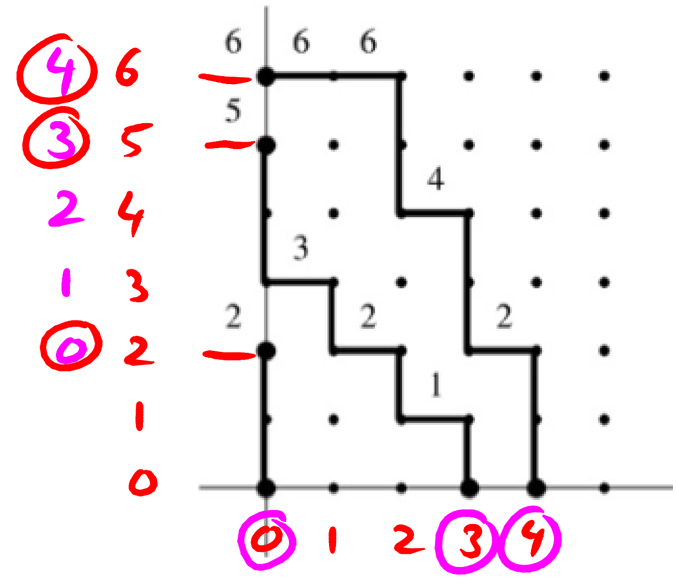

Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with $\Delta$ hole $\leftrightarrow$ Lattice Paths

# From DPP to Lattice Paths

(Krattenthaler '06)

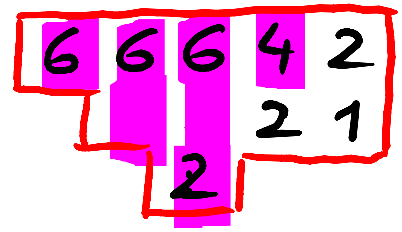


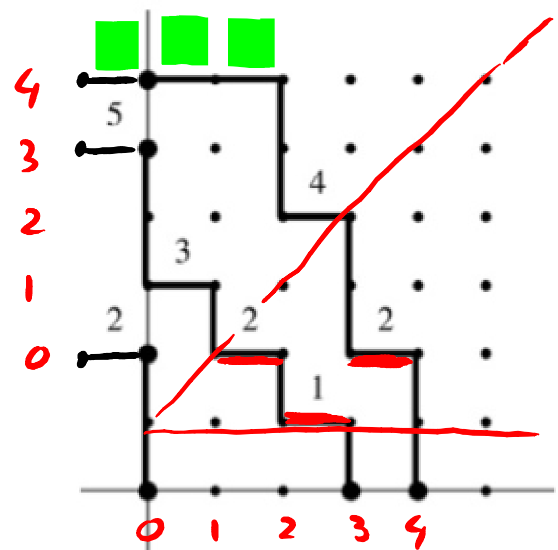

$\longleftrightarrow$



| | | | | |
|---|---|---|---|---|
| 6 | 6 | 6 | 4 | 2 |
| | 5 | 3 | 2 | 1 |
| | | 2 | | |

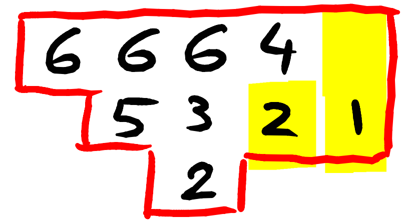
Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with $\Delta$ hole $\longleftrightarrow$ Lattice Paths horizontal steps — = non-special parts →




■ = parts = n

horizontal steps — = special parts →




} horizontal steps here do not count

**Lemma** $\det\limits_{n\times n}(I+M) = \sum\limits_{1\le i_1<\dots<i_k\le n} \det\left(M^{i_1\dots i_k}_{i_1\dots i_k}\right)$

M's minor with rows $i_1 \dots i_k$, cols $i_1 \dots i_k$

Here: $M_{ij}$ = Part. fctn (path $(i,0)\to(0,j)$)

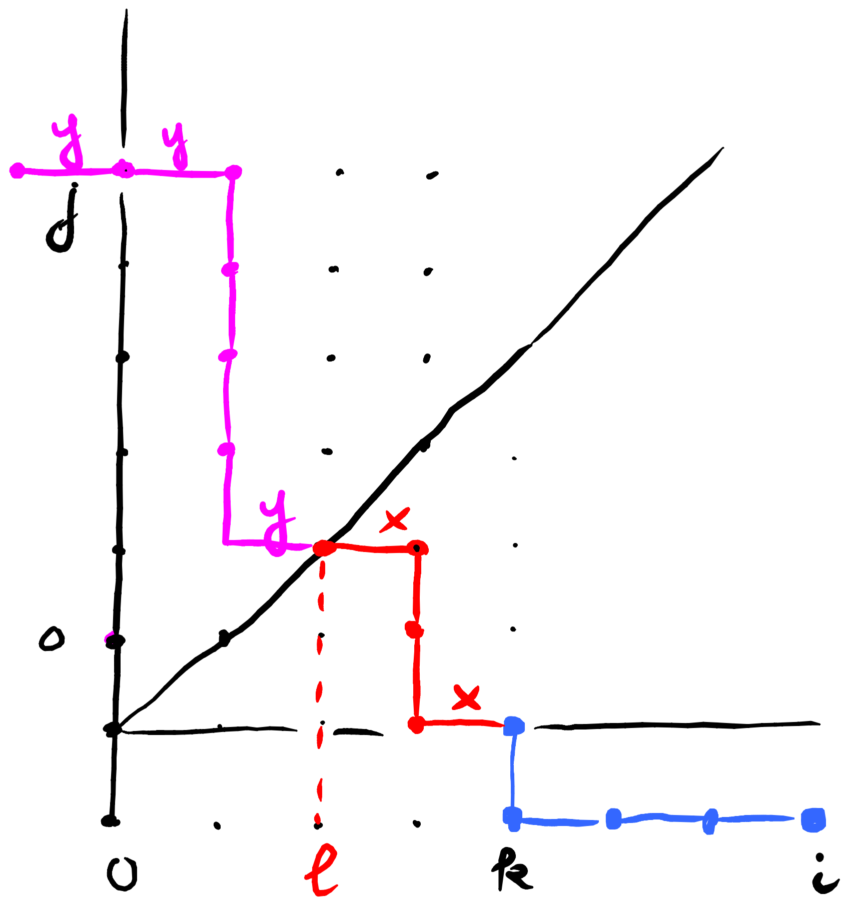
- By Gessel–Viennot theorem

$\det\left(M^{i_1\ldots i_k}_{i_1\ldots i_k}\right)$ = Partition fctn for families of $k$ non-intersecting paths starting at $(i_1,0)\ldots(i_k,0)$, ending at $(0,i_1)\cdots(0,i_k)$

- $M_{ij}$ = Partition Function for 1 path:

$$M_{ij} = \sum_{\text{paths } (i0)\to(0j)} x^{\#(-)}\, y^{\#(-)} \Big( \underset{\substack{\shortparallel \\ 1 \text{ for simplicity}}}{z}^{\#(\overset{n}{-})} \Big)$$

horizontal steps: lower wedge    upper wedge




$$Z^{(n)}_{DPP}(x, y) = \det(I + M)$$

↑ per special part ↑ per non-special part

$$M_{i,j} = \sum_{k=0}^{i} \sum_{\ell \geq 0} \binom{k}{\ell} x^{k-\ell} \binom{j+1}{\ell} y^{\ell+1}$$

## Generating function :

$$f_{DPP}(z,w) = \sum_{i,j \geq 0} z^i w^j \, (I + M)_{i,j}$$

**THM**

$$f_{DPP}(z,w) = \frac{1}{1 - zw} + \frac{1}{1-z} \, \frac{yz}{1 - xz - w - (y-x)zw}$$

weights : $x$ / special part $y$ / non-special part

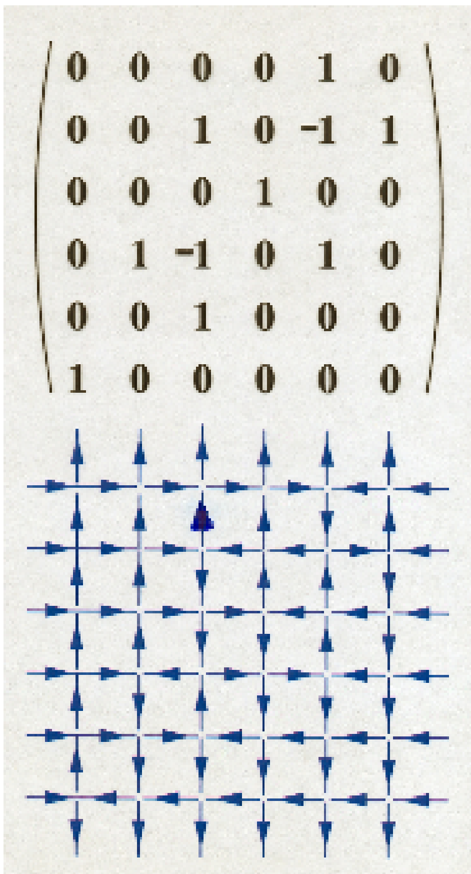
ASM

# From ASM to 6 Vertex model with Domain Wall Boundary conditions (Kuperberg)

$n \times n$ ASM

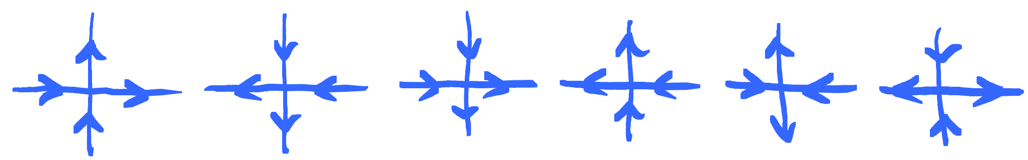
$\updownarrow$

6V + DWBC on $n \times n$ grid




Bijection:

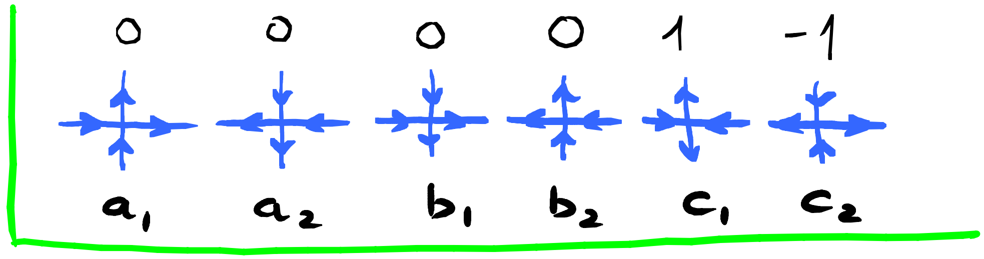
$$0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1$$



$$\underbrace{a_1 \quad a_2}_{qz - q^{-1}w} \qquad \underbrace{b_1 \quad b_2}_{q^{-1}z - qw} \qquad \underbrace{c_1 \quad c_2}_{(q^2 - q^{-2})\sqrt{zw}}$$

(integrable weights)

# Refinements :

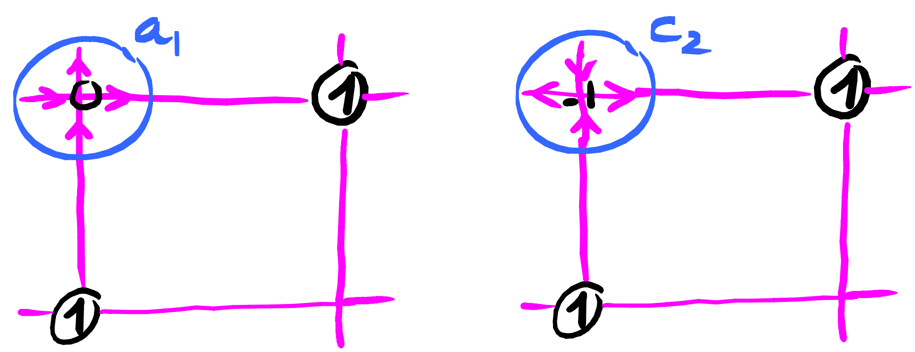



by symmetry :

$$\begin{cases} N_{a_1} = N_{a_2} = \frac{N_a}{2} & N_{b_1} = N_{b_2} = \frac{N_b}{2} \\ N_{c_1} = N_{c_2} + n & N_c = N_{c_1} + N_{c_2} \end{cases}$$

- $\boxed{\#(-1)} = N_{c_2} = \boxed{\frac{N_c - n}{2}}$
- $Inv(A) = N_{a_1} + N_{c_2}$

$$\Rightarrow \boxed{Inv(A) - \#(-1) = N_{a_1} = \frac{N_a}{2}}$$



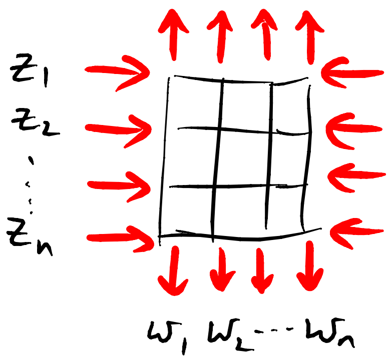
# Partition function

$$Z^{(n)}_{ASM}(x,y,z) = \sum_{\substack{\text{configs of } n\times n \\ 6V\ DWBC}} x^{\frac{N_c-n}{2}}\, y^{\frac{N_a}{2}}\, \underset{\substack{\shortparallel \\ 1}}{z}^{N_{a'}}$$

usually, one considers $Z^{(n)}_{6V}(a,b,c) = \sum_{\substack{\text{configs} \\ 6V\ DWBC}} a^{N_a}\, b^{N_b}\, c^{N_c} \times \underbrace{\left(\frac{a'}{a}\frac{b}{b'}\right)^{N_{a'}}}_{\substack{\shortparallel \\ 1}}$

$$Z^{(n)}_{6V}(a,b,c) = b^{n^2}\left(\frac{c}{b}\right)^{n} \sum \left(\frac{c}{b}\right)^{N_c-n} \left(\frac{a}{b}\right)^{N_a}$$

$$\Longleftrightarrow \boxed{x = \left(\frac{c}{b}\right)^2 \quad y = \left(\frac{a}{b}\right)^2}$$

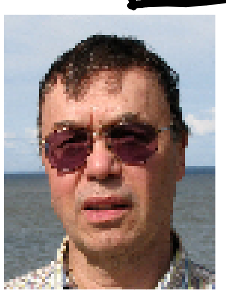



## Partition function of 6V+DWBC

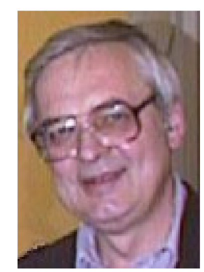
$$Z_n = \frac{\displaystyle\sum_{\substack{\text{configs}\\ \text{on grid}}} \prod_{\substack{\text{vertices}\\ (ij)}} \text{weights}(z_i,w_j)}{\displaystyle\prod_i c(z_i,w_i)}$$

**THM**

$$Z_n = \frac{\displaystyle\prod_{i,j} a(z_i,w_j)\, b(z_i,w_j)}{\displaystyle\prod_{i<j} (z_i-z_j)(w_i-w_j)} \det\left\{ \frac{1}{a(z_i,w_j)\, b(z_i,w_j)} \right\}_{1\le i,j\le n}$$



(Korepin – Izergin)



recursion relation + symmetries (from commutation of Transfer matrices).

Homogeneous limit: $\begin{cases} z_i \longrightarrow r & \forall i \\ w_j \longrightarrow r^{-1} & \forall j \end{cases}$

$$a(z_i, w_j) \longrightarrow qr - q^{-1}r^{-1} = a(r, r^{-1})$$

$$b(z_i, w_j) \longrightarrow q^{-1}r - qr^{-1} = b(r, r^{-1})$$

$$c(z_i, w_j) \longrightarrow q^2 - q^{-2} = c(r, r^{-1})$$

$$\boxed{Z_n(q,r) = \frac{(ab)^{n^2}}{c^n} \det_{0\le i,j\le n-1}\left( \left.\frac{\left(\frac{d}{du}\right)^i}{i!} \frac{\left(\frac{d}{dv}\right)^j}{j!}\left[\frac{c(u^{-1},v)}{a(u^{-1},v)\, b(u^{-1},v)}\right]\right|_{\substack{u=r \\ v=r^{-1}}} \right)}$$

by Taylor expansion around the limit

Note: $\dfrac{c}{a(u^{-1}v)\, b(u^{-1}v)} = \dfrac{1}{uv - q^{-2}} - \dfrac{1}{uv - q^{2}}$

Taylor-expand:

Define: $(A_+)_{i,j} = \dfrac{\left(\frac{d}{du}\right)^i}{i!} \dfrac{\left(\frac{d}{dv}\right)^j}{j!} \dfrac{1}{uv - q^2}\Big|_{\substack{u = r^{-1} \\ v = r^{-1}}}$ $\quad (A_-)_{i,j} = \text{idem } (q \to q^{-1})$

Introduce upper triangular matrix $U(\alpha, \beta)$

$$U(\alpha,\beta)_{i,j} = \begin{cases} \binom{j}{i} \alpha^i \beta^j & \text{if } i \le j \\ 0 & \text{if } i > j \end{cases} \qquad \begin{pmatrix} 1 & \beta & \beta^2 & \beta^3 & \cdots \\ 0 & \alpha\beta & 2\alpha\beta^2 & 3\alpha\beta^3 & \cdots \\ 0 & 0 & \alpha^2\beta^2 & 3\alpha^2\beta^3 \\ 0 & 0 & 0 & \alpha^3\beta^3 \\ \vdots & & & \ddots & \ddots \end{pmatrix}$$

$i, j \in \mathbb{Z}_+ \qquad \alpha, \beta \in \mathbb{C}^*$

THM

$$A_+ = -\frac{1}{r^{-2}-q^2}\left[U^t(\alpha,\beta)\right]^{-1} \bar{U}(\alpha',\beta')$$

with: $\alpha = \frac{1-q^2r^2}{r}$ $\beta = \frac{q^2-q^{-2}}{r^2-q^2}$ $\alpha' = -q^2r^2\beta$ $\beta' = -\frac{1}{\alpha}$

Proof:

1. generating function $(\bar{U}(\alpha\beta)) = \frac{1}{1-\beta w(1+\alpha z)}$
2. $U(\alpha,\beta)^{-1} = U(-1/\beta, -1/\alpha)$
3. generating function $(A_+) = \frac{1}{(r^{-1}+z)(r^{-1}+w)-q^2}$

Holds true for any finite truncation to $i,j \in [0,n-1]$

set $\quad V = U^t(\alpha, \beta) \qquad \bar{U} = U(\alpha', \beta')$

$\qquad \bar{V} = V(q \to q^{-1}) \qquad \bar{\bar{U}} = U(q \to q^{-1})$

then:
$$\boxed{\begin{aligned} A_+ &= \frac{1}{q^2 - r^{-2}} V^{-1} U \\ A_- &= \frac{1}{q^{-2} - r^{-2}} \bar{V}^{-1} \bar{U} \end{aligned}}$$

$$\det(A_- - A_+) = \det(A_-) \det\left[ I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} \bar{U}^{-1} \bar{V} V^{-1} U \right]$$

$$\propto \det\left[ I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} (\bar{V} V^{-1})(U \bar{U}^{-1}) \right]$$

$U \bar{U}^{-1} = U(-1, 1)$

$\bar{V} V^{-1} = U^t(-y, x)$

Collecting all prefactors, we get:

$$Z^{(n)}_{ASM}(x,y) = \det\big((1-\nu)I + \nu G\big)$$

$$\nu = \frac{r^{-2}-q^{-2}}{q^{2}-q^{-2}} \qquad G = U^{t}(-y,x)\,U(-1,1)$$

$$G_{ij} = \sum_{k\geq 0} \binom{i}{k} y^{k} \binom{j}{k} x^{i-k}$$

$\Rightarrow$ generating function of $(1-\nu)I+\nu G$ $= f_{ASM}(z,w)$

THM

$$\boxed{f_{ASM}(z,w) = \frac{1-\nu}{1-zw} + \frac{\nu}{1-zx-w-(y-x)zw}}$$

Final identity:

$$\boxed{Z^{(n)}_{DPP} = \det(I+M) = \det((1-v)I + vG) = Z^{(n)}_{ASM}}$$

**Proof:**

note that

$$(1-z)(1-(1-v)w)\, f_{DPP}(z,w) - \left(1-\frac{z}{1-v}\right)(1-w)\, f_{ASM}(z,w)$$

$$= \Big(\underbrace{x\, v(1-v) - y(1-v) - v}_{=0}\Big) \times \text{rational fraction}(z,w)$$

$$\left( x = \left(\frac{q^2-q^{-2}}{q^{-1}r-qr^{-1}}\right)^2,\ y = \left(\frac{qr-q^{-1}r^{-1}}{q^{-1}r-qr^{-1}}\right)^2,\ v = \left(\frac{r^{-2}-q^{-2}}{q^2-q^{-2}}\right),\ 1-v = \left(\frac{q^2-r^{-2}}{q^2-q^{-2}}\right) \right)$$

+ **Remark**: let $A = (A_{ij})_{i,j\geq 0}$ $F = \sum_{i,j\geq 0} A_{ij} z^i w^j$

then $\boxed{(1-\lambda z)(1-\mu w)\, F(z,w)}$ is the generating function for $(I-\lambda S)\, A\, (I-\mu S^t)$ where

$S_{i,j} = \delta_{i,j+1}$ "shift" matrix strictly lower triangular $\Rightarrow$ the determinant is unchanged.

Proof completed!

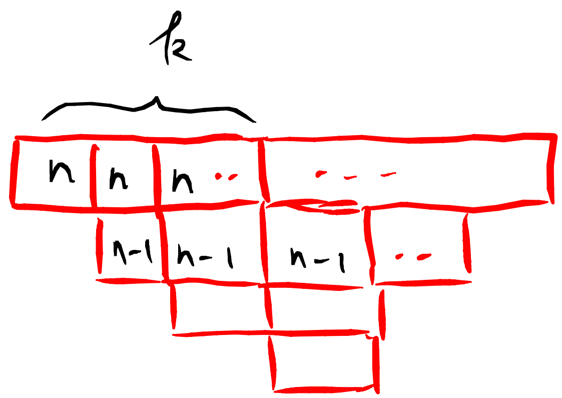
# CONCLUSION

$\boxed{\text{MRR PROVED}}$ → method of generating fctns for the matrix of which we take the det

→ Bijection ASM – DPP ? (– TSSCPP ?)
(– O(n) ?)

→ More refinements ?

$$\text{yes: } \begin{array}{c} k \\ \left(\begin{array}{c} \rightarrow 1 \\ \text{ASM} \\ 1 \leftarrow \end{array}\right) \\ e \end{array}$$




use Lewis Carroll identity (in progress)

→ Generalizations : DPP with symmetries

⟷ ASM with symmetries

→ q-deformation : $|D| = \sum a_{ij}$ for a DPP

$$\boxed{\sum_{D \in DPP(n)} q^{|D|} = \prod_{j=0}^{n-1} \frac{(3j+1)!_q}{(n+j)!_q}}$$

$[j]_q = \frac{1-q^j}{1-q}$

$j!_q = [1]_q [2]_q \cdots [j]_q$

pb = q-enumeration of ASMs?

→ Razumov-Stroganov for DPP?