Growth of Self-Similar Graphs

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Abstract: Locally finite self-similar graphs with bounded geometry and without bounded geometry as well as non-locally finite self-similar graphs are characterized by the structure of their cell graphs. Geometric properties concerning the volume growth and distances in cell graphs are discussed. The length scaling factor ν and the volume scaling factor μ can be defined similarly to the corresponding parameters of continuous self-similar sets. There are different notions of growth dimensions of graphs. For a rather general class of self-similar graphs, it is proved that all these dimensions coincide and that they can be calculated in the same way as the Hausdorff dimension of continuous self-similar fractals: $\dim X = \frac{\log \mu}{\log \nu}$. © 2004 Wiley Periodicals, Inc. J Graph Theory 45: 224–239, 2004

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1. INTRODUCTION

Self-similar sets are introduced in various ways. Usually they are defined as compact invariant sets of iterated function systems, see Hutchinson [6]. They are studied under different assumptions concerning their symmetries and the structure of the underlying space. Most important are the notions of nested fractals, see Lindstrøm [13], and post-critically finite self-similar sets, see Kigami [8] and the recent book [9].

Self-similar graphs can be seen as discrete versions of these self-similar sets. There exists a lot of literature on different examples of self-similar graphs. Especially the random walk on the Sierpiński graph was studied extensively, see [2], [5], and [7]. General connections between the volume growth and the transition probabilities of random walks were studied by Coulhon and Grigor'yan in [3]. Telcs studied connections between the growth dimension (also, fractal dimension), the random walk dimension, and the resistance dimension in [17], [18], and [19]. Barlow, Coulhon, and Grigor'yan used the growth dimension to give upper bounds for the heat kernel on graphs and manifolds in [1].

For a good introduction to the growth of finitely generated groups, the reader is referred to the book of de la Harpe, see [4].

One can define self-similarity of graphs, without using a given self-similar set, which is embedded into a complete metric space. A first axiomatic definition was stated by Malozemov and Teplyaev in [15]. Their graphs correspond to fractals such that the boundaries of their *cells,* see [13], contain exactly two points. With an axiomatic approach the author introduced the class of *symmetrically self-similar* graphs in [10]. In both papers, [10] and [15], the spectrum of the discrete Laplacian is studied. A similar approach to general self-similar graphs was chosen by Malozemov and Teplyaev in [16]. The definition of symmetrically self-similar graphs in [16] uses a stronger symmetry-condition than the definition in [10] or [12]. In [12], Teufl and the author calculated the asymptotic behavior of the transition probabilities of the simple random walk on symmetrically self-similar graphs. They generalized results of Grabner and Woess in [5] from the Sierpiński graph to these graphs.

Up to now, the class of symmetrically self-similar graphs is the biggest class of self-similar graphs where the simple random walk and consequently the Green functions as well as the spectrum of the Laplacian are understood well, see [10] and [12]. The class of graphs discussed in the present note contains the class of symmetrically self-similar graphs. Several results (e.g., Theorems 3.2 and 3.3 and Corollary 3.1) are relevant to these analytic studies.

After defining general self-similarity in Section 2, we reformulate the fixed point theorem for self-similar graphs, see Theorem 1 in [10]. This theorem can be interpreted as a graph theoretic analog to the Banach fixed-point theorem. For the more special class of homogeneously self-similar graphs, see Definition 2.2, we discuss some basic geometric properties concerning the so-called *n-cells*, see

Lemma 2.1. These n-cells correspond to n-cells and n-complexes in the sense of Lindstrøm, see [13].

Self-similar graphs of bounded geometry (the set of vertex degrees is bounded) correspond to finitely ramified fractals. In Section 3 it is proved that for homogeneously self-similar graphs having a *constant inner degree* (see Definition 3.1) there is a simple geometric equality relation between parameters, defined by the geometry of the graph, which is satisfied if and only if the graph has bounded geometry. Example 3.1 shows that in general this is not true for graphs without constant inner degree. The number of edges in the boundary of an *n-cell* is calculated explicitly. We give an example of a locally finite, homogeneously self-similar graph with constant inner degree and unbounded geometry.

Some basic properties of different growth dimensions are discussed in Section 4.

In Section 5 the diameter of the boundary of an n-cell in a homogeneously self-similar graph is computed. We give upper and lower bounds for the maximal distance between the boundary and vertices in the n-cell and bounds for the diameter of the whole n-cell. It is proved that for homogeneously self-similar graphs with bounded geometry, all growth dimensions can be computed by the same formula as the Hausdorff dimension of self-similar sets, which satisfy the open set, condition, namely,

$$\dim X = \frac{\log \mu}{\log \nu},$$

see, Hutchinson [6]. Here the *length scaling factor* ν is the diameter of the boundary of a 1-cell, and the *volume scaling factor* μ is the number of 1-cells, which are contained in a 2-cell. The result also holds if the diameter of a cell is greater than the length scaling factor ν.

2. SELF-SIMILAR GRAPHS

Graphs $X = (VX, EX)$ with vertex set VX and edge set EX are always connected, locally finite, infinite, without loops or multiple edges. We write $\deg_X x$ for the *degree* of a vertex x, which is number of vertices in VX being adjacent to x in X. A *path of length n* from x to y is an $(n+1)$-tuple of vertices

$$(z_0 = x, z_1, \ldots, z_n = y)$$

such that z_{i-1} is adjacent to z_i for $0 \leq i \leq n$. The distance $d_X(x, y)$ is the length of a shortest path from x to y. A path from x to y is *geodesic* if its length is $d_X(x, y)$. The *vertex boundary* or *boundary* θC of a set C of vertices in VX is the set of vertices in $VX \backslash C$ being adjacent to some vertex in C. The *closure* of C is defined as $\overline{C} = C \cup \theta C$. Let us write $\hat{C}$ for the subgraph of X, which is spanned by the closure of C. We call C *connected* if every pair of vertices in C can be connected by a path in X that does not leave C. The set of edges δC, which connect a vertex in C with a vertex in $VX \backslash C$, is the *edge boundary* of C.

For the convenience of the reader, we briefly repeat the definition of self-similar graphs and the respective point theorem, see Definitions 1 and 2 and Theorem 1 in [10].

Let F be a set of vertices in VX. Then $\mathcal{C}_X F$ denotes the set of connected components in $VX \backslash F$. We define the *reduced graph* X_F of X by setting $VX_F = F$ and connecting two vertices x and y in VX_F by an edge if and only if there exists a $C \in \mathcal{C}_X F$ such that x and y are in the boundary of C.

Definition 2.1. *X is self-similar with respect to F and $\psi : VX \to VX_F$ if*

(F1) *no vertices in F are adjacent in X,*
(F2) *the intersection of the closures of two different components in $\mathcal{C}_X F$ contains not more than one vertex, and*
(F3) *ψ is an isomorphism of X and X_F.*

We will also write ϕ instead of ψ^{-1}, F^n instead of $\psi^n F$ and we set $F^0 = VX$. Components of $\mathcal{C}_X F^n$ are *n-cells,* 1-cells are also just called *cells.* The subgraphs $\hat{C}_n$ of X, which are spanned by the closures of n-cells, are called *n-cell graphs* or *cell graphs* instead of 1-cell graphs. An origin cell is a cell C such that $\phi\theta C \subset \overline{C}$. A fixed point of ψ is called *origin vertex.*

The following theorem is a reformulation of the fixed point theorem for self-similar graphs. It is a consequence of Theorem 1 and Lemma 2 in [10].

Theorem 2.1. *Let X be self-similar with respect to $\tilde{F}$ and $\tilde{\psi}$. Then X is also self-similar with respect to $\tilde{F}^k$ and $\tilde{\psi}^k$ for any positive integer k. There is an integer n such that X, seen as self-similar graph with respect to $F = \tilde{F}^n$ and $\psi = \tilde{\psi}^n$, has either*

(i) *exactly one origin cell and no origin vertex, or*
(ii) *exactly one origin vertex o.*

In the latter case, the subgraphs X_A of X spanned by the closures $\overline{A}$ of components A in $\mathcal{C}_X\{o\}$ are self-similar graphs with respect to

$$F_A = F \cap \overline{A} \quad \text{and} \quad \psi_A = \psi|_{F_A}$$

and they have exactly one origin cell.

Definition 2.2. *A connected graph X which is self-similar with respect to F is called homogeneous if the following axioms are satisfied:*

(H1) *All cell graphs are finite and for any pair of cells C and D in $\mathcal{C}_X F$, there exists an isomorphism $\alpha : \hat{C} \to \hat{D}$ such that $\alpha\theta C = \theta D$.*
(H2) *Let v_1, v_2, v_3, and v_4 be vertices in the boundary θC of a cell C and $v_1 \neq v_2$ and $v_3 \neq v_4$, then $d_X(v_1, v_2) = d_X(v_3, v_4)$.*

In this section, X always denotes a homogeneously self-similar graph. The distance ν of two different vertices in the boundary of a cell is the *length scaling factor* of X. The number μ of 1-cells in a 2-cell is called *volume scaling factor* of X. We write δ_X instead of $|\delta C|$ and θ_X instead of $|\theta C|$ for some cell C in $\mathcal{C}_X F$. The diameter of a cell C is denoted by λ, and we set $\rho = \lambda - \nu$.

For homogeneously self-similar graphs, the numbers $\lambda, \mu, \nu, \rho, \delta_X$ and θ_X are independent of the choice of the cell C.

Example 2.1. Figure 1 shows a 2-cell graph of a self-similar tree. The diameter λ of a cell is greater than the length scaling factor ν. Vertices in F are drawn fat, the two vertices in F^2 are drawn fat and encircled. We have $\nu = \delta_X = \theta_X = 2, \lambda = 3$, and $\mu = 4$. See also Remark 1.

Lemma 2.1.

(i) *Let m and n be positive integers such that $n > m$ and let C_n be an n-cell. Then $\phi^m(C_n \cap F^m)$ is an $(n-m)$-cell.*
(ii) *The number of n-cells in an $(n+1)$-cell C_{n+1} is μ and $|\theta C_{n+1}| = \theta_X$.*
(iii) *Each cell graph $\hat{C}$ consists of μ copies of the complete graph K_{θ_X}. More precisely: The image $\phi\theta C$ of the boundary of a cell C spans a graph in X which is isomorphic to the complete graph K_{θ_X} with θ_X vertices.*

Proof.

(i) The set θC_n is the boundary of C_n in X as well as the boundary of $C_n \cap F^m$ in X_{F^m}. Since ϕ^m is an automorphism $X_{F^m} \to X$, the image $\phi^m \theta C_n$ is the boundary of $\phi^m(C_n \cap F^m)$ in X and it is contained in F^{n-m}. The set $C_n \cap F^m$ is connected in X_{F^m} and $\phi^m(C_n \cap F^m)$ is connected in X. It follows that $\phi^m \theta C_n$ is the boundary of the $(n-m)$-cell $\phi^m(C_n \cap F^m)$.
(ii) For $n = 1$ then the first part of the statement is clear. Suppose n is greater or equal to 2. Then $\phi^{n-1}(C_{n+1} \cap F^{n-1})$ is a 2-cell consisting of μ cells. These cells C correspond one-to-one to the n-cells D in C_{n+1} in the following way: $\phi^n(F^n \cap D) = C$ or $F^n \cap D = \psi^n C$.

The image $\phi^{n+1}\theta C_{n+1}$ is the boundary of a cell, hence $|\theta C_{n+1}| = \theta_X$.

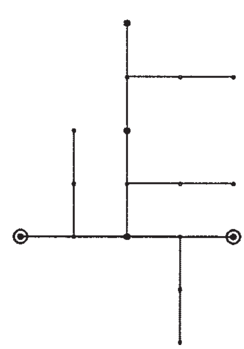

FIGURE 1.

(iii) By the definition of X_F, the vertices in the boundary of a cell in X are pairwise adjacent, thus they span a complete subgraph of X_F. Let C_2 be a 2-cell in X. Then $\overline{C_2} \cap F$ spans μ copies of the complete graph K_{θ_X} as subgraph of X_F. These copies constitute a cell graph in X_F. ■

3. BOUNDED GEOMETRY AND EDGE BOUNDARIES

Definition 3.1. *A graph X has bounded geometry if the set of vertex degrees is bounded. A number b is called constant inner degree if $b = \deg_{\hat{C}} v$ for any vertex v in the boundary of any cell C.*

Theorem 3.1. *Let X be a homogeneously self-similar graph with constant inner degree b, then*

$$|\delta C_n| = \left(\frac{b}{\theta_X - 1}\right)^{n-1} \delta_X$$

for any n-cell C_n.

Proof. For $n = 1$ the statement is clear. Let C_n be an n-cell and let the statement of the lemma be true for $n - 1$. The number of edges in $\delta(C_n \cap F)$ is $|\delta C_{n-1}|$, where $C_n \cap F$ is seen as $(n-1)$-cell in X_F and C_{n-1} is an arbitrary $(n-1)$-cell in X. Let C be a cell in X and let v be a vertex in F such that $C \subset C_n$ and $\theta(C_n \cap F) \cap \theta C = \{v\}$. Then v is adjacent in X_F to $\theta_X - 1$ vertices in θC. Thus each cell C in C_n corresponds to $\theta_X - 1$ edges in δC_{n-1} and $|\delta C_{n-1}|/(\theta_X - 1)$ is the number of cells C in C_n such that $\theta C \cap \theta C_n \neq \emptyset$. This implies

$$\delta C_n = \frac{|\delta C_{n-1}|}{\theta_X - 1} b.$$ ■

Theorem 3.2. *Let X be a homogeneously self-similar graph with constant inner degree b. Then the following conditions are equivalent:*

(i) *X has bounded geometry.*
(ii) $b = \theta_X - 1$.
(iii) *X is locally finite and $\deg_X v = \deg_{X_F} v$ for all $v \in F$.*
(iv) $\delta_X = |\delta C_n|$ *for any n-cell C_n.*
(v) *For any vertex v in the boundary of any n-cell C_n, there is exactly one cell C in C_n such that $v \in \theta C$.*
(vi) $\delta_X = \theta_X(\theta_X - 1)$.

Proof. The equivalence of (i), (ii), and (iii) is a slight generalization of Lemma 5 in [10], the proof remains the same. By Theorem 3.1, condition (iv) is equivalent to (ii). Condition (v) says that in any n-cell there are exactly θ_X different cells C such that $\theta C \cap \theta C_n \neq \emptyset$.

This implies $|\delta C_n| = \theta_X b$, then X must have bounded geometry and $\delta_X = \theta_X(\theta_X - 1)$. Condition (vi) implies $b = \theta_X - 1$. ■

As the following example shows, Theorem 3.2 is in general not true for homogeneously self-similar graphs without constant inner degree.

Example 3.1. The graph in Figure 2 is the 4-cell graph of a graph X, which is related to the modified Koch graph, see [14]. The graph X is homogeneously self-similar and it has a bounded geometry but

$$3 = \delta_X > \theta_X(\theta_X - 1) = 2.$$

There is no constant inner degree. Vertices in F are drawn fat, vertices in F^2 encircled, vertices in F^3 encircled twice, and vertices in F^4 encircled three times.

Theorem 3.3. *Let X be a homogeneously self-similar graph with constant inner degree b such that $b > \theta_X - 1$ and let v be a vertex in VX. Then the following statements are equivalent:*

(i) *The degree of v is infinite.*
(ii) *The vertex v is contained in F^n for any positive integer n.*
(iii) *The vertex v is an origin vertex.*

Proof. Let v be a vertex in the boundary of an n-cell C_n. Then Theorem 3.1 implies that v is adjacent to

$$\frac{\delta_X}{\theta_X}\left(\frac{b}{\theta_X - 1}\right)^{n-1}$$

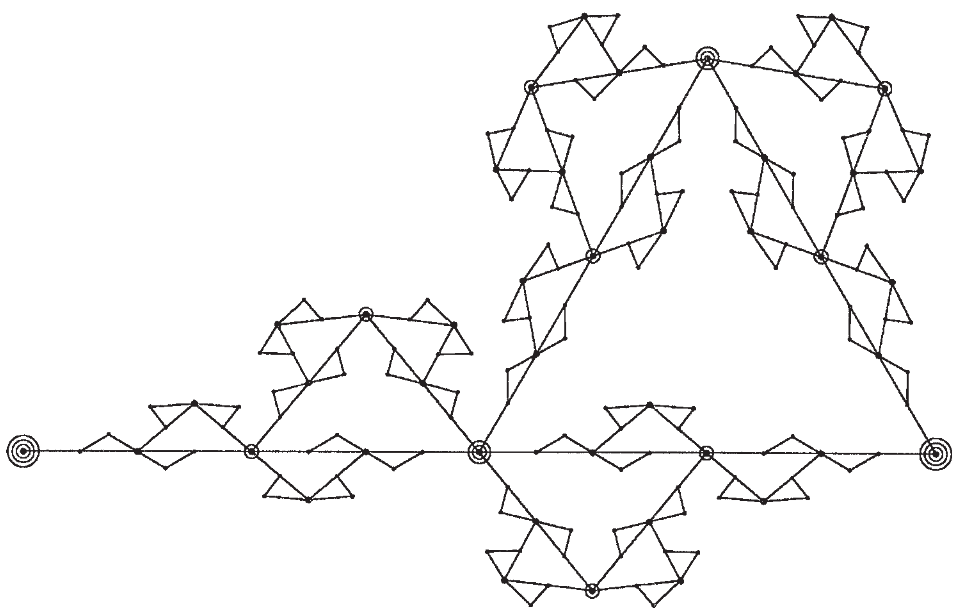

FIGURE 2.

vertices in C_n. If v is in F^n for any integer n then it must have infinite degree. Suppose $v \in F^n \backslash F^{n+1}$, then $\phi^n v$ is contained in $VX \backslash F$. Since all cell graphs are finite, the number of different complete graphs K_θ, which contain v, is finite. This is the same as the number of n-cells having v in their boundaries. Thus v has finite degree. The intersection

$$\bigcap_{n=1}^{\infty} F^n$$

cannot contain two different elements x and y, because ϕ is a bijective contraction and $d(\phi^n x, \phi^n y)$ would tend to zero, which is impossible. See also Theorem 5.2 (i). Since $\phi F^{n+1} = F^n$ for any positive integer, we have

$$\phi \bigcap_{n=1}^{\infty} F^n = \bigcap_{n=1}^{\infty} F^n$$

and a vertex lies in this intersection if and only if it is an origin cell. ■

As a consequence of Theorems 3.2 and 3.3 we obtain the following.

Corollary 3.1. *Let X be a homogeneously self-similar graph with constant inner degree. Then one of the following statements is true:*

(i) *The graph X has bounded geometry.*
(ii) *There exists no origin vertex and X is locally finite but has unbounded geometry.*
(iii) *There exists an origin vertex and X is non-locally finite.*

Example 3.2. The graph in Figure 3 is the 2-cell graph of a locally finite, homogeneously self-similar graph X with unbounded geometry. Again, vertices in F are drawn fat and vertices in F^2 encircled. The vertices v_3 and $\tilde{v}_3$ in F^3, which are encircled two times are only drawn as isolated vertices. The vertices v_1 and $\tilde{v}_1$ form the boundary of the origin cell. There is no origin vertex, $\phi v_{n+1} = v_n$ and

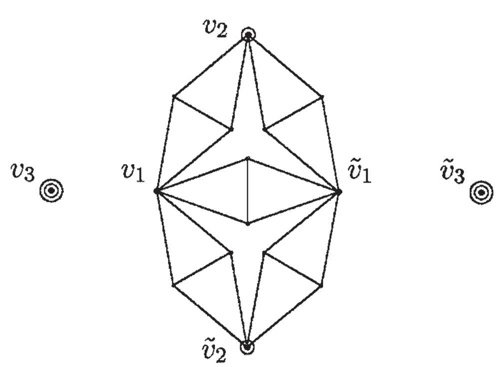

FIGURE 3.

$\phi\tilde{v}_{n+1} = \tilde{v}_n$ for any positive integer n. We have $b = 2, \theta_X = 2, \delta_X = 4$, thus $b > \theta_X - 1$ and $\delta_X > \theta_X(\theta_X - 1)$. Let C_n be an n-cell and let v_n be a vertex in θC_n. Then, according to Theorem 3.1, $|\delta C_n| = 2^{n+1}$. And, since v_n is in the boundary of three different n-cells, $\deg_X v_n = 3 \cdot 2^n$.

4. GROWTH DIMENSIONS

Definition 4.1. *For a vertex $x \in VX$ and an integer $r \in \mathbb{N}_0$ we call*

$$B(x, r) = \{y \in VX \mid d_X(y, x) \leq r\}$$

ball (or more precisely: closed d_X-ball) with centre x and radius r. Let $A \subset VX$ be a set of vertices. Then

$$\mathrm{Vol}_X A = \sum_{y \in A} \deg_X y,$$

is the volume of A. We write $\mathrm{Vol}\, X$ *instead of* $\mathrm{Vol}_X VX$.

Lemma 4.1. *Let X be any graph and let A be a set of vertices in VX. Then*

(i) $\mathrm{Vol}\, X = 2|EX|$ *and*
(ii) $\mathrm{Vol}\, \hat{A} = \mathrm{Vol}_X A + |\delta A|$

Proof. In the sum of the definition of the volume, each edge is counted twice. In $\mathrm{Vol}_X A$ the edges connecting two vertices in A are counted twice, the edges connecting a vertex in A with a vertex in $VX \backslash A$ are counted once. When we count these $|\delta A|$ edges a second time we obtain $\mathrm{Vol}_X A + |\delta A|$, twice the number of all edges in $E\hat{A}$, which is the same as $\mathrm{Vol}\, \hat{A}$. ■

Definition 4.2. *The growth function V_x at x is defined as*

$$V_x : \quad \mathbb{N}_0 \to \mathbb{N}_0 \cup \{\infty\}, \quad r \mapsto \mathrm{Vol}_X B(x, r).$$

We call

$$\underline{V}(r) = \inf\{V_x(r) \mid x \in VX\}$$

lower growth or lower global growth and

$$\bar{V}(r) = \sup\{V_x(r) \mid x \in VX\}$$

upper growth or upper global growth of X. The graph X has regular volume growth, or satisfies the doubling property, if there exists a constant c such that

$$V_x(2r) \leq c\ V_x(r)$$

for any vertex x and any integer r. We define

$$\underline{\dim}_G X = \liminf_{r\to\infty} \frac{\log \underline{V}(r)}{\log r},$$

the lower global growth dimension, and

$$\overline{\dim}_G X = \limsup_{r\to\infty} \frac{\log \bar{V}(r)}{\log r},$$

the upper global growth dimension of X.

Lemma 4.2. *Let x_1 and x_2 be any two vertices in a locally finite graph Y of regular volume growth. Then*

$$\liminf_{r\to\infty} \frac{\log V_{x_1}(r)}{\log\ r} = \liminf_{r\to\infty} \frac{\log V_{x_2}(r)}{\log\ r}$$

and

$$\limsup_{r\to\infty} \frac{\log V_{x_1}(r)}{\log\ r} = \limsup_{r\to\infty} \frac{\log V_{x_2}(r)}{\log\ r}.$$

Proof. Let r be an integer such that $r \geq d_X(x_1, x_2)$ and $r \geq 2$. Then

$$B(x_1, r) \subset B(x_2, d_X(x_1, x_2) + r) \subset B(x_2, 2r)$$

implies

$$V_{x_1}(r) \leq V_{x_2}(2r) \leq c\, V_{x_2}(r)$$

and

$$\frac{\log V_{x1}(r)}{\log r} \leq \frac{\log c}{\log r} + \frac{\log V_{x2}(r)}{\log r}. \qquad \blacksquare$$

This lemma gives reason for the following definition.

Definition 4.3. *Let x be a vertex of a graph Y of regular volume growth, then*

$$\underline{\dim}\, X = \liminf_{r\to\infty} \frac{\log V_x(r)}{\log r}$$

is the lower growth dimension (or lower local growth dimension) and

$$\overline{\dim}\, X = \limsup_{r\to\infty} \frac{\log V_x(r)}{\log r}$$

is the upper growth dimension (or upper local growth dimension) of X.

Lemma 4.3.

$$\underline{\dim}_G X \leq \underline{\dim}\, X \leq \overline{\dim}\, X \leq \overline{\dim}_G X.$$

Proof. Let x_0 be a vertex and $(r_n)_{n\in\mathbb{N}}$ be a sequence of integers such that

$$\lim_{n\to\infty} \frac{\log V_{x_0}(r_n)}{\log r_n} = \underline{\dim}\, X.$$

Then

$$\begin{aligned}\underline{\dim}_G X &= \liminf_{r\to\infty} \frac{\log \underline{V}(r)}{\log r} = \liminf_{r\to\infty} \frac{\log \inf\{V_x(r) \mid x \in VX\}}{\log r} \\ &\leq \liminf_{n\to\infty} \frac{\log \inf\{V_x(r_n) \mid x \in VX\}}{\log r_n} \leq \liminf_{n\to\infty} \frac{\log V_{x_0}(r_n)}{\log r_n} = \underline{\dim}\, X.\end{aligned}$$

The inequality relation between the upper growth dimensions follows analogously. ■

5. GROWTH OF HOMOGENEOUSLY SELF-SIMILAR GRAPHS

In this section, let X always be a homogeneously self-similar graph. Recall that $\hat{C}_n$ dentotes the graph spanned by $C_n \cup \theta C_n$.

Theorem 5.1. *Let C_n be an n-cell. Then*

$$\mathrm{Vol}_X C_n = \mathrm{Vol}\hat{C}_n - \delta_X = \mu^n \theta_X(\theta_X - 1) - \delta_X.$$

Proof. By Lemma 4.1 (i), the volume $\mathrm{Vol}\hat{C}_n$ can be calculated by counting the edges in $\hat{C}_n$ twice. Let C be a cell in X. The complete graph K_{θ_X} has $\binom{\theta_X}{2}$ edges, and Lemma 2.1 (iii) implies

$$|E\hat{C}| = \mu\binom{\theta_X}{2} \quad \text{and} \quad \mathrm{Vol}\,\hat{C} = \mu\theta_X(\theta_X - 1).$$

By Lemma 2.1 (ii), C_n contains μ disjoint $(n-1)$-cells $D_1, D_2, \ldots, D_\mu$ and

$$\bigcup_{k=1}^{\mu} \hat{D}_k = \hat{C}_n,$$

where this union means the union of graphs, not the usual set theoretic union. Thus

$$\mathrm{Vol}\,\hat{C}_n = \mu \mathrm{Vol}\,\hat{C}_{n-1} = \mu^{n-1}\mathrm{Vol}\,\hat{C} = \mu^n \theta_X(\theta_X - 1),$$

where C_{n-1} is any $(n-1)$-cell and C any cell. Lemma 4.1 (ii) implies the rest of the statement. ■

Theorem 5.2. *Let C_n be an n-cell. Then*

(i) diam $\theta C_n \nu^n$,
(ii) $\nu^n \leq \max\{d_X(x, v) \mid x \in \overline{C}_n, v \in \theta C_n\} \leq \nu^n + \frac{\rho\nu^n - 1}{\nu - 1}$ *and*
(iii) $\nu^n \leq \mathrm{diam}\ \overline{C}_n \leq \nu^n + \frac{\rho\nu^n - 1(\nu+1) - 2}{\nu - 1} < \nu^{n+\tilde{\kappa}}$
where $\tilde{\kappa} = \frac{\log(\nu + 3\rho)}{\log \nu} - 1.$

Proof.

(i) By the definition of the length scaling factor, $\theta C_1 = \nu$. Suppose $\theta C_{n-1} = \nu^{n-1}$ for all $(n-1)$-cells C_{n-1}.

Let π be a geodesic path connecting two vertices v and w in the boundary θC_n. In the intersection $\pi \cap F^{n-1}$, we can find vertices $v = x_0, x_1, \dots, w = x_n$, such that $\pi^* = (v = x_0, x_1, \dots, w = x_n)$ is a path in $X_{F^{n-1}}$ connecting v and w. The length of π^* is greater or equal ν. Each two consecutive vertices in π^* are starting and end point for a path in X connecting different vertices in the boundary of an $(n-1)$-cell. This means that π decomposes into at least ν paths, each of them with length of at least ν^{n-1}. Thus the length of π is greater or equal ν^n.

On the other hand, there exists a path β of length ν in $\overline{C_n} \cap F^{n-1}$, seen as cell in $X_{F^{n-1}}$, connecting two points in θC_n. Any pair of consecutive vertices in β can be connected by a path in X of length ν^{n-1}. Thus any two points in the boundary of an n-cell in X can be connected by a path of length less or equal ν^n.

(ii) For $n = 1$ we have $\nu + \rho = \lambda$. Suppose the statement is true for $n-1$. Let π be a geodesic path connecting a vertex v in θC_n and a vertex x in $\overline{C_n}$. The number of $(n-1)$-cells having vertices in common with π is at most λ. Otherwise the ϕ^{n-1}-projection of π would be a geodesic path in a cell whose length is greater than λ. The intersection of π with all of these $(n-1)$-cells except the $(n-1)$-cell whose closure contains x has at most length $(\lambda - 1)\nu^{n-1}$. The above statement for $n-1$ says that the intersection of π with the last cell has at most length $\nu^{n-1} + \rho\frac{\nu^{n-1}-1}{\nu-1}$. Thus the length of π is less or equal to

$$(\lambda - 1)\nu^{n-1} + \nu^{n-1} + \rho\frac{\nu^{n-1}-1}{\nu-1}$$

$$= \nu^n + \rho\nu^{n-1} + \rho\frac{\nu^{n-1}-1}{\nu-1} = \nu^n + \rho\frac{\nu^n - 1}{\nu - 1}.$$

(iii) We can copy the proof of (ii), but we now decompose a geodesic path π between any two vertices in $\overline{C_n}$ into at most $\lambda - 2$ paths connecting two vertices in the boundary of an $(n-1)$-cell, and the initial and the end part of π. The length of the latter ones is at most $\nu^{n-1} + \rho\frac{\nu^{n-1}-1}{\nu-1}$. Thus the length of π is less or equal

$$(\lambda - 2)\nu^{n-1} + 2\left(\nu^{n-1} + \rho\frac{\nu^{n-1}-1}{\nu-1}\right) = \nu^n + \rho\nu^{n-1} + 2\rho\frac{\nu^{n-1}-1}{\nu-1}$$

$$= \nu^n + \rho\frac{\nu^{n-1}(\nu+1) - 2}{\nu - 1} = \nu^n + \rho\nu^{n-1}\frac{(\nu+1) - \frac{2}{\nu^{n-1}}}{\nu-1}$$

$$< \nu^n + \rho\nu^{n-1}3 = \nu^n\frac{\nu + 3\rho}{\nu}.$$

Note that $\lambda = \nu + \rho$ and $\nu \geq 2$. The least real number $\tilde{\kappa}$ such that

$$\frac{\nu + 3\rho}{\nu} \leq \nu^{\tilde{\kappa}}$$

is

$$\tilde{\kappa} = \frac{\log(\nu + 3\rho)}{\log \nu} - 1.$$

The lower bounds in (ii) and (iii) are a consequence of (i).

Remark 1. For the self-similar tree in Example 2.1 the upper bound in Theorem 5.2 (ii), and the first upper bound for $\overline{C_n}$ in Theorem 5.2 (iii) are sharp.

Definition 5.1. *Let ${}_X v$ be the number of cells C, such that v is a vertex in θC and let c_X be*

$$\sup \text{cells } {}_X v \mid v \in F\}.$$

Let M_X be the supremum of degrees of vertices in VX. We write c and M instead of c_X and M_X if it is clear which graph is meant.

The following Lemma corresponds to Lemma 4 in [10].

Lemma 5.1.

$$\text{Cells } {}_X v \ (\theta_X - 1) = \deg_{X_F} v.$$

Corollary 5.1.

$$c_X(\theta_X - 1) = M_X.$$

Proof. Lemma 5.1 implies

$$c_X(\theta_X - 1) = M_{X_F}.$$

Since X and X_F are isomorphic, M_{X_F} equals M_X. ■

Note that homogeneously self-similar graphs have bounded geometry if and only if c is finite. Let κ be the least integer, which is greater or equal $\tilde{\kappa}$.

Theorem 5.3. *Let us write $r_n = \nu^n + \rho\frac{\nu^{n-1}(\nu+1)-2}{\nu-1}$ for a positive integer n. Then*

$$r_n^{\frac{\log \mu}{\log \nu}} \theta_X(\theta_X - 1)\mu^{-\kappa} \leq V(r_n) \leq \bar{V}(r_n)$$

$$\leq r_n^{\frac{\log \mu}{\log \nu}} \mu^{\kappa} \theta_X(\theta_X - 1)\big((c-1)\theta_X + 1\big) + \theta_X(\theta_X - 1)(c-1)(M-1).$$

Proof. According to Theorem 5.2 (iii) we have

$$r_n \leq \nu^{n+\kappa} \quad \text{and} \quad n \geq \frac{\log r_n}{\log \nu} - \kappa.$$

Let C_n be an n-cell and let x be a vertex in $\overline{C_n}$. Again by Theorem 5.2 (iii), C_n is a subset of $B(x, r_n)$. Theorem 5.1 implies

$$\underline{V}(r_n) \geq \operatorname{Vol} \hat{C}_n = \mu^n \theta_X(\theta_X - 1) \geq \mu^{\frac{\log r_n}{\log \nu} - \kappa} \theta_X(\theta_X - 1) = r_n^{\frac{\log \mu}{\log \nu}} \theta_X(\theta_X - 1)\mu^{-\kappa}.$$

On the other hand, let $C_{n+\kappa}$ be a $(n+\kappa)$-cell such that $x \in \overline{C_{n+\kappa}}$. Since $r_n \leq \nu^{n+\kappa}$, the ball $B(x, r_n)$ is contained in the union of $C_{n+\kappa}$ and the closures of all $(n+\kappa)$-cells which are adjacent to $C_{n+\kappa}$. There are at most $(c-1)\theta_X$ of $(n+\kappa)$-cells, being adjacent to $C_{n+\kappa}$. The volume of the union D of $C_{n+\kappa}$ and the closures of these $(n+\kappa)$-cells is at most

$$\big((c-1)\theta_X + 1\big)\mu^{n+\kappa}\theta_X(\theta_X - 1) + |\delta D|,$$

the twice the number of edges in the subgraph spanned by D, plus $|\delta D|$, see Lemma 4.1 and Theorem 5.1. In each boundary of one of these $(n+\kappa)$-cells, there are $\theta_X - 1$ vertices which are not in the boundary of $C_{n+\kappa}$, and these vertices have at most $M - 1$ edges in common with $VX \backslash D$. Thus

$$|\delta D| \leq (c-1)\theta_X(\theta_X - 1)(M - 1)$$

and

$$\bar{V}(r_n) \leq \operatorname{Vol}_X D \leq \big((c-1)\theta_X + 1\big)\mu^{n+\kappa}\theta_X(\theta_X - 1) + (c-1)\theta_X(\theta_X - 1)(M - 1).$$

Since $r_n \geq \nu^n$ we have

$$\mu^n \leq \mu^{\frac{\log r_n}{\log \nu}} = r_n^{\frac{\log \mu}{\log \nu}}$$

and finally

$$\bar{V}(r_n) \leq r_n^{\frac{\log \mu}{\log \nu}} \mu^{\kappa}\big((c-1)\theta_X + 1\big)\theta_X(\theta_X - 1) + (c-1)\theta_X(\theta_X - 1)(M - 1). \quad \blacksquare$$

The growth of a graph can be seen as the discrete analog to the Hausdorff dimension. The main difference is that the Hausdorff dimension of sets in metric spaces depends on the underlying metric. Whereas the growth of graphs is always determined by the natural geodesic graph metric. Thus it does only depend on the subject itself.

Theorem 5.4. *The global lower and upper growth dimensions of homogeneously self-similar graphs of bounded geometry are*

$$\underline{\dim}_G X = \overline{\dim}_G X = \frac{\log \mu}{\log \nu}.$$

This means that the global growth dimensions of homogeneously self-similar graphs of bounded geometry can be obtained by the same formula as the

Hausdorff dimension of self-similar sets, which satisfy the open set condition, see Hutchinson [6].

Proof. For a given radius r, we choose an integer n such that

$$\nu^n + \rho\frac{\nu^{n-1}(\nu+1)-2}{\nu-1} \le r \le \nu^{n+1} + \rho\frac{\nu^n(\nu+1)-2}{\nu-1}$$

$$= \nu\Big(\nu^n + \rho\frac{\nu^{n-1}(\nu+1)-2}{\nu-1}\Big) - \nu\rho\frac{\nu^{n-1}(\nu+1)-2}{\nu-1} + \rho\frac{\nu^n(\nu+1)-2}{\nu-1}$$

$$= \nu\Big(\nu^n + \rho\frac{\nu^{n-1}(\nu+1)-2}{\nu-1}\Big) + \frac{\rho}{\nu-1}\big(-\nu^n(\nu+1)+2\nu+\nu^n(\nu+1)-2\big)$$

$$= \nu\Big(\nu^n + \rho\frac{\nu^{n-1}(\nu+1)-2}{\nu-1}\Big) + 2\rho.$$

Then

$$\frac{r}{\nu} - \frac{2\rho}{\nu} \le \nu^n + \rho\frac{\nu^{n-1}(\nu+1)-2}{\nu-1} \le r \le \nu^{n+1} + \rho\frac{\nu^n(\nu+1)-2}{\nu-1} \le \nu r + 2\rho.$$

For the radii

$$r_n = \nu^n + \rho\frac{\nu^{n-1}(\nu+1)-2}{\nu-1} \quad \text{and} \quad r_{n+1} = \nu^{n+1} + \rho\frac{\nu^n(\nu+1)-2}{\nu-1}$$

we have

$$\underline{V}(r_n) \le \underline{V}(r) \le \bar{V}(r) \le \bar{V}(r_{n+1})$$

and by Theorem 5.3

$$\Big(\frac{r}{\nu} - \frac{2\rho}{\nu}\Big)^{\frac{\log\mu}{\log\nu}}\theta_X(\theta_X-1)\mu^{-\kappa} \le V(r) \le \bar{V}(r)$$

$$\le (\nu r + 2\rho)^{\frac{\log\mu}{\log\nu}}\theta_X(\theta_X-1)\big((c-1)\theta_X+1\big) + (c-1)\theta_X(\theta_X-1)(M-1)$$

for any integer r. It follows that

$$\liminf_{r\to\infty}\frac{\log \underline{V}(r)}{\log r} = \limsup_{r\to\infty}\frac{\log\bar{V}(r)}{\log r} = \frac{\log\mu}{\log\nu}. \qquad \blacksquare$$

Remark. This paper is based on parts of the author's PhD thesis [11].

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The coordinates for the 'Austria'-graph (fractal mountains looking like the shape of the country on a map) in Figure 2 were computed by a program for visualizing self-similar graphs, which was written by E. Teufl.

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