(67) $\dot{x} = x - y + \cos(t)$ $\qquad x(0) = 1,\ y(0) = 0$

$\dot{y} = 2x + 3y - \sin(t)$

$$p(\lambda) = \det\begin{pmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) + 2 = \lambda^2 - 4\lambda + 3 + 2 = \lambda^2 - 4\lambda + 5 \overset{!}{=} 0$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = \underline{2 \pm i}$$

Eigenvektor zu $\lambda_1 = 2+i$:

$$\begin{pmatrix} -1-i & -1 \\ 2 & 1-i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \vec{0} \implies \begin{matrix} -(1+i)u - v = 0 \\ 2u + (1-i)v = 0 \end{matrix} \quad \to v = -(1+i)u$$

wähle z.B. $u = 1$

$\to$ EV: $\underline{\begin{pmatrix} 1 \\ -1-i \end{pmatrix}}$

$\to$ komplexe Lösung $\hat{y}_1(t) = e^{(2+i)t}\begin{pmatrix} 1 \\ -1-i \end{pmatrix} =$

$$= e^{2t}\left(\cos(t) + i\sin(t)\right)\begin{pmatrix} 1 \\ -1-i \end{pmatrix} = e^{2t}\begin{pmatrix} \cos(t) + i\sin(t) \\ -\cos(t) + \sin(t) - i\sin(t) - i\cos(t) \end{pmatrix}$$

$\to$ reelle Lösungen: $y_1(t) = e^{2t}\begin{pmatrix} \cos(t) \\ \sin(t) - \cos(t) \end{pmatrix} \qquad y_2(t) = e^{2t}\begin{pmatrix} \sin(t) \\ -\sin(t) - \cos(t) \end{pmatrix}$

$$\to y_{hom}(t) = C_1 e^{2t}\begin{pmatrix} \cos(t) \\ \sin(t) - \cos(t) \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} \sin(t) \\ -\sin(t) - \cos(t) \end{pmatrix}$$

Spezielle Lösung:
$$b(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} \\ +\frac{i}{2}e^{it} - \frac{i}{2}e^{-it} \end{pmatrix} = \underbrace{e^{it}\begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \end{pmatrix}}_{b_1(t)} + e^{-it}\begin{pmatrix} \frac{1}{2} \\ -\frac{i}{2} \end{pmatrix}$$

Ansatz: $\hat{y}_{sp}(t) = e^{it}\begin{pmatrix} A \\ B \end{pmatrix}$ spezielle Lösung zu $b_1(t)$

$\hat{y}_{sp}'(t) = e^{it}\begin{pmatrix} Ai \\ Bi \end{pmatrix}$

Einsetzen & kürzen von e^{it}: $Ai = A - B + \frac{1}{2} \to B = A(1-i) + \frac{1}{2}$

$Bi = 2A + 3B + \frac{i}{2}$

$\implies A(1-i)i + \cancel{\frac{i}{2}} = 2A + 3A(1-i) + \frac{3}{2} + \cancel{\frac{i}{2}}$

$\implies -\frac{3}{2} = 2A + 3A - 3Ai - Ai \mp A = 4A - 4Ai$

$\implies A = -\frac{3}{2}\frac{1}{4-4i} = -\frac{3}{2}\frac{4+4i}{16+16} = \frac{-12-12i}{64} = \frac{-3}{16}(1+i)$

Spezielle Lösung zu $b(t)$

$\implies B = \frac{-12(1+i)(1-i)}{64} + \frac{1}{2} = -\frac{24}{64} + \frac{1}{2} = -\frac{6}{16} + \frac{1}{2} = -\frac{3}{8} + \frac{1}{2} = \frac{1}{8}$

$$\implies y_{sp}(t) = 2\,\mathrm{Re}\left(\hat{y}_{sp}(t)\right) = 2\,\mathrm{Re}\left(e^{it}\begin{pmatrix} -\frac{3}{16}(1+i) \\ \frac{1}{8} \end{pmatrix}\right) = 2\cdot\begin{pmatrix} -\frac{3}{16}\cos(t) + \frac{3}{16}\sin(t) \\ \frac{1}{8}\cos(t) \end{pmatrix} = \begin{pmatrix} -\frac{3}{8}\cos(t) + \frac{3}{8}\sin(t) \\ \frac{1}{4}\cos(t) \end{pmatrix}$$

$$\to \quad y_{all}(t) = C_1 e^{2t} \begin{pmatrix} \cos(t) \\ \sin(t) - \cos(t) \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \sin(t) \\ -\sin(t) - \cos(t) \end{pmatrix} + \begin{pmatrix} -\frac{3}{8}\cos(t) + \frac{3}{8}\sin(t) \\ \frac{1}{4}\cos(t) \end{pmatrix}$$

AWP:

$$1 = x(0) = C_1 - \frac{3}{8} \quad \to \quad C_1 = \frac{11}{8}$$

$$0 = y(0) = -C_1 - C_2 + \frac{1}{4}$$

$$\Rightarrow \quad C_2 = -C_1 + \frac{1}{4} = -\frac{11}{8} + \frac{2}{8} = -\frac{9}{8}$$

(68) $\dot{x} = 8x + 12y - 2z + t$

$\dot{y} = -3x - 4y + z - 1$

$\dot{z} = x + 2y + 2z + 2$

$$P(\lambda) = \det\begin{pmatrix} 8-\lambda & 12 & -2 \\ -3 & -4-\lambda & +1 \\ 1 & 2 & 2-\lambda \end{pmatrix} = (8-\lambda)(-4-\lambda)(2-\lambda) + 12 + 12$$
$$- 2(4+\lambda) - 2(8-\lambda) + 36(2-\lambda)$$
$$= (8-\lambda)(-8+2\lambda+\lambda^2) + \cancel{24-8} - \cancel{2\lambda} - \cancel{16} + \cancel{2\lambda} + 72 - 36\lambda$$
$$= -64 + 16\lambda + \underbrace{8\lambda^2} + 8\lambda \underbrace{- 2\lambda^2} \underbrace{- \lambda^3} + 72 - 36\lambda = -\lambda^3 + 6\lambda^2 - 12\lambda + 8$$

Rate $\lambda_1 = 2$ $\quad -8 + 24 - 24 + 8 = 0$ ✓

$$(-\lambda^3 + 6\lambda^2 - 12\lambda + 8) : (\lambda - 2) = -\lambda^2 + 4\lambda - 4 = -(\lambda^2 - 4\lambda + 4)$$
$$-(-\lambda^3 + 2\lambda^2)$$
$$4\lambda^2 - 12\lambda$$
$$-(4\lambda^2 - 8\lambda)$$
$$-4\lambda + 8$$
$$-(-4\lambda + 8)$$

$\Rightarrow \lambda_{2,3} = 2$

Also: $\mu(2) = 3$

Eigenvektor(en) zu $\lambda = 2$:

$$\begin{pmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \vec{0}$$

$\to w = 3u + 6v = -6v + 6v = 0$

$\to u = -2v$

$\to 6u + 12v = 0 \Downarrow -12v + 12v = 0$

Also: v bel. wählbar, etwa $v = 1$

$\Rightarrow$ EV: $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ also $\nu(2) = 1 <$

Finde 2 weitere Vektoren zu $\lambda = 2$:

i) $$\begin{pmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$\to 6u + 12v - 2 = -2 \to 0 = 0$

$\to w = 3u + 6v + 1 = -6v + 6v + 1 = 1$

$\to u = -2v$

Also v bel. wählbar etwa $v = 1$

$\to$ Neuer Vektor: $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

ii) $\begin{pmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$\rightarrow w = 1 + 3u + 6v = 1 + 3 - 6v + 6v = 4$

$\rightarrow u = 1 - 2v$

$\rightarrow -2 = 6u + 12v - 8 = 6 - 12v + 12v - 8 = -2$

v frei wählbar, etwa $v = 1$

$\rightarrow$ Neuer Vektor: $\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

$$\rightarrow \vec{y}_{hom}(t) = C_1 e^{2t} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \left(\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right) + C_3 e^{2t} \left(\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right)$$

Spezielle Lösung: $b(t) = \begin{pmatrix} t \\ -1 \\ 2 \end{pmatrix}$

Ansatz: $y_{sp}(t) = e^{0 \cdot t} \begin{pmatrix} A + Bt \\ C + Dt \\ E + Ft \end{pmatrix} = \begin{pmatrix} A + Bt \\ C + Dt \\ E + Ft \end{pmatrix}$

$\Longrightarrow y'_{sp}(t) = \begin{pmatrix} B \\ D \\ F \end{pmatrix}$

Einsetzen:

$B = 8A + 8Bt + 12C + 12Dt - 2E - 2Ft + t$

$D = -3A - 3Bt - 4C - 4Dt + E + Ft - 1$

$F = A + Bt + 2C + 2Dt + 2E + 2Ft + 2$

Koeff'vergleich:

I:

$B = 8A + 12C - 2E$

$D = -3A - 4C + E - 1$

$F = A + 2C + 2E + 2$

II:

$0 = 8B + 12D - 2F + 1$

$0 = -3B - 4D + F \longrightarrow F = 3B + 4D$

$0 = B + 2D + 2F$

Lösen von System II: in 1. Gleichung: $0 = 8B + 12D - 6B - 8D + 1 = 2B + 4D + 1$

$\Longrightarrow 2B = -1 - 4D$

$\Longrightarrow B = -2D - \frac{1}{2}$

in 3. Gleichung: $0 = \cancel{-2D} - \frac{1}{2} + \cancel{2D} + 6B + 8D$

$= -\frac{1}{2} - 12D - 3 + 8D = -4D - \frac{7}{2} \Longrightarrow 4D = \frac{-7}{2} \Longrightarrow D = -\frac{7}{8}$

$\Longrightarrow B = +\frac{7}{4} - \frac{1}{2} = \frac{5}{4}$, $F = \frac{15}{4} - \frac{7}{2} = \frac{1}{4}$

Lösen von System I:

$$\Longleftrightarrow \begin{cases} \frac{5}{4} = 8A + 12C - 2E \\ -\frac{7}{8} = -3A - 4C + E - 1 \\ \frac{1}{4} = A + 2C + 2E - 2 \end{cases} \qquad \rightarrow E = \frac{1}{8} + 3A + 4C$$

in 1. Gleichg: $\frac{5}{4} = 8A + 12C - \frac{1}{4} - 6A - 8C = 2A + 4C - \frac{1}{4}$

$$\Rightarrow 2A = \frac{3}{2} - 4C \Rightarrow A = \frac{3}{4} - 2C$$

in 3. Gleichg: $\frac{1}{4} = \frac{3}{4} - \cancel{2C} + \cancel{2C} + \frac{1}{4} + 6A + 8C - 2$

$$= -1 + 8C + \frac{9}{2} - 12C = -4C + \frac{7}{2}$$

$$\Rightarrow 4C = \frac{13}{4} \Rightarrow \underline{C = \frac{13}{16}}$$

$$\Rightarrow A = \frac{3}{4} - \frac{13}{8} = \underline{-\frac{7}{8}}, \quad E = \frac{1}{8} - \frac{21}{8} + \frac{13}{4} = \frac{1 - 21 + 26}{8} = \underline{\frac{3}{4}}$$

$$\Rightarrow y_{sp}(t) = \begin{pmatrix} -\frac{7}{8} + \frac{5}{4}t \\ \frac{13}{16} - \frac{7}{8}t \\ \frac{3}{4} + \frac{1}{4}t \end{pmatrix}$$

$$\Rightarrow \cancel{y_{allg}(t)} \; \underline{y_{all}(t) = y_{hom}(t) + y_{sp}(t)}$$

(69) $\dot{x} = 4x - 3y + 2z + e^{-t}$

$\dot{y} = 2x - y + 2z$

$\dot{z} = -2x + 3y - 2e^{-t}$

$$p(\lambda) = \begin{vmatrix} 4-\lambda & -3 & 2 \\ 2 & -1-\lambda & +2 \\ -2 & 3 & -\lambda \end{vmatrix} = (4-\lambda)(\lambda+\lambda^2) + 12 + 12 + 4(-1-\lambda) - 6(4-\lambda) - 6\lambda$$

$$= \cancel{4\lambda} + 4\lambda^2 - \lambda^2 - \lambda^3 + \cancel{24} - 4 - \cancel{4\lambda} - \cancel{24} + \cancel{6\lambda} - \cancel{6\lambda}$$

$$= \underline{-\lambda^3 + 3\lambda^2 - 4}$$

$\underline{\text{Rate } \lambda_1 = -1}$: $-(-1)^3 + 3\cdot 1^2 - 4 = 0$ ✓

$$(-\lambda^3 + 3\lambda^2 - 4) : (\lambda+1) = -\lambda^2 + 4\lambda - 4 = -(\lambda^2 - 4\lambda + 4)$$

$-(-\lambda^3 - \lambda^2)$

$4\lambda^2$

$-(4\lambda^2 + 4\lambda)$

$-4\lambda - 4$

$-(-4\lambda - 4)$

$-\!-$

$\Rightarrow \underline{\lambda_{2,3} = 2}$ $\quad \mu(2) = 2$

EV zu $\lambda_1 = -1$:

$$\begin{pmatrix} 5 & -3 & 2 \\ 2 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \vec{0} \longrightarrow u = -w$$

$$\rightarrow w = 2u - 3v = -2w - 3v \Rightarrow w = -v$$

$$\Rightarrow u = -w = +v$$

wähle z.B. $v = 1$ $\Rightarrow$ EV: $\underline{\underline{\begin{pmatrix} +1 \\ 1 \\ -1 \end{pmatrix}}}$

EV zu $\lambda_2 = 2$:

$$\underbrace{\begin{pmatrix} 2 & -3 & 2 \\ 2 & -3 & 2 \\ -2 & 3 & -2 \end{pmatrix}}_{} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \vec{0}$$

Rang $= 1$

$\rightarrow$ 2 Variablen frei wählbar

i) wähle z.B. $u = 1, v = 0 \Rightarrow 2w = -2u \rightarrow w = -u$

$\rightarrow$ 1. EV: $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

ii) wähle z.B. $u = 0, v = 1 \rightarrow 2w = 3v \rightarrow w = \frac{3}{2} v$

$\rightarrow$ 2. EV: $\begin{pmatrix} 0 \\ 1 \\ 3/2 \end{pmatrix}$

$$\Rightarrow y_{hom}(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 3/2 \end{pmatrix}$$

Spezielle Lösung: $b(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

Ansatz: $y_{sp}(t) = e^{-t} \begin{pmatrix} A + Bt \\ C + Dt \\ E + Ft \end{pmatrix}$

$$\Rightarrow y_{sp}'(t) = -e^{-t} \begin{pmatrix} A + Bt \\ C + Dt \\ E + Ft \end{pmatrix} + e^{-t} \begin{pmatrix} B \\ D \\ F \end{pmatrix} = e^{-t} \begin{pmatrix} B - A - Bt \\ D - C - Dt \\ F - E - Ft \end{pmatrix}$$

Einsetzen & Kürzen von e^{-t}:

$$B - A - Bt = 4A + 4Bt - 3C - 3Dt + 2E + 2Ft + 1$$

$$D - C - Dt = 2A + 2Bt - C - Dt + 2E + 2Ft$$

$$F - E - Ft = -2A - 2Bt + 3C + 3Dt - 2$$

Koeff. vergleich:

$$\text{I}: \quad \begin{aligned} B - A &= 4A - 3C + 2E + 1 \\ D - C &= 2A - C + 2E \\ F - E &= -2A + 3C - 2 \end{aligned} \qquad \text{II}: \quad \begin{aligned} -B &= 4B - 3D + 2F \\ -D &= 2B - D + 2F \\ -F &= -2B + 3D \end{aligned}$$

Löse System II:

$$\Leftrightarrow \begin{cases} 0 = 5B - 3D + 2F \\ 0 = 2B + 2F \quad \longrightarrow \underline{B = -F} \\ F = 2B - 3D \end{cases}$$

$$\longrightarrow F = -2F - 3D$$

$$\longrightarrow 3F = -3D \longrightarrow \underline{D = -F}$$

in 1. Gleichung: $0 = -5F + 3F + 2F = 0$ ✓ , F zunächst beliebig

Löse System I:

$$\Leftrightarrow \begin{cases} -F = 5A - 3C + 2E + 1 \\ -F = 2A + 2E \quad \longrightarrow F = -2A - 2E \\ F - E = -2A + 3C - 2 \end{cases}$$

3. Gleichung: $-\cancel{2A} - 2E - E = -\cancel{2A} + 3C - 2 \Leftrightarrow -3E = 3C - 2 \Rightarrow \underline{E = \frac{2}{3} - C}$

1. Gleichung: $2A + 2E = 5A - 3C + 2E + 1$

$\Leftrightarrow \cancel{2E} = 3A - 3C + \cancel{2E} + 1 \Leftrightarrow 3A = 3C - 1 \Rightarrow \underline{A = C - \frac{1}{3}}$

$\Rightarrow F = -2A - 2E = -\cancel{2C} + \frac{2}{3} - \frac{4}{3} + \cancel{2C} = \underline{-\frac{2}{3}} \Rightarrow B = D = \frac{2}{3}$

$\Rightarrow$ wähle z.B. $C = 0 \Rightarrow A = -\frac{1}{3}, \; E = \frac{2}{3}$

$$\Rightarrow y_{sp}(t) = e^{-t} \begin{pmatrix} -\frac{1}{3} + \frac{2}{3}t \\ \frac{2}{3}t \\ \frac{2}{3} - \frac{2}{3}t \end{pmatrix}$$

$$\underline{\underline{y_{allg}(t) = y_{hom}(t) + y_{sp}(t)}}$$

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$u_{xy}(x,y) - u_{xx}(x,y) + u(x,y) = 0$

<u>Produktansatz:</u> $u(x,y) = F(x) \cdot G(y)$

$u_x(x,y) = F'(x)G(y)$

$u_y(x,y) = F(x) \cdot G'(y)$

$u_{xx}(x,y) = F''(x)G(y)$

$u_{xy}(x,y) = F'(x)G'(y)$

$\Rightarrow F'(x)G'(y) - F''(x)G(y) + F(x)G(y) = 0$

$\Rightarrow F'(x)G'(y) = F''(x)G(y) - F(x)G(y) \quad | : G(y)$

$\Rightarrow F'(x)\frac{G'(y)}{G(y)} = F''(x) - F(x) \quad | : F'(x)$

$\Rightarrow \frac{G'(y)}{G(y)} = \frac{F''(x) - F(x)}{F'(x)} \overset{!}{=} k \leftarrow$ Separationskonstante

I) $\frac{G'(y)}{G(y)} = k \Rightarrow G'(y) = kG(y) \Rightarrow G'(y) - kG(y) = 0$

$P(\lambda) = \lambda - k \Rightarrow \underline{G(y) = C_1 e^{ky}}$

II) $\frac{F''(x) - F(x)}{F'(x)} = k \Rightarrow F''(x) - F(x) = kF'(x) \Rightarrow F''(x) - kF'(x) - F(x) = 0$

$P(\lambda) = \lambda^2 - k\lambda - 1 \Rightarrow \lambda_{1,2} = \frac{k \pm \sqrt{k^2+4}}{2}$ reelle Lösungen! <u>verschieden!</u>

$\Rightarrow F(x) = C_2 e^{\lambda_1 x} + C_3 e^{\lambda_2 x}$

$\Rightarrow \underline{u(x,y) = C_1 e^{ky} \cdot \left(C_2 e^{\lambda_1 x} + C_3 e^{\lambda_2 x}\right)}$