Upper and lower class separating sequences for Brownian motion with random argument

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Abstract

Let $\mathbf{X} = X_1, X_2, \ldots$ be a sequence of random variables, let W be a Brownian motion independent of $\mathbf{X}$ and let $Z_k = W(X_k)$. We call a numerical sequence (t_k) an *upper (lower) class sequence* for $\{Z_k\}$ if

$$P(Z_k > t_k \text{ for infinitely many } k) = 0 \text{ (or 1, respectively)}.$$

At a first look one might be tempted to believe that a "separating line" (t_k^0), say, between the upper and lower class sequences for $\{Z_k\}$ is directly related to the corresponding counter part (s_k^0) for the process $\{X_k\}$. E.g. by using the law of the iterated logarithm for the Wiener process a functional relationship

$$t_k^0 = \sqrt{2 s_k^0 \log\log s_k^0} \tag{1}$$

seems to be natural. When $X_k = |W_2(k)|$ for a second Brownian motion W_2 then we are dealing with an iterated Brownian motion, and it is known that the multiplicative constant $\sqrt{2}$ in (1) needs to be replaced by $2 \times 3^{-3/4}$, contradicting to this simple argument.

We will study this phenomenon from a different angle by letting $\{X_k\}$ be an i.i.d. sequence. It turns out that the relationship between the separating sequences (s_k^0) and (t_k^0) in the above sense, depends in an interesting way on the extreme value behavior of $\{X_k\}$.

1 Introduction

Let W_1^+, W_1^- and W_2 be independent Brownian motions, and set $W_1(t) = W_1^+(t)$ for $t \geq 0$ and $W_1(t) = W_1^-(-t)$ for $t < 0$. The process $\{W_1(W_2(t)),\ t \geq 0\}$ is called iterated Brownian motion, and was introduced by Burdzy [1]. It has been proven in this paper that

$$\limsup_{t\to 0} \frac{W_1(W_2(t))}{t^{1/4}(\log\log(1/t))^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.} \tag{2}$$

This has been significantly generalized by Csáki et al. [2], [3], who obtained results similar to (2) for a general class of iterated processes. They also proved a global version of (2),

$$\limsup_{t\to\infty} \frac{W_1(W_2(t))}{t^{1/4}(\log\log t)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.} \tag{3}$$

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and

$$\limsup_{t\to\infty} \frac{W_1(|W_2(t)|)}{t^{1/4}(\log\log t)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.}$$

(The asymptotic behavior of $W_1(W_2(t))$ and $W_1(|W_2(t)|)$ needs not always be the same as has been shown in [6] and [7] for the so-called "other law of the iterated logarithm".)

The interesting feature of relation (3) is the following: by the law of the iterated logarithm (LIL) for W_2 there exists for any $h > 0$ an almost surely (a.s.) finite random variable T_0 such that

$$W_2(t) \le (1+h)\sqrt{2t\log\log t} \quad \text{for all } t > T_0.$$

From this relation and the LIL for W_1 one obtains the upper bound

$$\limsup_{t\to\infty} \frac{W_1(W_2(t))}{t^{1/4}(\log\log t)^{3/4}} \le 2^{1/4} \qquad \text{a.s.,} \tag{4}$$

where $2^{1/4} \approx 1.189$, while $2^{5/4}3^{-3/4} \approx 1.043$. This shows that the LIL behavior of the two independent processes W_1 and W_2 cannot be simply combined to obtain a similar result for the process $W_1(W_2)$.

In this paper we try to explore the just described phenomenon from a different angle. To this end we switch to a discrete-time version of (3):

$$\limsup_{k\to\infty} \frac{W_1(W_2(k))}{k^{1/4}(\log\log k)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.,} \tag{5}$$

where k runs through the set of positive integers. Let $X_k = W_2(k)$, $k \ge 1$. Then $\{X_k\}$ is a (strongly dependent) sequence of random variables, having normal distribution with mean 0 and variance k, and (5) has the form

$$P\left(W_1(X_k) > (1+h)\frac{2^{5/4}}{3^{3/4}}\, k^{1/4}(\log\log k)^{3/4} \quad \text{i.o}\right) = 0 \text{ or } 1,$$

depending on $h > 0$ or $h < 0$, respectively. (Here i.o. stands for "infinitely often".) Letting

$$t_k^0 = \frac{2^{5/4}}{3^{3/4}}\, k^{1/4}(\log\log k)^{3/4}$$

one can say in other words that $\{(1+h)t_k^0\}$ belongs to the upper class of $\{W_1(X_k)\}$ if $h > 0$ and it belongs to the lower class if $h < 0$. Short we will write $(t_k) \in \mathcal{U}(\{W_1(X_k)\})$ for an upper class sequence and $(t_k) \in \mathcal{L}(\{W_1(X_k)\})$ for a lower class sequence. In this sense (t_k^0) is *a separating sequence, between upper and lower class sequences for* $\{W_1(X_k)\}$. Of course, the phrase "separating sequence" has to be given with much care. There exists not *the* separating sequence dividing upper and lower classes. E.g. for a Wiener process $\{W(k), k \ge 1\}$ the law of the iterated logarithm suggests as a candidate $s_k^0 = \sqrt{2k\log\log k}$ as a dividing line between $\mathcal{U}(\{W(k)\})$ and $\mathcal{L}(\{W(k)\})$. The Kolmogorov-Erdős-Petrovski integral test states that $\sqrt{k}\varphi(k)$ belongs to the upper or lower class of $\{W(k)\}$ according as

$$I(\varphi) := \int_1^\infty t^{-1}\varphi(t)e^{-\varphi(t)^2/2}dt < \infty \quad \text{or} \quad = \infty, \tag{6}$$

and gives thus a much sharper characterization of upper and lower class sequences than the LIL does (see e.g. Feller [4] and [5]). It implies e.g. that $(s_k^0) \in \mathcal{L}(\{W(k)\})$ and that (s_k^1) defined by

$$s_k^1 = \sqrt{2k \log\log k + 3(1+h)\log\log\log k}$$

is in $\mathcal{U}(\{W(k)\})$ if $h > 0$ and in $\mathcal{L}(\{W(k)\})$ if $h \le 0$. Adding further $\log_p k$ terms (where $\log_p$ is the p-times iterated logarithm) one can get sharper and sharper characterization of upper and lower class behavior. To clarify the usage of the notion "separating sequence" we introduce the following definition.

Definition 1 *Let $\{X_k\}$ be any random sequence and (a_k) a positive and non-decreasing sequence. We call (s_k^0) a $\mathcal{UL}$-separating sequence with respect to (a_k) for $\{X_k\}$ if for any $h > 0$ there exists $(s_k^u) \in \mathcal{U}(\{X_k\})$ and $(s_k^\ell) \in \mathcal{L}(\{X_k\})$ such that*

$$s_k^\ell \le s_k^0 \le s_k^u \quad \text{and} \quad \lim_k \frac{s_k^\ell}{s_k^u} a_k \ge \frac{1}{1+h}.$$

If $a_k = 1$ for all $k \ge 1$ we say that (s_k^0) is $\mathcal{UL}$-separating for $\{X_k\}$.

Roughly speaking, the sequence (a_k) tells us how sharp our separating line (s_k^0) is. For example $s_k^0 = \sqrt{2k \log\log k}$ defines an $\mathcal{UL}$-separating sequence for $\{W(k)\}$. (Choose $s_k^\ell = s_k^0$ and $s_k^u = (1+h)s_k^0$.) If $\{X_k\}$ is an i.i.d. sequence with $P(X_k > x) = x^{-1}$ for $x \ge 1$, then by the Borel-Cantelli lemma $s_k^0 = k \log k$ is $\mathcal{UL}$-separating for $\{X_k\}$ with respect to $((\log\log k)^{1+\gamma})$ for any $\gamma > 0$. (Choose $s_k^\ell = s_k^0$ and $s_k^u = s_k^0(\log\log k)^{1+\gamma}$.) More generally, $s_k^0 = k \prod_{p=1}^P \log_p k$ is $\mathcal{UL}$-separating for $\{X_k\}$ with respect to $((\log_{P+1} k)^{1+\gamma})$ for any $\gamma > 0$. Note also that if (s_k^0) is $\mathcal{UL}$-separating for $\{X_k\}$ with respect to some (a_k), then if $b_k \ge a_k$ for $k \ge 1$, we have that (s_k^0) is $\mathcal{UL}$-separating for $\{X_k\}$ with respect to (b_k).

In this paper we propose to study general random processes of the form $\{W(X_k)\}$, where W is a Brownian motion and $\{X_k\}$ is a sequence of random variables, independent of W. We are interested in finding a relation between sequences (s_k^0) and (t_k^0) which are $\mathcal{UL}$-separating for $\{X_k\}$ and $\{W(X_k)\}$, respectively. For example, if $X_k = W_2(k)$ then we have just seen that $s_k^0 = \sqrt{2k \log\log k}$ is $\mathcal{UL}$-separating for $\{X_k\}$. On the other hand we have by (5) that

$$t_k^0 = \frac{2}{3^{3/4}} \sqrt{s_k^0 \log\log s_k^0}\,. \tag{7}$$

is $\mathcal{UL}$-separating for $\{W_1(X_k)\}$.

Clearly, the behavior of $\{W(X_k)\}$ can be very involved in a general model, and we shall thus restrict ourselves in this attempt to the case when $\{X_k\}$ is an i.i.d. sequence. We will show that in this case the relationship between (s_k^0) and (t_k^0) depends on the tail structure of the X_k's. This leads to the field of extreme value theory (a classical monograph is e.g. Leadbetter, Lindgren, and Rootzén [9]). The arguably most important theorem in extreme value theory, known as Fisher-Tippet-Gnedenko-Theorem, states that if for a sequence $\{X_k\}$ of i.i.d. random variables with maximum $M_n = \max\{X_1, \ldots, X_n\}$ there exists a two-dimensional sequence $(a_n, b_n)_{n\ge1}$ such that

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} G \tag{8}$$

($\xrightarrow{d}$ denotes convergence in distribution) for some non-degenerate distribution function G, then G belongs either to the Gumbel, Fréchet or Weibull family of distributions (also called type I, type II and type III distributions). The Weibull distribution (or type III distribution) can only appear if the X_k's are bounded, which is not of interest in our situation. Type I and type II distributions can appear in different situations, but a typical case for which the (normalized) maximum has type I distribution is when the X_k's have exponential tails, and a typical case for the (normalized) maximum having type II distribution is when the X_k's have Pareto tails.

Roughly speaking, our Theorem 1 below shows, that the argument leading to (4) is optimal in the case when $\{\max_{1\le k\le n} X_k\}$ has type I limiting behavior. This is, when (s_k^0) is $\mathcal{UL}$-separating for $\{X_k\}$, then

$$t_k^0 = \sqrt{2s_k^0 \log\log s_k^0}$$

is $\mathcal{UL}$-separating for $\{W(X_k)\}$. One could say that this (t_k^0) is "natural" or "unbiased" in contrast to the (t_k^0) given in (7). Theorem 2 shows that the situation is radically different if the limit of $\{\max_{1\le k\le n} X_k\}$ is of type II. In this case it turns out that (t_k^0) is biased in the sense that

$$t_k^0 = \sqrt{s_k^0}\,.$$

2 Results

As we have pointed out in the introduction our results need to be related to results in extreme value theory, which we shall now briefly recall. Let $\{X_k\}$ be an i.i.d. sequence. If (8) holds, then the distribution G belongs to one of three types of so called *max-stable* distributions, which are given (up to location and scale) by

$$\begin{array}{ll}
\text{Type I:} & G(x) = \exp(-e^{-x}), \quad -\infty < x < \infty; \\
\text{Type II:} & G(x) = \begin{cases} 0, & x \le 0; \\ \exp(-x^{-\alpha}), \quad \text{for some } \alpha > 0, & x > 0; \end{cases} \\
\text{Type III:} & G(x) = \begin{cases} \exp(-(-x)^{\alpha}), \quad \text{for some } \alpha > 0, & x \le 0; \\ 1, & x > 0. \end{cases}
\end{array}$$

If the maximum of an i.i.d. sequence $\{X_k\}$ satisfies (8), then depending on which of the G's appears in the limit we say that $\{X_k\}$ belongs to type I, II or III. If $\{X_k\}$ belongs to type III, then $\{X_k\}$ needs to be bounded from above, and we are not interested in this case. Let $x_F = \sup\{x : F(x) < 1\}$. Then $\{X_k\}$ is
(A) of type I, if and only if there exists a strictly positive function $g(t)$ such that

$$\lim_{t\to x_F} \frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-x} \quad \text{for all } x > 0.$$

(B) of type II, if and only if $x_F = \infty$ and

$$\lim_{x\to\infty} t^{\alpha} P(X_1 > tx)/P(X_1 > x) = 1, \quad \text{for some } \alpha > 0 \text{ and for all } t > 0.$$

In case of (A) we write $\{X_k\} \in D_G$ and in case of (B) we write $\{X_k\} \in D_F$.

The classes D_G and D_F are slightly to general for our investigations. E.g. D_G still contains bounded sequences $\{X_k\}$ which we want to exclude from our analysis. We thus define the following subclasses D'_G and D'_F which exclude such cases and provide some technical simplifications for the proofs. We recall that a function $q(x)$ is slowly varying (at ∞) if

$$\lim_{x\to\infty} q(\lambda x)/q(x) = 1 \quad \text{for all } \lambda > 0.$$

Definition 2 *We say that $\{X_k\}$ belongs to D'_G if there is an $\alpha > 0$ and a slowly varying function $q(x)$, such that $P(X_1 > x) = \exp(-x^\alpha q(x))$ for $x > 0$. We say that $\{X_k\}$ belongs to D'_F if there is an $\alpha > 0$ and a slowly varying function $q(x)$ such that*

$$P(X_1 > x) = x^{-\alpha} q(x)$$

and

$$\lim_{x\to\infty} \sup_{t\in[1,(\log x)^{2/\alpha}]} q(tx)/q(x) = 1. \tag{9}$$

Remark 1 *It is obvious that $D'_F \subset D_F$ and it is not hard to prove that $D'_G \subset D_G$. We would also like to stress that D'_G and D'_F are not very restrictive and contain all practically relevant examples of sequences $\{X_k\}$ belonging to D_G or D_F (see for example Corollaries 1-3).*

To simplify the presentation we assume throughout this paper that $W(t) = 0$ for $t < 0$. Anyway, only minor changes are required to obtain exactly the same results for $W(t) = \mathbf{1}_{(-\infty,0)}(t)W^-(-t) + \mathbf{1}_{[0,\infty)}(t)W^+(t)$, where W^- and W^+ are independent Brownian motions. For the sake of simplicity we call the resulting W, defined now on the whole real line, again a Brownian motion. Furthermore, throughout this paper $\log x$ is meant as $\max(1, \log x)$.

We are now ready to formulate our first result.

Theorem 1 *Let $\mathbf{X} = X_1, X_2, \ldots$ be a system of i.i.d. random variables, and let W be a Brownian motion independent of $\mathbf{X}$. Assume that the X_k's have a continuous distribution function and that $\{X_k\} \in \mathcal{D}'_G$. Then*

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{2m_k \log\log\log k}} = 1 \qquad \text{a.s.,} \tag{10}$$

where

$$m_k = \min\left\{x \in \mathbb{R} : F(x) = 1 - \frac{1}{k}\right\}. \tag{11}$$

It is not difficult to show (see Section 3.2) that under the assumptions of Theorem 1

$$\limsup_{k\to\infty} \frac{X_k}{m_k} = 1 \quad \text{a.s.,}$$

and thus (s_k^0) given by $s_k^0 = m_k$ is a $\mathcal{UL}$-separating sequence for $\{X_k\}$. It is also quite easy to show that under the assumptions of Theorem 1 we always have

$$\lim_{k\to\infty} \frac{\log\log m_k}{\log\log\log k} = 1,$$

and (10) can be replaced by

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{2s_k^0 \log\log s_k^0}} = 1 \qquad \text{a.s.},$$

showing that $t_k^0 = \sqrt{2s_k^0 \log\log s_k^0}$ is a $\mathcal{UL}$-separating sequence for $\{W(X_k)\}$.

Here are two special cases of Theorem 1.

Corollary 1 (Normal distribution) *Let* $\mathbf{X} = X_1, X_2, \ldots$ *be a system of i.i.d. random variables having normal distribution with mean* μ *and variance* σ*. Let* W *be a Brownian motion independent of* $\mathbf{X}$*. Then*

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{2\,(\log k)^{1/2} \log\log\log k}} = \left(2\sigma^2\right)^{1/4} \qquad \text{a.s.}$$

Corollary 2 (Exponential distribution) *Let* $\mathbf{X} = X_1, X_2, \ldots$ *be a system of i.i.d. random variables having exponential distribution with parameter* λ*. Let* W *be a Brownian motion independent of* $\mathbf{X}$*. Then*

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{2\log k \log\log\log k}} = \frac{1}{\sqrt{\lambda}} \qquad \text{a.s.}$$

Theorem 2 describes the behavior of $W(X_k)$ in the case of the X_k's having polynomial tails, which corresponds to type II behavior of $\max_{1\le k\le N} X_k$ in the sense of extreme value theory:

Theorem 2 *Let* $\mathbf{X} = X_1, X_2, \ldots$ *be a system of i.i.d. random variables, and let* W *be a Brownian motion independent of* $\mathbf{X}$*. Assume that the* X_k*'s have a continuous distribution function and that* $\{X_k\} \in \mathcal{D}'_F$*, and let* α *be defined like in (9). Then*

$$\left(\sqrt{m_k (\log k)^{1/\alpha} (\log\log k)^{1/\alpha+\varepsilon}}\right)_{k\ge 1} \in \begin{cases} \mathcal{U}(\{W(X_k)\}) & \text{if } \varepsilon > 0, \\ \mathcal{L}(\{W(X_k)\}) & \text{if } \varepsilon \le 0, \end{cases} \tag{12}$$

where m_k $(k \ge 1)$ *is defined as in* (11)*.*

Theorem 2 shows that the sequence (t_k^0) defined by

$$t_k^0 = \sqrt{m_k (\log k)^{1/\alpha} (\log\log k)^{1/\alpha}},$$

is $\mathcal{UL}$-separating for $\{W(X_k)\}$ with respect to (a_k), when $a_k = (\log\log k)^{\varepsilon}$, with arbitrary $\varepsilon > 0$. Furthermore under the assumptions of Theorem 2 we can show

$$k^{1/\alpha-\varepsilon} \le m_k \le k^{1/\alpha+\varepsilon}$$

(for arbitrary $\varepsilon > 0$ and sufficiently large k). Thus we may also choose

$$t_k^0 = \sqrt{m_k (\log m_k)^{1/\alpha} (\log\log m_k)^{1/\alpha}},$$

and similarly the left-hand side in (12) can be replaced accordingly. A routine application of the Borel-Cantelli lemma together with our assumptions on the tails of the distribution of the X_k's shows that

$$s_k^0 = m_k(\log m_k)^{1/\alpha}(\log\log m_k)^{1/\alpha}$$

is $\mathcal{UL}$-separating for $\{X_k\}$ with respect to (a_k), when $a_k = (\log\log k)^{\varepsilon}$, with arbitrary $\varepsilon > 0$. This shows the relationship

$$t_k^0 = \sqrt{s_k^0},$$

which is radically different from the one obtained in Theorem 1.

Here is a simple example for Theorem 2.

Corollary 3 (Pareto distribution) *Let $\mathbf{X} = X_1, X_2, \dots$ be a system of i.i.d. random variables, and let W be a Brownian motion independent of $\mathbf{X}$. Assume that the distribution function $F(x)$ of the X_k's is*

$$F(x) = \begin{cases} 1 - \left(\frac{x_0}{x}\right)^{\alpha} & \text{for} \quad x \geq x_0 \\ 0 & \text{for} \quad x < x_0 \end{cases}$$

for some $x_0 > 0$ and $\alpha > 0$. Then

$$\left(\sqrt{k^{1/\alpha}(\log k)^{1/\alpha}(\log\log k)^{1/\alpha+\varepsilon}}\right)_{k\geq 1} \in \begin{cases} \mathcal{U}(\{W(X_k)\}) & \text{if} \quad \varepsilon > 0, \\ \mathcal{L}(\{W(X_k)\}) & \text{if} \quad \varepsilon \leq 0, \end{cases}$$

The following table summarizes possible relationships between the $\mathcal{UL}$-separating sequences (s_k^0) for the different sequence $\{X_k\}$ we have seen in this paper and a $\mathcal{UL}$-separating sequences (t_k^0) for $\{W(X_k)\}$:

$X_k = W_2(k)$	$t_k^0 = \frac{2}{3^{3/4}}\sqrt{s_k^0 \log\log s_k^0}$;
$\{X_k\} \in D'_G$	$t_k^0 = \sqrt{2 s_k^0 \log\log s_k^0}$;
$\{X_k\} \in D'_F$	$t_k^0 = \sqrt{s_k^0}$.

Table 1: Relationship between (s_k^0) and (t_k^0).

Remark 2 *It is important to note that we are talking here about a possible relationship between (s_k^0) and (t_k^0) in Table 1. As we have seen $\mathcal{UL}$-separating sequences are not unique, and hence the transformation s_k^0 to t_k^0 is also not unique.*

3 Proofs

For the proofs we will use the following standard notation: $[x]$ is the integer-part of some real x. We write $a_n \ll b_n$ if $\limsup_{n\to\infty} |a_n/b_n| < \infty$.

3.1 Proof of the upper bound in Theorem 1

We have for $k \geq 1$ and for $\varepsilon > 0$

$$P\left(X_k \geq m_{[k^{1+\varepsilon}]}\right) = \frac{1}{[k^{1+\varepsilon}]}.$$

Therefore by the Borel-Cantelli lemma

$$\limsup_{k\to\infty} \frac{X_k}{m_{[k^{1+\varepsilon}]}} \leq 1 \qquad \text{a.s.}$$

Our assumptions imply that $m_k^\alpha q(m_k) = \log k$, with slowly varying q. One easily obtains

$$\frac{m_{[k^{1+\varepsilon}]}}{m_k} \to (1+\varepsilon)^{1/\alpha}, \quad k \to \infty.$$

As ε can be chosen arbitrarily small we conclude that

$$\limsup_{k\to\infty} \frac{X_k}{m_k} \leq 1 \qquad \text{a.s.},$$

and consequently by the law of the iterated logarithm for W we have

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{2 m_k \log\log m_k}} \leq 1 \qquad \text{a.s.}$$

3.2 Proof of the lower bound in Theorem 1

Let $\varepsilon > 0$ be arbitrary, but fixed. We choose $\theta > 1$ so large that

$$2\varepsilon^{-\alpha} \leq \theta \tag{13}$$

(and, to shorten notations, we will assume throughout this section that θ is an integer). Set

$$i_n = e^{(\theta^n)} \qquad \text{and} \qquad I_n = \{k: \ i_{n-1} < k \leq i_n\},$$

and

$$M_n = \min\left\{x \in \mathbb{R}: \ F(x) = 1 - \frac{1}{i_n}\right\}.$$

Then for sufficiently large n,

$$(\log i_n)^{\frac{1}{\alpha}-\varepsilon} \leq M_n \leq (\log i_n)^{\frac{1}{\alpha}+\varepsilon}.$$

Since for sufficiently large x

$$-(\varepsilon^{-1}x)^\alpha q\left(\varepsilon^{-1}x\right) \geq -2\varepsilon^{-\alpha} x^\alpha q(x),$$

by (13) for sufficiently large n

$$\begin{aligned}
1 - F(\varepsilon^{-1}M_n) &\geq e^{-(\varepsilon^{-1}M_n)^\alpha q\left(\varepsilon^{-1}M_n\right)} \\
&\geq e^{-2\varepsilon^{-\alpha} M_n^\alpha q(M_n)} \\
&\geq (1 - F(M_n))^{-2\varepsilon^{-\alpha}} \\
&= \left(\frac{1}{i_n}\right)^{2\varepsilon^{-\alpha}} \\
&\gg \frac{1}{i_{n+1}},
\end{aligned}$$

and

$$M_{n+1} \geq \varepsilon^{-1} M_n. \tag{14}$$

Set further

$$\varphi(n) = \sqrt{(1-4\varepsilon)2M_n \log\log\log i_n}$$

and

$$\begin{aligned} t_n &= (1+\varepsilon)M_{n-1}, \\ B(n) &= [(1-\varepsilon)M_n, (1+\varepsilon)M_n]. \end{aligned}$$

Informally speaking, we will show that with large probability $\max_{k \in I_n} X_k \in B(n)$, and prove a lower bound for $\limsup_{n\to\infty} \frac{W((1-\varepsilon)M_n)-W(t_n)}{\varphi(n)}$. To obtain this lower bound we will use the fact that the random variables $W((1-\varepsilon)M_n) - W(t_n)$, $n \geq 1$, are independent. Finally, we will show that $W(\max_{k\in I_n} X_k)$ is almost of the same size as $W((1-\varepsilon)M_n)$, provided $\max_{k\in I_n} X_k \in B(n)$. Combining these results will prove Theorem 1.

There exists an $n_0 \geq 1$, such that all intervals $B(n)$, $n \geq n_0$ are disjoint. We define events

$$A_n = \left\{ \max_{k\in I_n} X_k \in B(n) \right\} \cap \{W((1-\varepsilon)M_n) - W(t_n) \geq \varphi(n)\}, \qquad n \geq n_0.$$

Then these events are independent, since the sets I_n are disjoint and since $t_{n+1} > (1-\varepsilon)M_n$.

The events $(\max_{k\in I_n} X_k \in B(n))$ and $(W((1-\varepsilon)M_n) - W(t_n \geq \varphi(n))$ are also independent for $n \geq n_0$, which implies

$$P(A_n) = P\left(\max_{k\in I_n} X_k \in B(n) \right) \times P\left(W(1-\varepsilon)M_n) - W(t_n) \geq \varphi(n) \right). \tag{15}$$

We have

$$P\left(\max_{k\in I_n} X_k \in B(n) \right) \;=\; P\left(\max_{k\in I_n} X_k \leq (1+\varepsilon)M_n \right) - P\left(\max_{k\in I_n} X_k < (1-\varepsilon)M_n \right).$$

Since q is slowly varying, for sufficiently large x,

$$q\left((1+\varepsilon)x\right) \geq \frac{1}{(1+\varepsilon)^{\alpha/2}} q(x).$$

Therefore, for sufficiently large n,

$$\begin{aligned} 1 - F((1+\varepsilon)M_n) &\leq e^{-(1+\varepsilon)^{\alpha/2}(M_n^\alpha q(M_n))} \\ &\leq (1-F(M_n))^{(1+\varepsilon)^{\alpha/2}} \\ &= \left(\frac{1}{i_n}\right)^{\left((1+\varepsilon)^{\alpha/2}\right)}, \end{aligned}$$

and, since

$$\frac{i_n - i_{n-1}}{i_n^{\left((1+\varepsilon)^{\alpha/2}\right)}} \to 0 \quad \text{as} \quad n \to \infty,$$

we obtain

$$
\begin{aligned}
P\left(\max_{k\in I_n} X_k \le (1+\varepsilon)M_n\right) &\ge \left(1-\left(\frac{1}{i_n}\right)^{\left((1+\varepsilon)^{\alpha/2}\right)}\right)^{i_n-i_{n-1}} \\
&\ge \left(\left(1-\left(\frac{1}{i_n}\right)^{\left((1+\varepsilon)^{\alpha/2}\right)}\right)^{i_n^{\left((1+\varepsilon)^{\alpha/2}\right)}}\right)^{\frac{i_n-i_{n-1}}{i_n^{\left((1+\varepsilon)^{\alpha/2}\right)}}} \\
&\ge \frac{3}{4} \qquad (16)
\end{aligned}
$$

for sufficiently large n.

Similarly, since q is slowly varying, for sufficiently large x,

$$
q\left((1-\varepsilon)x\right) \le \frac{1}{(1-\varepsilon)^{\alpha/2}} q(x).
$$

Thus

$$
\begin{aligned}
1-F((1-\varepsilon)M_n) &\ge \left(e^{-(M_n^{\alpha})q(M_n)}\right)^{\left((1-\varepsilon)^{\alpha/2}\right)} \\
&= \left(\frac{1}{i_n}\right)^{\left((1-\varepsilon)^{\alpha/2}\right)},
\end{aligned}
$$

and, since

$$
\frac{i_n - i_{n-1}}{i_n^{\left((1-\varepsilon)^{\alpha/2}\right)}} \to \infty \quad \text{as} \quad n \to \infty,
$$

we obtain

$$
\begin{aligned}
P\left(\max_{k\in I_n} X_k \le (1-\varepsilon)M_n\right) &\le \left(\left(1-\left(\frac{1}{i_n}\right)^{\left((1-\varepsilon)^{\alpha/2}\right)}\right)^{i_n^{\left((1-\varepsilon)^{\alpha/2}\right)}}\right)^{\frac{i_n}{i_n^{\left((1-\varepsilon)^{\alpha/2}\right)}}} \\
&\le \frac{1}{4} \qquad (17)
\end{aligned}
$$

for sufficiently large n.

Combining (16) and (17) we have shown

$$
P\left(\max_{k\in I_n} X_k \in B(n)\right) \ge \frac{1}{2} \qquad (18)
$$

for sufficiently large n.

For sufficiently large n we have by (14)

$$\begin{aligned}
P\left(W(1-\varepsilon)M_n) - W(t_n) \geq \varphi(n)\right) &= P\left(W(1-\varepsilon)M_n - (1+\varepsilon)M_{n-1}) \geq \varphi(n)\right) \\
&\geq P\left(W((1-3\varepsilon)M_n) \geq \varphi(n)\right) \\
&= P\left(W(1) \geq \sqrt{2(1-4\varepsilon)(1-3\varepsilon)^{-1}\log\log\log i_n}\right) \\
&\gg \frac{e^{-(1-4\varepsilon)(1-3\varepsilon)^{-1}\log\log\log i_n}}{\sqrt{\log\log\log i_n}} \\
&\gg \frac{1}{n^{\frac{1-4\varepsilon}{1-3\varepsilon}}\sqrt{\log n}}.
\end{aligned}$$

Combined with (15) and (18) this yields

$$P(A_n) \gg \frac{1}{n^{\frac{1-4\varepsilon}{1-3\varepsilon}}\sqrt{\log n}}, \tag{19}$$

and hence

$$\sum_{n=n_0}^{\infty} P(A_n) = \infty.$$

Thus we have shown that, by the second Borel-Cantelli lemma, with probability 1 infinitely many events A_n occur.

Next we want to replace $W(1-\varepsilon)M_n)$ by $W(\max_{k\in I_n} X_k)$. We have

$$\begin{aligned}
& P\left(\left|\min_{t\in B(n)} W(t) - W((1-\varepsilon)M_n)\right| \geq 2\sqrt{\varepsilon}\varphi(n)\right) \\
= \;& P\left(\max_{t\in[0,2\varepsilon M_n]} W(t) \geq 2\sqrt{\varepsilon}\varphi(n)\right) \\
= \;& 2P(W(2\varepsilon M_n) \geq 2\sqrt{\varepsilon}\varphi(n)) \\
= \;& 2P(W(1) \geq \sqrt{2(1-4\varepsilon)\log\log\log i_n}) \\
\ll \;& n^{-2(1-4\varepsilon)}.
\end{aligned}$$

We can assume w.l.o.g. that $1-4\varepsilon > 1/2$, and have, by the first Borel-Cantelli lemma, that with probability 1 only finitely many events

$$\left(\left|\min_{t\in B(n)} W(t) - W((1-\varepsilon)M_n)\right| \geq 2\sqrt{\varepsilon}\varphi(n)\right)$$

occur.

To replace $W((1-\varepsilon)M_n) - W(t_n)$ by $W((1-\varepsilon)M_n)$ we consider the following: since by (14) for sufficiently large n

$$t_n \leq \varepsilon M_n$$

we have

$$\begin{aligned}
P(W(t_n) \geq \sqrt{2\varepsilon}\varphi(n)) &\leq P(W(M_n) \geq \sqrt{2}\varphi(n)) \\
&= P(W(1) \geq \sqrt{2(1-4\varepsilon)\log\log\log i_n}) \\
&\ll n^{-2(1-4\varepsilon)}.
\end{aligned}$$

Thus, assuming again w.l.o.g. that $1-4\varepsilon > 1/2$, by the first Borel-Cantelli lemma with probability 1 only finitely many events

$$\left(W(t_n) \geq \sqrt{2\varepsilon}\varphi(n)\right)$$

occur.

This means, that with probability 1 infinitely many events

$$\begin{aligned}&\left\{\max_{k\in I_n} X_k \in B(n)\right\} \ \cap \ \{W((1-\varepsilon)M_n)) - W(t_n) \geq \varphi(n)\} \\ &\cap\left\{\left|\min_{t\in B(n)} W(t) - W((1-\varepsilon)M_n))\right| \leq 2\sqrt{\varepsilon}\varphi(n)\right\} \cap \left\{W(t_n) \leq \sqrt{2\varepsilon}\varphi(n)\right\}\end{aligned}$$

occur. Therefore, with probability one, also infinitely many events

$$\left\{W\left(\max_{k\in I_n} X_k\right) \geq (1-4\sqrt{\varepsilon})\varphi(n)\right\}$$

occur. Thus we have

$$\limsup_{n\to\infty} \frac{W\left(\max_{k\in I_n} X_k\right)}{(1-4\sqrt{\varepsilon})\varphi(n)} \geq 1 \quad \text{a.s.},$$

which implies

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{2m_n \log\log\log i_n}} \geq (1-4\sqrt{\varepsilon})\sqrt{1-4\varepsilon} \quad \text{a.s.}$$

Since ε can be chosen arbitrarily small, this proves Theorem 1.

3.3 Proof of the upper bound in Theorem 2

Let $\theta > 1$ be arbitrary, but fixed, and set

$$i_n = [\theta^n] \quad \text{and} \quad I_n = \{k:\ 1 \leq k \leq i_n\},$$

and

$$M_n = \min\left\{x \in \mathbb{R}:\ F(x) = 1 - \frac{1}{i_n}\right\}.$$

Let $\varepsilon > 0$ be fixed and set

$$\varphi(n) = \sqrt{M_n(\log i_n)^{1/\alpha}(\log\log i_n)^{1/\alpha+\varepsilon}}.$$

Then for any $n \geq 1$ and

$$\begin{aligned} S_n &= 2M_n(\log i_n)^{1/\alpha}(\log\log i_n)^{1/\alpha-1+\varepsilon}, \\ T_n &= (1+\alpha)M_n(\log i_n)^{1/\alpha}(\log\log i_n)^{1/\alpha+\varepsilon}(\log\log\log i_n)^{-1}. \end{aligned}$$

we have

$$\begin{aligned} & P\left(\max_{k\in I_n} W(X_k) \geq \varphi(n)\right) \\ \leq\ & P\left(\max_{t\in[0,S_n]} W(t) \geq \varphi(n)\right) && (20) \\ & +P\left(\left\{\max_{k\in I_n} X_k \geq S_n\right\} \cap \left\{\max_{t\in[0,T_n]} W(t) \geq \varphi(n)\right\}\right) && (21) \\ & +P\left(\max_{k\in I_n} X_k \geq T_n\right). && (22) \end{aligned}$$

The term (20) is bounded by

$$2P\left(W(S_n) \geq \varphi(n)\right) = 2P(W(1) \geq \sqrt{2\log\log i_n}) \ll \frac{1}{(\log i_n)^2} \tag{23}$$

For sufficiently large x and $y \in [1, (\log x)^{2/\alpha}]$, by (9),

$$q(yx) \leq (1+\varepsilon)q(x).$$

Thus for sufficiently large n for all $y \in [1, (\log M_n)^{2/\alpha}]$,

$$\begin{aligned} 1 - F(yM_n) &\leq \frac{1}{y^\alpha M_n^\alpha} q(yM_n) \\ &\leq (1+\varepsilon)\left(\frac{1}{y^\alpha}\frac{1}{M_n^\alpha} q(M_n)\right) \\ &\leq (1+\varepsilon)\frac{1}{y^\alpha}\left(1 - F(M_n)\right) \\ &= (1+\varepsilon)\frac{1}{y^\alpha}\frac{1}{i_n}, \end{aligned}$$

and

$$\begin{aligned} P\left(\max_{k\in I_n} X_k \geq yM_n\right) &\leq 1 - \left(1 - (1+\varepsilon)\frac{1}{y^\alpha}\frac{1}{i_n}\right)^{i_n} \\ &\leq 1 - \left(\frac{1}{e}\right)^{\frac{(1+\varepsilon)^2}{y^\alpha}} \\ &\leq \frac{(1+\varepsilon)^2}{y^\alpha} \end{aligned} \tag{24}$$

Since $S_n \leq T_n$ and $T_n \leq M_n(\log M_n)^{2/\alpha}$ for sufficiently large n, the term (21) is bounded by

$$\begin{aligned} & 2P\left(\max_{k\in I_n} X_k \geq S_n\right) P\left(W(T_n) \geq \varphi(n)\right) \\ \ll\ & \frac{1}{\log i_n (\log\log i_n)^{\alpha(1/\alpha - 1 + \varepsilon)}} \frac{1}{(\log\log i_n)^{1+\alpha}}. \end{aligned} \tag{25}$$

The term (22) is bounded by

$$\ll \frac{(\log\log\log i_n)^\alpha}{\log i_n (\log\log i_n)^{1+\alpha\varepsilon}}. \tag{26}$$

Combining the estimates (23), (25) and (26) for (20), (21) and (25), we obtain

$$P\left(\max_{k\in I_n} W(X_k) \geq \varphi(n)\right) \ll \frac{1}{\log i_n (\log\log i_n)^{1+\varepsilon_1}}$$

for some appropriate (small) $\varepsilon_1 > 0$. In particular

$$\sum_{n\geq n_0} P\left(\max_{k\in I_n} W(X_k) \geq \varphi(n)\right) < \infty,$$

and by the first Borel-Cantelli lemma with probability 1 only finitely many events

$$\left\{\max_{k\in I_n} W(X_k) \geq \varphi(n)\right\}$$

occur. Since

$$\limsup_{n\to\infty} \frac{\varphi(n+1)}{\varphi(n)} < \infty,$$

this implies

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{m_k(\log k)^{1/\alpha}(\log\log k)^{1/\alpha+\varepsilon}}} < \infty \qquad \text{a.s.}$$

3.4 Proof of the lower bound in Theorem 2

We will use the following version of the second Borel-Cantelli lemma (which is due to Kochen and Stone [8] and Spitzer [11]; cf. also [10]):

Lemma 1 *Let $A_1, A_2, \ldots$ be events such that*

$$\sum_{n=1}^{\infty} P(A_n) = \infty.$$

If additionally

$$\liminf_{n\to\infty} \frac{\sum_{k,l=1}^{n} P(A_k A_l)}{\left(\sum_{k=1}^{n} P(A_k)\right)^2} = L,$$

then

$$P(\limsup_{n\to\infty} A_n) \geq \frac{1}{L}.$$

Let $\varepsilon > 0$ be given. Choose $\theta > 1$ such that

$$\theta > \frac{8}{\varepsilon \min(\alpha, 1)} \tag{27}$$

and set

$$i_n = [\theta^n] \qquad \text{and} \qquad I_n = \{k : \ i_{n-1} < k \leq i_n\}.$$

Set further

$$M_n = \min\left\{x \in \mathbb{R} : \ F(x) = 1 - \frac{1}{i_n}\right\}$$

and

$$\varphi(n) = \sqrt{M_n(\log i_n)^{1/\alpha}(\log\log i_n)^{1/\alpha}(\log\log\log i_n)^{1/\alpha}}.$$

and

$$\begin{aligned} T_n &= M_n(\log i_n)^{1/\alpha}(\log\log i_n)^{1/\alpha}(\log\log\log i_n)^{1/\alpha}, \\ B(n) &= [T_n, (1+\varepsilon)^{7/\alpha} T_n]. \end{aligned}$$

Then for some appropriate $n_0 \geq 1$ the intervals $B(n)$, $n \geq n_0$ are disjoint, and by (9) and (27) we have

$$\frac{M_n}{M_{n+1}} \leq \varepsilon, \quad \frac{T_n}{T_{n+1}} \leq \varepsilon \quad \text{and} \quad \frac{\varphi(n)}{\varphi(n+1)} \leq \sqrt{\varepsilon} \tag{28}$$

for sufficiently large n.

Define events

$$A_n = \left\{ \max_{k \in I_n} X_k \in B(n) \right\} \cap \{ W(t) \in [\varphi(n), 2\varphi(n)] \text{ for all } t \in B(n) \}, \qquad n > n_0.$$

Then the events $A_n,\ n > n_0$ are *not* independent, but the events

$$\left\{ \max_{k \in I_n} X_k \in B(n) \right\}, \quad n \geq n_0$$

are independent since the sets $I_n,\ n \geq n_0$ are disjoint.

For sufficiently large x and $y \in [1, (\log x)^{2/\alpha}]$ by (9)

$$q(yx) \geq (1-\varepsilon) q(x).$$

Thus for sufficiently large n for all $y \in [1, (\log M_n)^{2/\alpha}]$,

$$\begin{aligned} 1 - F(yM_n) &\geq \frac{1}{y^\alpha M_n^\alpha} q(yM_n) \\ &\geq (1-\varepsilon) \left(\frac{1}{y^\alpha} \frac{1}{M_n^\alpha} q(M_n) + \frac{1}{y^\alpha} \right) \\ &\geq \frac{(1-\varepsilon)}{y^\alpha} (1 - F(yM_n)) \\ &= (1-\varepsilon) \frac{1}{y^\alpha} \frac{1}{i_n}, \end{aligned}$$

and, since by (27) for sufficiently large n

$$\frac{i_n - i_{n-1}}{i_n} \geq (1-\varepsilon),$$

we obtain (if w.l.o.g. ε is sufficiently small)

$$\begin{aligned} P\left(\max_{k \in I_n} X_k \geq yM_n \right) &\geq 1 - \left(1 - (1-\varepsilon) \frac{1}{y^\alpha} \frac{1}{i_n} \right)^{i_n - i_{n-1}} \\ &\geq 1 - \left(\frac{1}{e} \right)^{\frac{(1-\varepsilon)^3}{y^\alpha}} \\ &\geq \frac{(1-\varepsilon)^6}{y^\alpha} \\ &\geq \frac{1}{(1+\varepsilon)^4 y^\alpha}. \end{aligned}$$

Similar to (24) we can get

$$P\left(\max_{k \in I_n} X_k \geq yM_n \right) \leq \frac{(1+\varepsilon)^2}{y^\alpha},$$

and therefore we get (since $T_n \le M_n(\log M_n)^{2/\alpha}$ for sufficiently large n)

$$\begin{aligned}
& P\left(\max_{k\in I_n} X_k \in B(n)\right) \\
\geq\ & \frac{1}{(1+\varepsilon)^4 \log i_n \log\log i_n} - \frac{(1+\varepsilon)^2}{(1+\varepsilon)^7 \log i_n \log\log i_n \log\log\log i_n} \\
\geq\ & \underbrace{\left(\frac{1}{(1+\varepsilon)^4} - \frac{1}{(1+\varepsilon)^5}\right)}_{>0} \frac{1}{\log i_n \log\log i_n \log\log\log i_n}.
\end{aligned}$$

for sufficiently large n. On the other hand, it is easy to see that

$$\begin{aligned}
& P\left(W(t) \in [\varphi(n), 2\varphi(n)] \text{ for all } t \in B(n)\right) \\
\geq\ & P\left(W(T_n) \in [5/4\varphi(n), 7/4\varphi(n)]\right) - P\left(\max_{t\in B(n)} W(t) - W(T_n) \geq \frac{1}{4}\varphi(n)\right) \\
=\ & P\left(W(1) \in [5/4, 7/4]\right) - 2P\left(W((1+\varepsilon)^{7/\alpha}) \geq \frac{1}{4}\right) \\
\geq\ & \frac{1}{20},
\end{aligned} \tag{29}$$

if we assume (w.l.o.g.) that ε is sufficiently small.

Thus

$$P(A_n) \gg \frac{1}{\log i_n \log\log i_n \log\log\log i_n}$$

and

$$\sum_{n>n_0} P(A_n) = \infty. \tag{30}$$

Let $n_1 < n_2$ be two positive integers. Then

$$\begin{aligned}
& P(A_{n_1} A_{n_2}) \\
=\ & P\left(\left\{\max_{k\in I_{n_1}} X_k \in B(n_1)\right\} \cap \{W(t) \in [\varphi(n_1), 2\varphi(n_1)] \text{ for all } t \in B(n_1)\} \cap \right. \\
& \quad \left. \cap \left\{\max_{k\in I_{n_2}} X_k \in B(n_2)\right\} \cap \{W(t) \in [\varphi(n_2), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\}\right) \\
=\ & P\left(\max_{k\in I_{n_1}} X_k \in B(n_1)\right) \times P\left(\max_{k\in I_{n_2}} X_k \in B(n_2)\right) \times \\
& \quad \times P\left(\{W(t) \in [\varphi(n_1), 2\varphi(n_1)] \text{ for all } t \in B(n_1)\} \cap \right. \\
& \qquad \left. \cap \{W(t) \in [\varphi(n_2), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\}\right),
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
& P\left(\{W(t) \in [\varphi(n_1), 2\varphi(n_1)] \text{ for all } t \in B(n_1)\} \cap \right. \\
& \quad \left. \cap \{W(t) \in [\varphi(n_2), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\}\right) \\
\leq \; & P\left(\{W(t) \in [\varphi(n_1), 2\varphi(n_1)] \text{ for all } t \in B(n_1)\} \cap \right. \\
& \quad \left. \cap \left\{W(t) - W((1+\varepsilon)^{7/\alpha} T_{n_1}) \in [\varphi(n_2) - 2\varphi(n_1), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\right\}\right) \\
= \; & P\left(W(t) \in [\varphi(n_1), 2\varphi(n_1)] \text{ for all } t \in B(n_1)\right) \times \\
& \times P\left(W(t) - W((1+\varepsilon)^{7/\alpha} T_{n_1}) \in [\varphi(n_2) - 2\varphi(n_1), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\right). \quad (32)
\end{aligned}$$

By (28) for sufficiently large n_1, n_2

$$\frac{T_{n_1}}{T_{n_2}} \leq \varepsilon \quad \text{and} \quad \frac{\varphi(n_1)}{\varphi(n_2)} \leq \sqrt{\varepsilon},$$

and if (w.l.o.g.) ε is sufficiently small

$$(1+\varepsilon)^{7/\alpha} \leq \frac{8}{\alpha}\varepsilon,$$

which implies by (27)

$$(1+\varepsilon)^{7/\alpha} T_{n_1} \leq \varepsilon T_{n_2}.$$

Therefore, if we assume w.l.o.g. that ε is so small that

$$2P(|W(1)| \geq \alpha^{1/2} 8^{-1/2} \varepsilon^{-1/4}) \leq \varepsilon^{1/4} \quad \text{and} \quad P\left(W(1) \geq \varepsilon^{-1/4}\right) \leq \varepsilon^{1/4},$$

we get, using (29),

$$\begin{aligned}
& P\left(W(t) - W((1+\varepsilon)^{7/\alpha} T_{n_1}) \in [\varphi(n_2) - 2\varphi(n_1), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\right) \\
\leq \; & P\left(W(t) \in [(1 - 3\varepsilon^{1/4})\varphi(n_2), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\right) \\
& + P\left(W((1+\varepsilon)^{7/\alpha} T_{n_1}) \geq \varepsilon^{1/4} \varphi(n_2)\right) \\
\leq \; & P\left(W(t) \in [\varphi(n_2), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\right) \\
& + P\left(W(T_{n_2}) \in [(1 - 4\varepsilon^{1/4})\varphi(n_2), (1 + \varepsilon^{1/4})\varphi(n_2)]\right) \\
& + P\left(\max_{t \in B(n_2)} |W(t) - W(T_{n_2})| \geq \varepsilon^{1/4} \varphi(n_2)\right) \\
& + P\left(W(\varepsilon T(n_2)) \geq \varepsilon^{1/4} \varphi(n_2)\right) \\
\leq \; & P\left(W(t) \in [\varphi(n_2), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\right) \\
& + P\left(W(1) \in [(1 - 4\varepsilon^{1/4}), (1 + \varepsilon^{1/4})]\right) \\
& + P\left(\max_{t \in [0, 8\varepsilon/\alpha]} |W(t)| \geq \varepsilon^{1/4}\right) \\
& + P\left(W(1) \geq \varepsilon^{-1/4}\right) \\
\leq \; & P\left(W(t) \in [\varphi(n_2), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\right) + 7\varepsilon^{1/4} \\
\leq \; & \left(1 + 140\varepsilon^{1/4}\right) P\left(W(t) \in [\varphi(n_2), 2\varphi(n_2)] \text{ for all } t \in B(n_2)\right). \quad (33)
\end{aligned}$$

Thus, combining (31), (32) and (33), we have

$$P(A_{n_1}A_{n_2}) \leq \left(1 + 140\varepsilon^{1/4}\right) P(A_{n_1})P(A_{n_2}).$$

By Lemma 1 and (30), with probability $\geq \left(1 + 140\varepsilon^{1/4}\right)^{-1}$ infinitely many events A_n occur. Therefore with probability $\geq \left(1 + 140\varepsilon^{1/4}\right)^{-1}$

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{m_n(\log k)^{1/\alpha}(\log\log k)^{1/\alpha}(\log\log\log k)^{1/\alpha}}} \geq 1.$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{m_n(\log k)^{1/\alpha}(\log\log k)^{1/\alpha}(\log\log\log k)^{1/\alpha}}} \geq 1 \quad \text{a.s.},$$

and

$$\limsup_{k\to\infty} \frac{W(X_k)}{\sqrt{m_n(\log k)^{1/\alpha}(\log\log k)^{1/\alpha}}} = \infty \quad \text{a.s.},$$

which proves Theorem 2.

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