Irregular discrepancy behavior of lacunary series

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Abstract

In 1975 Philipp showed that for any increasing sequence (n_k) of positive integers satisfying the Hadamard gap condition $n_{k+1}/n_k > q > 1, k \geq 1$, the discrepancy D_N of $(n_k x)$ mod 1 satisfies the law of the iterated logarithm

$$1/4 \leq \limsup_{N\to\infty} N D_N(n_k x)(N \log\log N)^{-1/2} \leq C_q \quad \text{a.e.}$$

Recently, Fukuyama computed the value of the lim sup for sequences of the form $n_k = \theta^k$, $\theta > 1$, and in a preceding paper the author gave a Diophantine condition on (n_k) for the value of the limsup to be equal to 1/2, the value obtained in the case of i.i.d. sequences. In this paper we utilize this number-theoretic connection to construct a lacunary sequence (n_k) for which the lim sup in the LIL for the star-discrepancy D_N^* is not a constant a.e. and is not equal to the lim sup in the LIL for the discrepancy D_N.

1 Introduction and statement of result

Given a sequence $(x_k)_{k\geq 1}$ of real numbers, the values

$$D_N = D_N(x_1, \dots, x_N) = \sup_{0\leq a<b<1} \left| \frac{\sum_{k=1}^N \mathbb{1}_{[a,b)}(\langle x_k \rangle)}{N} - (b-a) \right|,$$

$$D_N^* = D_N^*(x_1, \dots, x_N) = \sup_{0\leq a<1} \left| \frac{\sum_{k=1}^N \mathbb{1}_{[0,a)}(\langle x_k \rangle)}{N} - a \right|$$

are called the "discrepancy", resp. "star discrepancy" of the first N elements of $(x_k)_{k\geq 1}$. Here $\mathbb{1}_{[a,b)}$ denotes the indicator function of the interval $[a,b)$, and $\langle \cdot \rangle$ denotes fractional part. A sequence $(x_k)_{k\geq 1}$ is called uniformly distributed mod 1 if $D_N(x_1,\dots,x_N) \to 0$ as $N \to \infty$. By a classical result of H. Weyl [14], for any increasing sequence $(n_k)_{k\geq 1}$ of positive integers, $(n_k x)_{k\geq 1}$ is uniformly distributed mod 1 for almost all real x in the sense of the Lebesgue measure. The precise order of magnitude of the discrepancy of $(n_k x)$ is known only for a few sequences (n_k), for example for $n_k = k$ (Khinchin [9], Kesten [8]), and for lacunary sequences (n_k) (Philipp [10]). Specifically, Philipp proved that if (n_k) satisfies the Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1, \qquad k = 1, 2 \dots$$

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then

$$\frac{1}{4\sqrt{2}} \le \limsup_{N\to\infty} \frac{ND_N(n_kx)}{\sqrt{2N\log\log N}} \le C_q \qquad \text{a.e.,} \tag{1}$$

where C_q is a positive number depending on q. (For simplicity, we use the notation $D_N(n_kx)$ instead of $D_N(n_1x,\dots,n_Nx)$.) Comparing with the Chung-Smirnov law of the iterated logarithm

$$\limsup_{N\to\infty} \frac{ND_N(\xi_k)}{\sqrt{2N\log\log N}} = 1/2 \qquad \text{a.e.} \tag{2}$$

(see e.g. [11], p. 504) for the discrepancy of i.i.d. sequences $(\xi_k)_{k\ge1}$, Philipp's result shows that the sequence $(\langle n_kx\rangle)_{k\ge1}$ behaves like a sequence of i.i.d. uniform random variables in $[0,1)$. The analogy between $(\langle n_kx\rangle)_{k\ge1}$ and independent r.v.'s is, however, not complete. Erdős and Fortet (see [7], p. 646) showed that for $n_k = 2^k - 1$ and $f(x) = \cos 2\pi x + \cos 4\pi x$ the normed sum $N^{-1/2}\sum_{k=1}^N f(n_kx)$ has a nongaussian limit distribution and it is easy to see that the sequence $f(n_kx)$ also fails the LIL, namely

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N f(n_kx)}{\sqrt{2N\log\log N}} = \sqrt{2}|\cos\pi x| \qquad \text{a.e.}$$

As the theory shows, the asymptotic behavior of the sequence $(f(n_kx))_{k\ge1}$ is determined by an interplay between the probabilistic properties of $(\langle n_kx\rangle)$, the analytic properties of f and the number-theoretic properties of the sequence (n_k). This makes the study of $\sum_{k=1}^N f(n_kx)$ a very difficult problem; in particular, the value of the limsup in (1) has been found only for a few sequences (n_k) and it is an open question if the limsup is always a constant a.e.

Very recently, Fukuyama [6] computed the value of the lim sup for the sequences $n_k = \theta^k$, where $\theta > 1$, not necessarily integer. Put

$$\Sigma_\theta = \limsup_{N\to\infty} \frac{ND_N(\theta^kx)}{\sqrt{2N\log\log N}} \quad \text{and} \quad \Sigma_\theta^* = \limsup_{N\to\infty} \frac{ND_N^*(\theta^kx)}{\sqrt{2N\log\log N}}.$$

Among others, Fukuyama proved that

$$\begin{aligned}
\Sigma_\theta &= \Sigma_\theta^* = \frac{1}{2} \quad \text{a.e.} \quad \text{if } \theta^r \notin \mathbb{Q} \text{ for all } r \in \mathbb{N},\\
\Sigma_2 &= \Sigma_2^* = \frac{\sqrt{42}}{9} \quad \text{a.e.,}\\
\Sigma_\theta &= \Sigma_\theta^* = \frac{\sqrt{(\theta+1)\theta(\theta-2)}}{2\sqrt{(\theta-1)^3}} \quad \text{a.e.} \quad \text{if } \theta \ge 4 \text{ is an even integer,}\\
\Sigma_\theta &= \Sigma_\theta^* = \frac{\sqrt{\theta+1}}{2\sqrt{\theta-1}} \quad \text{a.e.} \quad \text{if } \theta \ge 3 \text{ is an odd integer.}
\end{aligned}$$

In our recent paper [1] we proved that if $(n_k)_{k\ge1}$ satisfies a number theoretic condition slightly stronger than what is required for the validity of the central limit theorem for $\sum_{k=1}^N f(n_kx)$, see [2], the limsup in (1) is equal to $1/2$. This covers, for example, the case when $n_{k+1}/n_k \to \theta$, where θ^r is irrational for $r = 1, 2, \dots$. Note that in all cases, where the exact value of the lim sup has been explicitly calculated, it is a constant a.e. and it is the same for the discrepancy and the star discrepancy.

We want to mention that the best possible lower bound in (1) is still unknown. In all examples where the exact value of the lim sup is known, it is $\geq 1/2$ a.e. for both the discrepancy and the star discrepancy, and it is unclear if this always has to be the case for lacunary (n_k). It is also worth mentioning that recently Fukuyama [5] constructed a sequence of positive integers (n_k) satisfying the linear growth condition $1 \leq n_{k+1} - n_k \leq 5$, for which the value of the lim sup in the LIL for the discrepancy of $(n_k x)$ is not a constant a.e.

The purpose of this paper is to construct a Hadamard lacunary sequence $(n_k)_{k\geq 1}$ such that

$$\limsup_{N\to\infty} \frac{ND_N^*(n_k x)}{\sqrt{2N \log\log N}}$$

is not a constant a.e. Specifically, let

$$n_{2k-1} = 2^{k^2}, \quad n_{2k} = 2^{k^2+1} - 1 \quad (k = 1, 2, \ldots) \tag{3}$$

Then $(n_k) = (2, 3, 16, 31, 512, 1023, \ldots)$. It is easy to see that $n_{k+1}/n_k > 3/2$, $k = 1, 2, \ldots$. We prove

Theorem 1 *For the sequence $(n_k)_{k\geq 1}$ defined by (3) we have*

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N \log\log N}} = \frac{3}{4\sqrt{2}} \quad \text{a.e.}$$

and

$$\limsup_{N\to\infty} \frac{ND_N^*(n_k x)}{\sqrt{2N \log\log N}} = \Psi^*(x) \quad \text{a.e.},$$

where

$$\Psi^*(x) = \begin{cases} \dfrac{3}{4\sqrt{2}}, & 0 \leq x \leq 3/8,\ 5/8 \leq x \leq 1 \\ \dfrac{\sqrt{(4x(1-x) - x)}}{\sqrt{2}}, & 3/8 \leq x \leq 1/2 \\ \dfrac{\sqrt{(4x(1-x) - (1-x))}}{\sqrt{2}}, & 1/2 \leq x \leq 5/8. \end{cases}$$

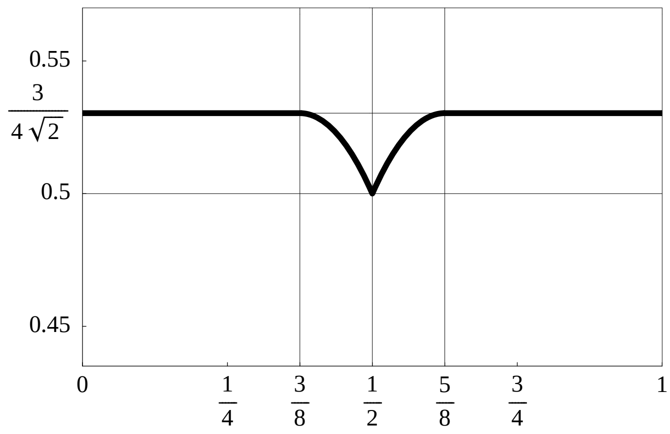

Figure 1: $\Psi^*(x)$

Interestingly, the limsup in the LIL for the discrepancy $D_N(n_k x)$ is a constant, leaving the possibility open that this happens for all Hadamard lacunary sequences (n_k). On the other hand, Theorem 1 disproves a conjecture of Philipp [10, p. 242]. Philipp conjectured that for any lacunary sequence (n_k)

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N \log\log N}} \leq C \qquad \text{a.e.},$$

where

$$C^2 = \sup_{0\leq a<b\leq 1} \limsup_{N\to\infty} \frac{\int_0^1 \left(\sum_{k=1}^N \left(\mathbb{1}_{[a,b)}(n_k x) - (b-a)\right)\right)^2 dx}{N}.$$

However, for the sequence (n_k) defined in (3) for any $0 \leq a < b \leq 1$

$$\lim_{N\to\infty} \frac{\int_0^1 \left(\sum_{k=1}^N \left(\mathbb{1}_{[a,b)}(n_k x) - (b-a)\right)\right)^2 dx}{N} = \left\|\mathbb{1}_{[a,b)} - (b-a)\right\|_2^2 = (b-a)((1-(b-a)),$$

and therefore

$$C^2 = \sup_{0\leq a<b\leq 1} (b-a)((1-(b-a)) = \frac{1}{4}.$$

This would imply

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N \log\log N}} \leq \frac{1}{2} \qquad \text{a.e.},$$

contrary to the result of Theorem 1.

To prove our result we use a "discrepancy splitting" technique due to Fukuyama [6], together with methods developed in one of our earlier papers [1], which utilize the connection between the number of solutions of certain Diophantine equations and the probabilistic behavior of $D_N(n_k x)$.

2 Main section

Lemma 1 *For any function f satisfying*

$$f(x+1) = f(x) \quad \textit{and} \quad \int_0^1 f(x)\ dx = 0$$

we have

$$\left| \int_a^b f(\lambda x)\ dx \right| \le \frac{2}{\lambda} \int_0^1 |f(x)|\ dx \le \frac{2}{\lambda} \|f\|_\infty$$

for any real numbers $a < b$ and any $\lambda > 0$. In particular

$$\left| \int_a^b \cos(2\pi\lambda x)\ dx \right| \le \frac{2}{\lambda} \quad \textit{and} \quad \left| \int_a^b \sin(2\pi\lambda x)\ dx \right| \le \frac{2}{\lambda}.$$

Proof. The lemma follows from

$$\int_a^b f(\lambda x)\ dx = \frac{1}{\lambda} \int_{\lambda a}^{\lambda b} f(x)\ dx. \quad \square$$

For $N \ge 1, j \ge 1, j' \ge 1, \nu \in \mathbb{Z}$ we define

$$A(N, j, j', \nu) = \#\{1 \le k, k' \le N, (j,k) \ne (j', k') : j n_k - j' n_{k'} = \nu\}.$$

Lemma 2 *For $1 \le j' \le j \le d$*

$$A(N, j, j', \nu) = \begin{cases} \frac{N}{2} + \mathcal{O}(1) & \textit{if } j = 2j' \textit{ and } j' = \nu \\ \mathcal{O}(1) & \textit{otherwise} \end{cases}$$

where the implicit constant may only depend on d.

The proof of this lemma will be given in Section 3.

We write $\mathbf{I}_{[a,b)}$ for the indicator function of the interval $[a,b)$, extended with period 1 and centered at expectation, i.e.

$$\mathbf{I}_{[a,b)}(x) = \mathbb{1}_{[a,b)}(\langle x \rangle) - (b-a), \quad x \in \mathbb{R}.$$

For nonnegative $r \in \mathbb{Z}$, $N \ge 1$ and a sequence $(x_k)_{k \ge 1}$ of reals we define

$$\begin{aligned} D_N^{(\le 2^{-r})}(x_1, \ldots, x_N) &= \sup_{0 \le a < b < 1,\ b-a \le 2^{-r}} \left| \frac{\sum_{k=1}^N \mathbf{I}_{[a,b)}(x_k)}{N} \right|, \\ D_N^{(\ge 2^{-r})}(x_1, \ldots, x_N) &= \max_{a_1, a_2 \in \mathbb{Z}, 0 \le a_1 < a_2 \le 2^r} \left| \frac{\sum_{k=1}^N \mathbf{I}_{[a_1 2^{-r}, a_2 2^{-r})}(x_k)}{N} \right|. \end{aligned}$$

and

$$D_N^{*\,(\ge 2^{-r})}(x_1, \ldots, x_N) = \max_{a_1 \in \mathbb{Z}, 0 < a_1 \le 2^r} \left| \frac{\sum_{k=1}^N \mathbf{I}_{[0, a_1 2^{-r})}(x_k)}{N} \right|.$$

It is easy to see that always

$$D_N^{(\geq 2^{-r})} \leq D_N \leq D_N^{(\geq 2^{-r})} + 2D_N^{(\leq 2^{-r})} \tag{4}$$

and

$$D_N^{*\,(\geq 2^{-r})} \leq D_N^* \leq D_N^{*\,(\geq 2^{-r})} + D_N^{(\leq 2^{-r})}. \tag{5}$$

The idea to split the discrepancy D_N into two parts $D_N^{(\leq 2^{-r})}$ and $D_N^{(\geq 2^{-r})}$ to prove an exact LIL for the discrepancy of $(n_k x)$ is due to Fukuyama [6].

We write

$$\mathbf{I}_{[a,b)}(x) \sim \sum_{j=1}^{\infty} a_j(a,b) \cos 2\pi j x + b_j(a,b) \sin 2\pi j x$$

for the Fourier series of $\mathbf{I}_{[a,b)}$, and

$$p_{[a,b),d}(x) = \sum_{j=1}^{d} a_j(a,b) \cos 2\pi j x + b_j(a,b) \sin 2\pi j x$$

for the d-th partial sum of this Fourier series. Then

$$a_j = a_j(a,b) = 2 \int_0^1 \mathbf{I}_{[a,b)}(x) \cos 2\pi j x \; dx = \frac{\sin 2\pi j b - \sin 2\pi j a}{\pi j}, \quad j \geq 1,$$

and

$$b_j = b_j(a,b) = 2 \int_0^1 \mathbf{I}_{[a,b)}(x) \sin 2\pi j x \; dx = \frac{-\cos 2\pi j b + \cos 2\pi j a}{\pi j}, \quad j \geq 1.$$

In particular

$$|a_j| \leq \frac{1}{j}, \quad |b_j| \leq \frac{1}{j}, \quad j \geq 1. \tag{6}$$

For $0 \leq a < b \leq 1,\ x \in \mathbb{R}$ and $d \geq 1$ we define functions

$$\begin{aligned}
\sigma_{[a,b),d}(x) &= \frac{1}{2} \left(\sum_{j=1}^{d} \left(a_j^2 + b_j^2\right) + \sum_{j=1}^{\lfloor d/2 \rfloor} (a_j a_{2j} + b_j b_{2j}) \cos 2\pi j x + (a_j b_{2j} - a_{2j} b_j) \sin 2\pi j x \right), \\
\sigma_{[0,a)}(x) &= a(1-a) - \frac{1}{2} \int_0^1 \mathbf{I}_{[0,1-a)}(t) \cdot \mathbf{I}_{[0,\langle 2a \rangle)}(x-t) \; dt, \\
\sigma_{[a,b)}(x) &= \sigma_{[0,b-a)}(x-a).
\end{aligned}$$

We assert some properties of these functions:

Lemma 3 *For $x, y \in [0,1]$ and $0 \le a < b \le 1$*

- $\sigma_{[0,a)}(x) = \sigma_{[0,1-a)}(1-x),$ (7)
- $\sigma_{[0,a)}(x) = 2a(1-a) - \frac{1}{2}\int_0^1 \mathbb{1}_{[0,1-a)}(t) \cdot \mathbb{1}_{[0,2a)}(\langle x-t \rangle)dt, \quad a \le 1/2$ (8)
- $\sigma_{[a,b)}(x) > 0,$ (9)
- $|\sigma_{[a,b)}(x) - \sigma_{[a,b)}(y)| \le |x-y|,$ (10)
- $|\sigma_{[0,a+\delta)}(x) - \sigma_{[0,a)}(x)| \le 20|\delta|, \qquad 0 \le a+\delta \le 1,$ (11)
- $|\sigma_{[a+\delta_1,b+\delta_2)}(x) - \sigma_{[a,b)}(x)| \le 21(|\delta_1| + |\delta_2|), \qquad 0 \le a+\delta_1 < b+\delta_2 \le 1$ (12)
- $\sup_{0 \le a < b < 1} \sqrt{\sigma_{[a,b)}(x)} = \lim_{r\to\infty} \max_{i,j\in\mathbb{Z}, 0 \le i < j \le 2^r} \sqrt{\sigma_{[i2^{-r}, j2^{-r})}(x)},$ (13)
- $\sup_{0 < a < 1} \sqrt{\sigma_{[0,a)}(x)} = \lim_{r\to\infty} \max_{i\in\mathbb{Z}, 0 < i \le 2^r} \sqrt{\sigma_{[0,i2^{-r})}(x)},$ (14)
- $\sup_{0 \le a < b < 1} \sqrt{\sigma_{[a,b)}(x)} = \frac{3}{4\sqrt{2}},$ (15)
- $\sup_{0 < a < 1} \sqrt{\sigma_{[0,a)}(x)} = \Psi^*(x)$ (16)

Proof of (7): Using the identity $\mathbf{I}_{[0,a)}(x) = -\mathbf{I}_{[0,1-a)}(1-x)$ we get

$$\begin{aligned}\sigma_{[0,1-a)}(1-x) &= (1-a)(1-(1-a)) - \frac{1}{2}\int_0^1 \mathbf{I}_{[0,1-(1-a))}(t)\cdot \mathbf{I}_{[0,\langle 2(1-a)\rangle)}((1-x)-t)\,dt \\ &= a(1-a) - \frac{1}{2}\int_0^1 \mathbf{I}_{[0,1-a)}(1-t)\cdot \mathbf{I}_{[0,\langle 2a\rangle)}(x+t)\,dt \\ &= a(1-a) - \frac{1}{2}\int_0^1 \mathbf{I}_{[0,1-a)}(s)\cdot \mathbf{I}_{[0,\langle 2a\rangle)}(x-s)\,ds \;=\; \sigma_{[0,a)}(x).\end{aligned}$$

Proof of (8): Since by assumption $a \le 1/2$,

$$\begin{aligned}\sigma_{[0,a)}(x) &= a(1-a) - \frac{1}{2}\int_0^1 \mathbf{I}_{[0,1-a)}(t)\cdot \mathbf{I}_{[0,\langle 2a\rangle)}(x-t)\,dt \\ &= a(1-a) - \frac{1}{2}\int_0^1 \big(\mathbb{1}_{[0,1-a)}(t) - (1-a)\big)\cdot\big(\mathbb{1}_{[0,2a)}(\langle x-t\rangle) - 2a\big)\,dt \\ &= a(1-a) + \frac{1}{2}(1-a)2a - \frac{1}{2}\int_0^1 \mathbb{1}_{[0,1-a)}(t)\cdot \mathbb{1}_{[0,2a)}(\langle x-t\rangle)\,dt.\end{aligned}$$

Proof of (9): In view of (7) it suffices to prove (9) for $\sigma_{[0,a)}, 0 < a \le 1/2$. Using (8) we have

$$\begin{aligned}\sigma_{[0,a)}(x) &= 2a(1-a) - \frac{1}{2}\int_0^1 \mathbb{1}_{[0,1-a)}(t)\cdot \mathbb{1}_{[0,2a)}(\langle x-t\rangle)\,dt \\ &\ge 2a(1-a) - \frac{1}{2}\min\{(1-a), 2a\} \;>\; 0.\end{aligned}$$

Proof of (10): By the definition of $\sigma_{[a,b)}$ and (7) it suffices to consider $\sigma_{[0,a)}, a \le 1/2$.

$$|\sigma_{[0,a)}(x) - \sigma_{[0,a)}(y)| \le \frac{1}{2}\int_0^1 \big|\mathbb{1}_{[0,2a)}(\langle x-t\rangle) - \mathbb{1}_{[0,2a)}(\langle y-t\rangle)\big|\,dt \le |x-y|.$$

Proof of (11): Assuming $0 \le a \le 1/2$ and $0 \le a+\delta \le 1/2$ we have

$$\begin{aligned} & |\sigma_{[0,a+\delta)}(x) - \sigma_{[0,a)}(x)| \\ \le\ & 2|(a+\delta)(1-a-\delta) - a(1-a)| \\ +\ & \frac{1}{2}\int_0^1 \left|\mathbb{1}_{[0,1-a-\delta)}(t)\cdot \mathbb{1}_{[0,2a+2\delta)}(\langle x-t\rangle) - \mathbb{1}_{[0,1-a)}(t)\cdot \mathbb{1}_{[0,2a)}(\langle x-t\rangle)\right| \, dt \\ \le\ & 2|\delta|\cdot|2a+\delta-1| + 2|\delta| \ \le\ 10|\delta|. \end{aligned}$$

If $0 \le a \le 1/2$ and $1/2 < a+\delta \le 1$, then

$$\begin{aligned} \left|\sigma_{[0,a+\delta)}(x) - \sigma_{[0,a)}(x)\right| &\le \left|\sigma_{[0,1/2)}(x) - \sigma_{[0,a)}(x)\right| + \left|\sigma_{[0,1/2)}(x) - \sigma_{[0,a+\delta)}(x)\right| \\ &\le 10|\delta| + \left|\sigma_{[0,1/2)}(1-x) - \sigma_{[0,1-a-\delta)}(1-x)\right| \\ &\le 20|\delta|. \end{aligned}$$

Together with property (7) this yields (11).
Proof of (12): Using (10) and (11) we get

$$\begin{aligned} & |\sigma_{[a+\delta_1,b+\delta_2)}(x) - \sigma_{[a,b)}(x)| \\ =\ & |\sigma_{[0,b+\delta_2-a-\delta_1)}(x-a-\delta_1) - \sigma_{[0,b-a)}(x-a)| \\ \le\ & \left|\sigma_{[0,b+\delta_2-a-\delta_1)}(x-a-\delta_1) - \sigma_{[0,b-a)}(x-a-\delta_1)\right| \\ & + \left|\sigma_{[0,b-a)}(x-a-\delta_1) - \sigma_{[0,b-a)}(x-a)\right| \\ \le\ & 20(|\delta_1|+|\delta_2|) + |\delta_1| \ \le\ 21(|\delta_1|+|\delta_2|). \end{aligned}$$

Proof of (13) and (14): These equations are consequences of (12).
Proof of (15): Again we assume $a \le 1/2$. For $x \le 2a$ we have

$$\begin{aligned} \sigma_{[0,a)}(x) &= 2a(1-a) - \frac{1}{2}\int_0^1 \mathbb{1}_{[0,1-a)}(t)\cdot \mathbb{1}_{[0,2a)}(\langle x-t\rangle)\, dt \\ &= 2a(1-a) - \frac{1}{2}\int_0^1 \mathbb{1}_{[0,1-a)}(t)\cdot \mathbb{1}_{\{[0,x)\cup[x+1-2a,1)\}}(t)\, dt \\ &= 2a(1-a) - 1/2\ \mathbb{P}\left\{[0,1-a)\cap([0,x)\cup[x+1-2a,1))\right\} \\ &= 2a(1-a) - 1/2\ \left(\min(1-a,x) + \max(0,a-x)\right) \qquad (17) \\ &\le 2a(1-a) - a/2 \le \frac{9}{32} \end{aligned}$$

(here and in the sequel $\mathbb{P}$ denotes the Lebesgue measure on $(0,1)$), and for $x \ge 2a$ we get

$$\begin{aligned} \sigma_{[0,a)}(x) &= 2a(1-a) - 1/2\ \left(\min(1-a,x) - (x-2a)\right) \qquad (18) \\ &\le 2a(1-a) - a/2 \le \frac{9}{32} \end{aligned}$$

in a similar way. In view of (7) and the definition of $\sigma_{[a,b)}$ this yields

$$\sigma_{[a,b)}(x) \le \frac{9}{32}, \quad 0 \le a < b \le 1,\ x \in [0,1],$$

which implies

$$\sup_{0\le a<b<1} \sqrt{\sigma_{[a,b)}(x)} \ \le\ \frac{3}{4\sqrt{2}}. \qquad (19)$$

On the other hand, for $0 \le x \le 3/8$, using (17) we find

$$\sigma_{[0,3/8)}(x) = \frac{9}{32}. \tag{20}$$

Therefore, since by definition $\sigma_{[a,b)}(x) = \sigma_{[0,b-a)}(x-a)$,

$$\begin{aligned} \sup_{0\le a<b<1} \sqrt{\sigma_{[a,b)}(x)} &\ge \sqrt{\max\left(\sigma_{[0,3/8)}(x), \sigma_{[3/8,6/8)}(x), \sigma_{[5/8,1)}(x)\right)} \\ &= \sqrt{\max\left(\sigma_{[0,3/8)}(x), \sigma_{[0,3/8)}(x-3/8), \sigma_{[0,3/8)}(x-5/8)\right)} \\ &= \frac{3}{4\sqrt{2}}, \quad x \in [0,1]. \end{aligned} \tag{21}$$

Combining (19) and (21) proves (15).
Proof of (16): By (17), (18) and (20)

$$\sup_{0<a<1} \sqrt{\sigma_{[0,a)}(x)} = \frac{3}{4\sqrt{2}} = \Psi^*(x)$$

for $0 \le x \le 3/8$ and $5/8 \le x \le 1$. For the other values of x, let us, for example, assume $x \le 2a$ and $x \in [3/8, 1/2]$, $a \in [0, 1/2]$. Then we have

$$\begin{aligned} \sigma_{[0,a)}(x) &= 2a(1-a) - 1/2\ (\min(1-a, x) + \max(0, a-x)) \\ &= 2a(1-a) - x/2 - \max(0, a-x)/2 \\ &\le 2x(1-x) - x/2, \end{aligned}$$

with "=" instead of "≤" in the last line if $a = x$. Considering all possible cases for a and x finally yields

$$\sup_{0<a<1} \sigma_{[0,a)}(x) = \begin{cases} 9/32 & \text{if } 0 \le x \le 3/8 \text{ or } 5/8 \le x \le 1 \\ 2x(1-x) - x/2 & \text{if } 3/8 \le x \le 1/2 \\ 2x(1-x) - (1-x)/2 & \text{if } 1/2 \le x \le 5/8 \end{cases}$$

which implies (16). □

Lemma 4

$$\left\|\sigma_{[a,b),d} - \sigma_{[a,b)}\right\|_\infty \le 3d^{-1}$$

Proof: After some calculations, using some standard trigonometric identities, we find

$$\begin{aligned} &(a_j a_{2j} + b_j b_{2j}) \cos 2\pi jx + (a_j b_{2j} - a_{2j} b_j) \sin 2\pi jx \\ = &\frac{4}{j^2\pi^2} \left(\cos 2\pi j(a-x) + \cos 2\pi j(b-x)\right) \left(\sin(a-b)j\pi\right)^2. \end{aligned}$$

This shows

$$\sigma_{[a,b),d}(x) = \sigma_{[0,b-a),d}(x-a).$$

Also, by definition we have $\sigma_{[a,b)}(x) = \sigma_{[0,b-a)}(x-a)$. Thus for the proof of the lemma it suffices to consider intervals of the form $[0,a)$. We observe that

$$a_{2j}(0,a) = \frac{\sin 2\pi 2ja}{\pi 2j} = \frac{a_j(0, \langle 2a \rangle)}{2}, \quad j \ge 1,$$

and similarly $b_{2j}(0,a) = b_j(0,\langle 2a\rangle)/2$. By a classical result in Fourier analysis ("convolution theorem"), for

$$f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi jx + b_j \sin 2\pi jx$$

and

$$g(x) \sim \sum_{j=1}^{\infty} c_j \cos 2\pi jx + d_j \sin 2\pi jx,$$

the function

$$\frac{1}{2}\sum_{j=1}^{\infty}(a_jc_j - b_jd_j)\cos 2\pi jx + (a_jd_j + b_jc_j)\sin 2\pi x$$

is the Fourier series of

$$\int_0^1 f(t)g(x-t)\ dt.$$

Thus, observing that

$$a_j(0,1-a) = -a_j(0,a) \quad \text{and} \quad b_j(0,1-a) = b_j(0,a), \quad a \in (0,1), j \geq 1,$$

we see that

$$\begin{aligned} & \frac{1}{2}\sum_{j=1}^{\lfloor d/2\rfloor} \big(a_j(0,a)a_{2j}(0,a) + b_j(0,a)b_{2j}(0,a)\big)\cos 2\pi jx \\ & +\big(a_j(0,a)b_{2j}(0,a) - a_{2j}(0,a)b_j(0,a)\big)\sin 2\pi jx \\ = & -\frac{1}{4}\sum_{j=1}^{\lfloor d/2\rfloor} \big(a_j(0,1-a)a_j(0,\langle 2a\rangle) - b_j(0,1-a)b_j(0,\langle 2a\rangle)\big)\cos 2\pi jx \\ & +\big(a_j(0,1-a)b_j(0,\langle 2a\rangle) + a_j(0,\langle 2a\rangle)b_j(0,1-a)\big)\sin 2\pi jx \end{aligned}$$

is the $\lfloor d/2\rfloor$-th partial sum of the Fourier series of

$$-\frac{1}{2}\int_0^1 \mathbf{I}_{[0,1-a)}(t)\mathbf{I}_{[0,\langle 2a\rangle)}(x-t)\ dt = \sigma_{[0,a)}(x) - a(1-a). \tag{22}$$

Thus, writing a_j and b_j for $a_j(0,a)$ and $b_j(0,a)$, respectively, and using (6) we have

$$\begin{aligned} \|\sigma_{[0,a),d} - \sigma_{[0,a)}\|_\infty & \leq \frac{1}{2}\left(\sum_{j=d+1}^{\infty}\left(a_j^2 + b_j^2\right) + \sum_{j=\lfloor d/2\rfloor+1}^{\infty} |a_ja_{2j}| + |b_jb_{2j}| + |a_jb_{2j}| + |a_{2j}b_j|\right) \\ & \leq \frac{1}{2}\left(\sum_{j=d+1}^{\infty} 2j^{-2} + \sum_{j=\lfloor d/2\rfloor+1}^{\infty} 2j^{-2}\right) \\ & \leq 3d^{-1}. \quad \square \end{aligned}$$

Lemma 5 *Let $r(x)$ be a function of the form*

$$r(x) = \sum_{j=d+1}^{\infty} a_j \cos 2\pi jx + b_j \sin 2\pi jx,$$

where

$$|a_j| \le j^{-1} \quad \textit{and} \quad |b_j| \le j^{-1}, \qquad j \ge d+1.$$

Then

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N r(n_k x)\right|}{\sqrt{2N\log\log N}} \le C\|r\|_2 \le Cd^{-1/4} \qquad \text{a.e.},$$

where C is a positive constant.

This is Lemma 3.1 of [1].

Lemma 6 *For any fixed $r \ge 0$*

$$\limsup_{N\to\infty} \frac{ND_N^{(\le 2^{-r})}(n_k x)}{\sqrt{2N\log\log N}} \le Cr^{-1} \qquad \text{a.e.},$$

where C is a positive constant.

This is Lemma 3.3 of [1].

Lemma 7 *For $0 \le a < b \le 1$ and sufficiently large d*

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p_{[a,b),d}(n_k x)\right|}{\sqrt{2N\log\log N}} = \sqrt{\sigma_{[a,b),d}(x)} \qquad \text{a.e.}$$

The proof of this lemma will be given in Section 4.

Corollary 1

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|}{\sqrt{2N\log\log N}} = \sqrt{\sigma_{[a,b)}(x)} \qquad \text{a.e.}$$

Corollary 2 *For any fixed $r \ge 0$*

$$\left|\limsup_{N\to\infty} \frac{ND_N^{(\ge 2^{-r})}(n_k x)}{\sqrt{2N\log\log N}} - \frac{3}{4\sqrt{2}}\right| \le C2^{-r/2} \qquad \text{a.e.}$$

and

$$\left|\limsup_{N\to\infty} \frac{ND_N^{*\,(\ge 2^{-r})}(n_k x)}{\sqrt{2N\log\log N}} - \Psi^*(x)\right| \le C2^{-r/2} \qquad \text{a.e.},$$

where C is a positive constant.

Any function $\mathbf{I}_{[a,b)}(x)$ can be written as a sum of a trigonometric polynomial $p_{[a,b),d}(x)$ and a remainder function $r(x)$, both of which satisfy the conditions in Lemma 7 and Lemma 5, respectively. Lemma 4, Lemma 5 and Lemma 7 imply

$$\begin{aligned}
\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|}{\sqrt{2N\log\log N}} &\le \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p_{[a,b),d}(n_k x)\right|}{\sqrt{2N\log\log N}} + \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N r(n_k x)\right|}{\sqrt{2N\log\log N}} \\
&\le \sqrt{\sigma_{[a,b),d}(x)} + C_1 d^{-1/4} \quad \text{a.e.} \\
&\le \sqrt{\sigma_{[a,b)}(x)} + C_2 d^{-1/4} \quad \text{a.e.}
\end{aligned}$$

and

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|}{\sqrt{2N\log\log N}} \geq \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p_{[a,b),d}(n_k x)\right|}{\sqrt{2N\log\log N}} - \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N r(n_k x)\right|}{\sqrt{2N\log\log N}}$$
$$\geq \sqrt{\sigma_{[a,b),d}(x)} - C_3 d^{-1/4} \quad \text{a.e.}$$
$$\geq \sqrt{\sigma_{[a,b)}(x)} - C_4 d^{-1/4} \quad \text{a.e.}$$

for constants C_1, C_2, C_3, C_4. This yields Corollary 1, since d can be chosen arbitrarily.

Corollary 2 follows from Corollary 1, Lemma 3 and the definitions of $D_N^{(\geq 2^{-r})}$ and $D_N^{*\,(\geq 2^{-r})}$. In fact, by Corollary 1

$$\limsup_{N\to\infty} \frac{ND_N^{(\geq 2^{-r})}(n_k x)}{\sqrt{2N\log\log N}} = \max_{i,j\in\mathbb{Z},0\leq i<j\leq 2^r} \sqrt{\sigma_{[i2^{-r},j2^{-r})}(x)} \quad \text{a.e.},$$

and by (12) and (15)

$$\left|\frac{3}{4\sqrt{2}} - \max_{i,j\in\mathbb{Z},0\leq i<j\leq 2^r} \sqrt{\sigma_{[i2^{-r},j2^{-r})}(x)}\right|$$
$$= \left|\sup_{0\leq a<b<1} \sqrt{\sigma_{[a,b)}(x)} - \max_{i,j\in\mathbb{Z},0\leq i<j\leq 2^r} \sqrt{\sigma_{[i2^{-r},j2^{-r})}(x)}\right|$$
$$\leq \sqrt{44\cdot 2^{-r}},$$

which proves the first part of Corollary 2. The second part, concerning $D_N^{*\,(\geq 2^{-r})}$, is deduced in the same way.

Corollary 2 and Lemma 6, together with (4) and (5), prove Theorem 1, since

$$\left|\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} - \frac{3}{4\sqrt{2}}\right|$$
$$\leq \left|\limsup_{N\to\infty} \frac{ND_N^{(\geq 2^{-r})}(n_k x)}{\sqrt{2N\log\log N}} - \frac{3}{4\sqrt{2}}\right| + 2\limsup_{N\to\infty} \frac{ND_N^{(\leq 2^{-r})}(n_k x)}{\sqrt{2N\log\log N}}$$
$$\leq C2^{-r/2} + 2Cr^{-1} \to 0 \quad \text{a.e.} \qquad \text{as} \quad r\to\infty,$$

and similarly

$$\left|\limsup_{N\to\infty} \frac{ND_N^*(n_k x)}{\sqrt{2N\log\log N}} - \Psi^*(x)\right|$$
$$\leq \left|\limsup_{N\to\infty} \frac{ND_N^{*\,(\geq 2^{-r})}(n_k x)}{\sqrt{2N\log\log N}} - \Psi^*(x)\right| + \limsup_{N\to\infty} \frac{ND_N^{(\leq 2^{-r})}(n_k x)}{\sqrt{2N\log\log N}}$$
$$\leq C2^{-r/2} + Cr^{-1} \to 0 \quad \text{a.e.} \qquad \text{as} \quad r\to\infty.$$

Therefore it remains only to proof Lemma 2 and Lemma 7. The proof of Lemma 7 (given in Section 4) is crucial. Lemma 2 will be shown in the following Section 3.

3 Proof of Lemma 2

The proof of Lemma 2 is simple number theory. We subdivide it into three parts (Lemmas 8-10), which together yield the desired result.

Lemma 8 *For* $1 \le j' \le j \le d$ *and all* $N \ge 1$

$$A(N, j, j', 0) \le C.$$

(Remark: Here and in the sequel, C denotes appropriate positive numbers, not always the same, that may only depend on d).
Proof: We call two indices a "pair" if they can be written in the form

$$2l - 1,\ 2l \qquad \text{for some} \qquad l \ge 1. \tag{23}$$

Then for $k, k' \ge k_0(d)$ we have

$$\frac{n_k}{n_{k'}} > d \quad \text{or} \quad \frac{n_k}{n_{k'}} < \frac{1}{d}$$

if $k \ne k'$ are *not* a pair. Thus for $k, k' \ge k_0(d)$

$$jn_k = j'n_{k'}, \quad \text{where} \quad 1 \le j' \le j \le d,\ k \ne k', \tag{24}$$

is impossible, if k and k' are not a pair. If they are a pair, (24) implies $k' > k$, i.e. $k = 2l - 1, k' = 2l$. Then

$$0 = jn_k - j'n_{k'} = j2^{l^2} - j'\left(2^{l^2+1} - 1\right)$$

implies

$$\frac{j'}{j} = \frac{2^{l^2}}{2^{l^2+1} - 1},$$

which is possible only finitely many times for bounded j, j', since the denominator of the (irreducible) fraction on the right hand side increases as $l \to \infty$. $\square$

Lemma 9 *For* $\nu \ne 0$ *and* $1 \le j' \le j \le d$

$$A(N, j, j', \nu) < C,$$

provided

$$j \ne 2j' \qquad \text{or} \qquad j' \ne \nu. \tag{25}$$

Proof: For fixed $j' \le j$ and assume

$$\frac{j}{j'} \ne 2. \tag{26}$$

We define

$$(m_k)_{k\ge1} = (m_k(j, j'))_{k\ge1} = (jn_k)_{k\ge1} \cup (j'n_k)_{k\ge1},$$

i.e. (m_k) is the sequence that contains all the numbers

$$\left(\bigcup_{k\ge1} \{jn_k\}\right) \cup \left(\bigcup_{k\ge1} \{j'n_k\}\right),$$

sorted in increasing order. Then there exists an $k_0 = k_0(j, j')$ such that the sequence

$$(m_k)_{k \geq k_0}$$

is lacunary. In fact, assume (m_k) is not lacunary, i.e. there exist increasing sequences $(k_i)_{i \geq 1}$ and $(l_i)_{i \geq 1}$ with $l_i > k_i, i \geq 1$ such that

$$\frac{m_{l_i}}{m_{k_i}} \to 1 \quad \text{as} \quad i \to \infty.$$

This implies there exist increasing sequences $(r_i)_{i \geq i}$ and $(s_i)_{i \geq 1}$ such that

$$\frac{n_{s_i}}{n_{r_i}} \to \frac{j}{j'} \quad \text{as} \quad i \to \infty,$$

which is impossible by (26) since

$$\frac{n_{2k}}{n_{2k-1}} \to 2, \quad \frac{n_{2k+1}}{n_{2k}} \to 0, \quad \frac{n_{k+2}}{n_k} \to 0 \quad \text{as} \quad k \to \infty.$$

Thus $(m_k(j, j'))_{k \geq k_0(d)}$, where $k_0(d) = \max_{j,j'} k_0(j, j')$, is lacunary for all (j, j'), i.e. there exists an $q = q(d) > 1$ such that

$$\frac{m_{k+1}(j, j')}{m_k(j, j')} > q, \qquad k \geq k_0(d),$$

for all j, j' satisfying (26) and $1 \leq j' \leq j \leq d$. By a well-known number-theoretic property of lacunary sequences (see Zygmund [15, p. 203]) this implies

$$\#\{(k, k'), k \neq k' : \ m_k(j, j') \pm m_{k'}(j, j') = \nu\} \leq C, \tag{27}$$

uniformly in $\nu \in \mathbb{Z}$ for all j, j' satisfying (26). Now assume (26) does not hold, i.e.

$$\frac{j}{j'} = 2, \tag{28}$$

and that we have

$$j' \neq \nu \tag{29}$$

instead. Then

$$jn_k - j'n_{k'} = jn_k - \frac{j}{2} n_{k'} = \nu$$

implies

$$2n_k - (n_{k'} + 1) = \frac{2\nu}{j} - 1. \tag{30}$$

By (28) and (29) the right-hand side of (30) is not equal zero. It is easy to see that

$$(2n_k)_{k \geq 1} \cup (n_k + 1)_{k \geq 1}$$

is lacunary, which proves the lemma. $\square$

Lemma 10 *For $1 \leq j' \leq j \leq d$ assume*

$$j = 2j' \qquad \text{and} \qquad j' = \nu.$$

Then

$$\left| A(N, j, j', \nu) - \frac{N}{2} \right| \leq C,$$

Proof: We have to estimate the number of solutions (k, k'), $1 \leq k, k' \leq N$ of the equation

$$2n_k - (n_{k'} + 1) = 0. \tag{31}$$

Since $2n_k$ and $n_{k'}+1$ are of the form 2^{l^2+1} or $2^{l^2+2}-2$ and $2^{m^2}+1$ or 2^{m^2+1} for some positive l and m, respectively, it is easy to see that (31) is only valid if $k = 2l-1$ and $k' = 2l$ for some $l \geq 1$, i.e. if k' and k are a pair and $k' \geq k$. The number of pairs with indices bounded by N is $\lfloor N/2 \rfloor$, which proves the lemma. $\square$

4 Proof of Lemma 7

The following lemma is a slight modification of Corollary 4.5 of Strassen [12]:

Lemma 11 *Let $(Y_i, \mathcal{F}_i, i \geq 1)$ be a martingale difference sequence with finite fourth moments, let $V_M = \sum_{i=1}^M \mathbb{E}(Y_i^2 | \mathcal{F}_{i-1})$ and assume $V_1 = \mathbb{E}Y_1^2 > 0$ and $V_M \to \infty$. Assume additionally*

$$\liminf_{M \to \infty} \frac{V_M}{r_M} \geq 1 \qquad \text{a.s.} \tag{32}$$

with some sequence of positive real numbers $r_M \to \infty$ such that

$$\sum_{M=1}^{\infty} \frac{(\log r_M)^{10}}{r_M^2} \mathbb{E}Y_M^4 < +\infty. \tag{33}$$

Let $\phi(t)$ be a positive real function such that $t^{-1/2}\phi(t)$ is non-decreasing. Then

$$\mathbb{P}\left(Y_1 + \dots + Y_M < \phi(V_M) \text{ eventually as } M \to \infty\right) = 1 \quad \text{or} \quad 0,$$

according to

$$\int_1^{\infty} t^{-3/2} \phi(t) e^{-\phi(t)^2/2t} \, dt < \infty \quad \text{or} \quad = \infty. \tag{34}$$

The Beppo Levi theorem and (33) imply the a.s. convergence of $\sum_{M=1}^{\infty} (\log r_M)^{10} r_M^{-2} \mathbb{E}(Y_M^4 | \mathcal{F}_{M-1})$, and by (32) the series $\sum_{M=1}^{\infty} (\log V_M)^{10} V_M^{-2} \mathbb{E}(Y_M^4 | \mathcal{F}_{M-1})$ is also a.s. convergent. Hence

$$\begin{aligned}
& \sum_{M=1}^{\infty} \frac{(\log V_M)^5}{V_M} \int_{x^2 > V_M (\log V_M)^{-5}} x^2 \, dP(Y_M < x | \mathcal{F}_{M-1}) \\
\leq & \sum_{M=1}^{\infty} \frac{(\log V_M)^{10}}{V_M^2} \int_{-\infty}^{+\infty} x^4 \, dP(Y_M < x | \mathcal{F}_{M-1}) \\
= & \sum_{M=1}^{\infty} \frac{(\log V_M)^{10}}{V_M^2} \mathbb{E}(Y_M^4 | \mathcal{F}_{M-1}) \; < \; \infty \quad \text{a.s.,}
\end{aligned}$$

Thus Lemma 11 follows from Corollary 4.5 of [12]. Choosing the function $\phi(t)$ in Lemma 11 as

$$\phi(t) = \sqrt{Kt \log\log t}$$

for a constant K, it is easy to see that

$$\begin{aligned} \int_1^\infty t^{-3/2}\phi(t)e^{-\phi(t)^2/2t}\, dt &= \int_1^\infty t^{-3/2}(Kt\log\log t)^{1/2}e^{-(Kt\log\log t)/2t}\, dt \\ &= \sqrt{K}\int_1^\infty t^{-1}(\log\log t)^{1/2}(\log t)^{-K/2}\, dt \\ & \quad < \infty \quad \text{or} \quad = \infty, \text{ according to } K > 2 \text{ or } K \leq 2. \end{aligned}$$

As a consequence we get

Corollary 3 *Under the same assumptions as in Lemma 11, plus the additional assumption*

$$\lim_{M\to\infty} \frac{V_M}{s_M} = 1 \quad \text{a.s.}$$

for some sequence of functions $(s_M)_{M\geq 1}$ *we have*

$$\limsup_{M\to\infty} \frac{\left|\sum_{i=1}^M Y_i\right|}{\sqrt{2s_M \log\log s_M}} = 1 \quad \text{a.s.}$$

Lemma 12 (Berkes, Philipp [4]**)** *For any interval* $[a,b) \subset [0,1)$

$$\int_0^1 \left(\sum_{k=N_1+1}^{N_1+N_2} p_{[a,b),d}(n_k x)\right)^4 dx \leq CN_2^2$$

for all integers $N_1, N_2 \geq 0$.

This is [4, Lemma 2.2].

In this section we assume that a nonempty interval $[a,b)$ and a positive integer d are fixed, and we write p for $p_{[a,b),d}$.

We divide the set of positive integers into consecutive blocks $\Delta_1', \Delta_1, \Delta_2', \Delta_2, \ldots, \Delta_i', \Delta_i, \ldots$ of lengths $2\lceil 2\log_{3/2} i\rceil$ and $2i$, respectively. Then $\frac{n_{(i-1)^+}}{n_{i^-}}$, where i^- denotes the smallest and i^+ denotes the largest integer in Δ_i, is at most i^{-4}, since

$$\frac{n_{(i-1)^+}}{n_{i^-}} \leq (3/2)^{-4\log_{3/2} i} = i^{-4}, \quad i \geq 2. \tag{35}$$

For $k \in \bigcup_{i=1}^\infty \Delta_i$ define $i = i(k)$ by $k \in \Delta_i$, put $m(k) = \lceil \log_2 n_k + 2\log_2 i\rceil$, and approximate the functions $p(n_k x)$ by discrete function $\varphi_k(x)$ such that the following properties are satisfied:

(P1) $\varphi_k(x)$ is constant for $\frac{v}{2^{m(k)}} \leq x < \frac{v+1}{2^{m(k)}}, \quad v = 0, 1, \ldots, 2^{m(k)} - 1$
(P2) $|\varphi_k(x) - p(n_k x)| \leq Ci^{-2}, \quad x \in [0,1)$
(P3) $\mathbb{E}(\varphi_k(x)|\mathcal{F}_{i-1}) = 0$

Here $Y_i = \sum_{k \in \Delta_i} \varphi_k(x)$ and $\mathcal{F}_i$ denotes the σ-field generated by the intervals

$$\left[\frac{v}{2^{m(i^+)}}, \frac{v+1}{2^{m(i^+)}}\right), \quad 0 \le v < 2^{m(i^+)}.$$

We have

$$\begin{aligned} |p(n_k x) - p(n_k x')| &\le 2 \sum_{j=1}^{d} j^{-1} 2\pi j n_k |x - x'| \\ &\le C n_k 2^{-m(k)} \\ &\le C i^{-2} \quad \text{for} \quad \frac{v}{2^{m(k)}} \le x, x' \le \frac{v+1}{2^{m(k)}}, \quad 0 \le v < 2^{m(k)}. \end{aligned}$$

Thus it is possible to approximate $p(n_k x)$ by discrete functions $\hat{\varphi}_k(x)$ that satisfy (P1) and (P2) only. Then for $k \in \Delta_i$ and any interval I of the form

$$\left[\frac{v}{2^{m((i-1)^+)}}, \frac{v+1}{2^{m((i-1)^+)}}\right), \quad 0 \le v \le 2^{m((i-1)^+)}, \tag{36}$$

letting $|I|$ denote the length of I,

$$\begin{aligned} \frac{1}{|I|} \left| \int_I \hat{\varphi}_k(x)\ dx \right| &\le \frac{1}{|I|} \left(\left| \int_I p(n_k x)\ dx \right| + \int_I \frac{C}{i^2}\ dx \right) \\ &\le 2^{m((i-1)^+)}\ \frac{2\|p\|_\infty}{n_{i^-}} + C\ i^{-2} \\ &\le C\ \frac{i^2 n_{(i-1)^+}}{n_{i^-}} + C\ i^{-2} \ \le\ C\ i^{-2}, \end{aligned}$$

by (35) and since $\|p\|_\infty \le C$. For every $x \in [0,1)$ we can find an interval of type I for some v such that $x \in I$, and we put $\varphi_k(x) = \hat{\varphi}_k(x) - |I|^{-1} \int_I \hat{\varphi}_k(t)\ dt$. Then these functions $\varphi_k(x)$ satisfy (P1), (P2) and (P3), since $\mathcal{F}_{i-1}$ is generated by the intervals of type (36).
We set

$$T_i = \sum_{k \in \Delta_i} p(n_k x) \qquad T_{i'} = \sum_{k \in \Delta_i'} p(n_k x), \qquad V_M = \sum_{i=1}^{M} \mathbb{E}(Y_i^2 | \mathcal{F}_{i-1}).$$

Letting $|\Delta_i|$ denote the number of integers in Δ_i, we set

$$s_M(x) = \sigma_{[a,b),d}(x) \sum_{i=1}^{M} |\Delta_i|.$$

In the following equation, to shorten formulas, we write $\mathrm{c}(v)$ and $\mathrm{s}(v)$ for $\cos 2\pi vx$ and $\sin 2\pi vx$, respectively. We have

$$\begin{aligned}
& T_i(x)^2 - \sigma_{[a,b),d}(x)|\Delta_i| \\
= & \left(\sum_{k\in\Delta_i} \sum_{j=1}^{d} a_j\,\mathrm{c}(jn_k) + b_j\,\mathrm{s}(jn_k) \right)^2 \\
& -|\Delta_i| \sum_{j=1}^{d} \frac{a_j^2+b_j^2}{2} - \frac{|\Delta_i|}{2} \sum_{j=1}^{\lfloor d/2 \rfloor} (a_j a_{2j} + b_j b_{2j})\,\mathrm{c}(j) + (a_j b_{2j} - a_{2j} b_j)\,\mathrm{s}(j) \\
= & \sum_{k,k'\in\Delta_i} \sum_{1\le j,j'\le d} \frac{a_j a_{j'} - b_j b_{j'}}{2}\,\mathrm{c}(jn_k + j'n_{k'}) + \frac{b_j a_{j'} + a_j b_{j'}}{2}\,\mathrm{s}(jn_k + j'n_{k'}) && (37) \\
& + \sum_{k\in\Delta_i} \sum_{j=1}^{d} \frac{a_j^2+b_j^2}{2}\,\mathrm{c}(jn_k - jn_k) + \frac{2a_j b_j}{2}\,\mathrm{s}(jn_k - jn_k) && (38) \\
& -|\Delta_i| \sum_{j=1}^{d} \frac{a_j^2+b_j^2}{2} && (39) \\
& + \sum_{k,k'\in\Delta_i, k\neq k'} \sum_{j=1}^{d} \frac{a_j^2+b_j^2}{2}\,\mathrm{c}(jn_k - jn_{k'}) && (40) \\
& +2 \underbrace{\sum_{k,k'\in\Delta_i} \sum_{1\le j'<j\le d}}_{\neg((j=2j')\wedge(\exists\ell:k=2\ell-1,k'=2\ell))} \frac{a_j a_{j'} + b_j b_{j'}}{2}\,\mathrm{c}(jn_k - j'n_{k'}) + \frac{b_j a_{j'} - a_j b_{j'}}{2}\,\mathrm{s}(jn_k - j'n_{k'}) && (41) \\
& +2 \underbrace{\sum_{k,k'\in\Delta_i} \sum_{1\le j'<j\le d}}_{(j=2j')\wedge(\exists\ell:k=2\ell-1,k'=2\ell)} \frac{a_j a_{j'} + b_j b_{j'}}{2}\,\mathrm{c}(jn_k - j'n_{k'}) + \frac{b_j a_{j'} - a_j b_{j'}}{2}\,\mathrm{s}(jn_k - j'n_{k'}) && (42) \\
& -\frac{|\Delta_i|}{2} \sum_{j=1}^{\lfloor d/2 \rfloor} (a_j a_{2j} + b_j b_{2j})\,\mathrm{c}(j) + (a_j b_{2j} - a_{2j} b_j)\,\mathrm{s}(j). && (43)
\end{aligned}$$

Now (38) and (39) cancel out, and (42) and (43) cancel out as well. Thus $T_i(x)^2 - \sigma_{[a,b),d}(x)|\Delta_i|$ is the sum of the functions in (37), (40) and (41). If we collect all trigonometric functions in $T_i(x)^2 - \sigma_{[a,b),d}(x)|\Delta_i|$ with frequency less than $n_{(i-1)^+}$ in a function U_i, and all trigonometric functions with frequency at least $n_{(i-1)^+}$ in a function W_i, we can write

$$T_i(x)^2 - \sigma_{[a,b),d}(x)|\Delta_i| = U_i(x) + W_i(x),$$

where

$$U_i = \sum_{0\le u<n_{(i-1)^+}} c_u \cos 2\pi ux + d_u \sin 2\pi ux, \qquad W_i = \sum_{u\ge n_{(i-1)^+}} c_u \cos 2\pi ux + d_u \sin 2\pi ux.$$

It is easy to see that

$$\sum_{u\ge n_{i^-}} |c_u| + |d_u| \le \left(\sum_{k\in\Delta_i} \sum_{j=1}^{d} \frac{2}{j} \right)^2 \le Ci^2. \tag{44}$$

For fixed j,k the number of solutions (j',k'), $k'\geq k$ of $|jn_k - j'n_{k'}| < n_{(i-1)^+}$ is at most C. In fact, $k' \geq k + \log_{3/2}(d+1)$ implies

$$j'n_{k'} - jn_k \geq n_{k'} - dn_k \geq \left(\frac{3}{2}\right)^{\log_{3/2}(d+1)} n_k - dn_k \geq n_k \geq n_{(i-1)^+}.$$

Thus

$$\sum_{0\leq u<n_{i^-}} |c_u| + |d_u| \leq \sum_{k\in\Delta_i}\sum_{j=1}^{d} C \leq Ci.$$

Additionally, Lemma 2 implies

$$\max_{0\leq u<n_{i^-}} |c_u| \leq C, \qquad \max_{0\leq u<n_{i^-}} |d_u| \leq C.$$

In particular

$$\|U_i\|_\infty \leq Ci \qquad \text{and} \qquad \|W_i\|_\infty \leq Ci^2. \tag{45}$$

By Minkowski's inequality,

$$\|Y_i^2 - T_i^2\|_\infty \;\leq\; \|Y_i - T_i\|_\infty \, \|Y_i + T_i\|_\infty \;\leq\; C|\Delta_i|\; i^{-2}\; |\Delta_i| \;\leq\; C \tag{46}$$

and

$$\left\|\mathbb{E}\left(\sigma_{[a,b),d}|\mathcal{F}_{i-1}\right) - \sigma_{[a,b),d}\right\|_\infty \leq C2^{-m((i-1)^+)} \leq \frac{C}{i^2 n_{(i-1)^+}} \leq C(3/2)^{-i}$$

we get

$$\begin{aligned}
& \|V_M - s_M\|_2 \\
\leq\; & \left\|\sum_{i=1}^{M}\left(\mathbb{E}(T_i^2|\mathcal{F}_{i-1}) - |\Delta_i|\sigma_{[a,b),d}\right)\right\|_2 + \sum_{i=1}^{M} C \\
\leq\; & \left\|\sum_{i=1}^{M}\mathbb{E}\left(\left(T_i^2 - |\Delta_i|\sigma_{[a,b),d}\right)\big|\mathcal{F}_{i-1}\right)\right\|_2 + C\left(\sum_{i=1}^{M}|\Delta_i|(3/2)^{-i}\right) + CM \\
\leq\; & \left\|\sum_{i=1}^{M}\mathbb{E}(U_i|\mathcal{F}_{i-1})\right\|_2 \qquad (47)\\
+\; & \left\|\sum_{i=1}^{M}\mathbb{E}(W_i|\mathcal{F}_{i-1})\right\|_2 \qquad (48)\\
+\; & CM.
\end{aligned}$$

First we estimate the term (48). We observe

$$\left(\sum_{i=1}^{M}\mathbb{E}(W_i|\mathcal{F}_{i-1})\right)^2 \leq 2\sum_{1\leq i\leq i'\leq M}\mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i'-1}).$$

By Lemma 1, Minkowski's inequality and the Jensen inequality

$$\begin{aligned}
\|\mathbb{E}\left(W_i|\mathcal{F}_{i-1}\right)\|_2 &\leq \left\|\mathbb{E}\left(\sum_{u\geq n_{i-}} c_u\cos 2\pi ux + d_u \sin 2\pi ux \Big| \mathcal{F}_{i-1}\right)\right\|_\infty \\
&\quad + \left\| \sum_{n_{(i-1)^+}\leq u < n_{i-}} c_u \cos 2\pi ux + d_u \sin 2\pi ux \right\|_2 \\
&\leq \left(2^{m((i-1)^+)} \sum_{u\geq n_{i-}} 2\frac{|c_u|+|d_u|}{u}\right) + \left(\sum_{n_{(i-1)^+}\leq u<n_{i-}} c_u^2+d_u^2\right)^{1/2} \\
&\leq C\left(\frac{i^2 n_{(i-1)^+}}{n_{i^-}} \sum_{u\geq n_{i-}} |c_u|+|d_u|\right) + Ci^{1/2} \\
&\leq Ci^{1/2}
\end{aligned}$$

and therefore

$$\mathbb{E}\left(\sum_{i=1}^M (\mathbb{E}(W_i|\mathcal{F}_{i-1}))^2\right) \leq \sum_{i=1}^M Ci \leq CM^2. \tag{49}$$

For fixed $i<i'$, since $\mathbb{E}(W_i|\mathcal{F}_{i-1})$ is $\mathcal{F}_{i-1}$ measurable,

$$\begin{aligned}
\left|\mathbb{E}\Big(\mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i'-1})\Big|\mathcal{F}_{i-1}\Big)\right| &= |\mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})| \\
&\leq \|W_i\|_\infty \left|\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})\right| \ \leq\ Ci^2 \left|\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})\right|.
\end{aligned}$$

Writing $W_{i'}$ in the form

$$\sum_{u\geq n_{(i'-1)^+}} c_u \cos 2\pi ux + d_u \sin 2\pi ux,$$

where $\sum_u |c_u|+|d_u| \leq C(i')^2$, and using Lemma 1 we get

$$\begin{aligned}
|\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})| &\leq 2^{m((i-1)^+)} \sum_{u\geq n_{(i'-1)^+}} 2(|c_u|+|d_u|)u^{-1} \\
&\leq C\frac{i^2\ n_{(i-1)^+}}{n_{(i'-1)^+}}(i')^2 \\
&\leq C\ i^2\ (i')^2\ (3/2)^{(i-1)^+-(i'-1)^+} \\
&\leq C\ i^2\ (i')^2\ (3/2)^{-(i'-1)}.
\end{aligned} \tag{50}$$

Combining the estimates (49) and (50) we see that the term (48) is at most

$$\left(CM^2 + \sum_{1\leq i<i'\leq M} C\ i^4\ (i')^2\ (3/2)^{-(i'-1)}\right)^{1/2} \ \leq\ CM. \tag{51}$$

Next we estimate the term (47). Writing

$$U_i = \sum_{u<n_{(i-1)^+}} c_u\cos 2\pi ux + d_u \sin 2\pi ux,$$

the fluctuation of U_i on any atom of $\mathcal{F}_{i-1}$ is at most

$$\begin{aligned}\sum_{u<n_{(i-1)^+}} (|c_u|+|d_u|)2\pi u\ 2^{-m((i-1)^+)} &\leq C \sum_{u<n_{(i-1)^+}} (|c_u|+|d_u|)n_{(i-1)^+}\ \left(i^2 n_{(i-1)^+}\right)^{-1} \\ &\leq Ci^{-1}\end{aligned}$$

and consequently

$$|\mathbb{E}(U_i|\mathcal{F}_{i-1}) - U_i| \leq Ci^{-1},$$

which gives

$$\left\|\sum_{i=1}^{M} \mathbb{E}(U_i|\mathcal{F}_{i-1})\right\|_2 \leq \left\|\sum_{i=1}^{M} \mathbb{E}(U_i)\right\|_2 + C\log M. \tag{52}$$

Writing

$$\sum_{i=1}^{M} U_i(x) = \sum_{u<n_{(M-1)^+}} c_u \cos 2\pi ux + d_u \sin 2\pi ux,$$

where by Lemma 2

$$|c_u| \leq C, \quad |d_u| \leq C$$

and $\sum_u(|c_u|+|d_u|) \leq \sum_{i=1}^{M} Ci \leq CM^2$, the right-hand side of (52) is at most

$$\left(\sum_{u<n_{(M-1)^+}} c_u^2 + d_u^2\right)^{1/2} + C\log M \leq CM. \tag{53}$$

Assembling the estimates (51) and (53) for (47) and (48) we obtain

$$\|V_M - s_M\|_2 \leq CM. \tag{54}$$

If d is large enough, $\sigma_{[a,b),d}(x) > 0$, and thus $s_M \geq CM^2$. Defining sets

$$A_M = \left\{x \in (0,1) :\ |V_M - s_M| > s_M^{7/8}\right\},$$

this implies

$$\mathbb{P}\left(A_M\right) \leq CM^2 s_M^{-14/8} \leq CM^2 M^{-28/8} \leq CM^{-12/8},$$

and by the Borel-Cantelli-Lemma

$$\frac{V_M}{s_M} \to 1 \quad \text{a.e.} \tag{55}$$

We define a numerical sequence $(r_M)_{M\geq 1}$ by

$$r_M = \min_{x\in(0,1)} s_M(x).$$

Then

$$\liminf_{M\to\infty} \frac{V_M}{r_M} \geq \lim_{M\to\infty} \frac{V_M}{s_M} = 1 \quad \text{a.e.}$$

By Lemma 12 and property (P2)

$$\mathbb{E}Y_M^4 \leq C|\Delta_M|^2 \leq CM^2,$$

and thus $r_M \geq CM^2, M \geq 1$ yields

$$\sum_{M=1}^{\infty} \frac{(\log r_M)^{10}}{r_M^2} \mathbb{E} Y_M^4 \leq \sum_{M=1}^{\infty} C \frac{(\log M)^{10} M^2}{M^4} < +\infty.$$

Now by (55), Lemma 11 and Corollary 3

$$\limsup_{M\to\infty} \frac{\left|\sum_{i=1}^{M} Y_i\right|}{\sqrt{2 s_M \log\log s_M}} = 1 \quad \text{a.e.}$$

We add the sums of "short blocks" T_i', change from Y_i to T_i, where $|Y_i - T_i| \leq C|\Delta_i| i^{-2} \leq C i^{-1}$, and get

$$\limsup_{M\to\infty} \frac{\sum_{k=1}^{M^+} p(n_k x)}{\sqrt{2 s_M \log\log s_M}} = \limsup_{M\to\infty} \frac{\left|\sum_{i=1}^{M} (T_i + T_i')\right|}{\sqrt{2 s_M \log\log s_M}} = 1 \quad \text{a.e.}$$

Now we break into the blocks of sums. Since

$$\max_{N \in \Delta_i \cup \Delta_i'} \left| \sum_{k \in \Delta_i \cup \Delta_i', k \leq N} p(n_k x) \right| \leq C|\Delta_i| \leq Ci$$

and $s_M \geq CM^2$,

$$\limsup_{M\to\infty} \frac{\left|\max_{(M-1)^+ < N \leq M^+} \sum_{k \leq N} p(n_k x)\right|}{\sqrt{2 s_M \log\log s_M}} = 1 \quad \text{a.e.} \tag{56}$$

For $N \geq 1$ we define $M(N)$ as the index m, for which N is contained in $\Delta_m \cup \Delta_m'$. Then (56) can be rewritten in the form

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} p(n_k x)\right|}{\sqrt{2 s_{M(N)} \log\log s_{M(N)}}} = 1 \quad \text{a.e.}$$

Observing

$$\frac{s_{M(N)}}{N} \to \sigma_{[a,b),d}(x) \quad \text{as} \quad N \to \infty,$$

and finally arrive at

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} p(n_k x)\right|}{\sqrt{2N \log\log N}} = \sqrt{\sigma_{[a,b),d}(x)} \quad \text{a.e.},$$

which is Lemma 7.

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