# Diophantine equations and the LIL for the discrepancy of sub-lacunary sequences

Christoph Aistleitner *

*Dedicated to the memory of Professor Walter Philipp.*

**Abstract**

Let $(n_k)_{k\geq 1}$ be a lacunary sequence, i.e. a sequence of positive integers satisfying the Hadamard gap condition $n_{k+1}/n_k \geq q > 1,\ k \geq 1$. By a classical result of Philipp (1975), the discrepancy $D_N$ of $(n_k x)_{k\geq 1}$ mod 1 satisfies the law of the iterated logarithm, i.e. we have $1/(4\sqrt{2}) \leq \limsup_{N\to\infty} N D_N(n_k x)(2N \log\log N)^{-1/2} \leq C_q$ for almost all $x \in (0,1)$, where $C_q$ is a constant depending on $q$. Recently, Fukuyama computed the exact value of the lim sup for $n_k = \theta^k$, where $\theta > 1$, not necessarily an integer, and the author showed that for a large class of lacunary sequences the value of the lim sup is the same as in the case of i.i.d. random variables. In the sublacunary case, the situation is much more complicated. Using methods of Berkes, Philipp and Tichy, we prove an exact law of the iterated logarithm for a large class of sub-lacunary growing sequences $(n_k)_{k\geq 1}$, characterized in terms of the number of solutions of certain Diophantine equations, and show that the value of the lim sup is the same as in the case of i.i.d. random variables.

## 1 Introduction and statement of results

Given a sequence $(x_k)_{k\geq 1}$ of real numbers, we call the value

$$D_N = D_N(x_1,\ldots,x_N) = \sup_{0\leq a<b<1} \left| \frac{\sum_{k=1}^N \mathbb{1}_{[a,b)}(\langle x_k\rangle)}{N} - (b-a) \right|,$$

where $\mathbb{1}_{[a,b)}$ is the indicator function of the interval $[a,b)$ and $\langle\cdot\rangle$ denotes the fractional part, the discrepancy of the first $N$ elements of $(x_k)_{k\geq 1}$, and we call the value

$$D_N^* = D_N^*(x_1,\ldots,x_N) = \sup_{0<a<1} \left| \frac{\sum_{k=1}^N \mathbb{1}_{[0,a)}(\langle x_k\rangle)}{N} - a \right|$$

the star discrepancy of the first $N$ elements of this sequence. It is easy to see that always $D_N^* \leq D_N \leq 2D_N^*$.

*Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. Research supported by the Austrian Research Foundation (FWF), Project S9603-N13.

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In 1975 Philipp [17] proved a law of the iterated logarithm (LIL) for the discrepancy of lacunary sequences of integers, i.e. for sequences $(n_k)_{k\geq 1}$ satisfying the Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1, \quad k \geq 1. \tag{1}$$

He showed that for such sequences we have

$$\frac{1}{4\sqrt{2}} \leq \limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} \leq C_q \quad \text{a.e.,} \tag{2}$$

where $C_q$ is a number depending on $q$. A comparison with the Chung-Smirnov law of the iterated logarithm (cf. [18], p. 504) shows that under (1) $(\langle n_k x\rangle)_{k\geq 1}$ behaves like a sequence of independent, identically distributed (i.i.d.) random variables. However, the value $L$ of the limsup in (2) can be different from that for i.i.d. random variables and $L$ remained unknown until very recently, when Fukuyama [10] succeeded in calculating its value for $n_k = \theta^k$, where $\theta > 1$, not necessarily an integer. In particular, he showed

$$\limsup_{N\to\infty} \frac{ND_N(2^k x)}{\sqrt{2N\log\log N}} = \limsup_{N\to\infty} \frac{ND_N^*(2^k x)}{\sqrt{2N\log\log N}} = \frac{\sqrt{42}}{9} \quad \text{a.e.}$$

and

$$\limsup_{N\to\infty} \frac{ND_N(\theta^k x)}{\sqrt{2N\log\log N}} = \limsup_{N\to\infty} \frac{ND_N^*(\theta^k x)}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.e.,}$$

if $\theta$ is a real number such that $\theta^r$ is irrational for $r = 1, 2, \ldots$. In [2] we showed that for general lacunary $(n_k)_{k\geq 1}$ the value of the lim sup in (2) is intimately connected with the number of solutions of Diophantine equations of the type

$$j_1 n_{k_1} \pm j_2 n_{k_2} = b, \qquad j_1, j_2, b \in \mathbb{Z}, \tag{3}$$

subject to

$$1 \leq k_1, k_2 \leq N.$$

If the number of solutions of this equation is "not too large", we have

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} = \limsup_{N\to\infty} \frac{ND_N^*(n_k x)}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.e.,} \tag{4}$$

and the value 1/2 in (4) is the same as in the Chung-Smirnov LIL. In Aistleitner and Berkes [3] a necessary and sufficient condition, also in terms of the Diophantine equation (3), was given for the the validity of the central limit theorem for $(f(n_k x))_{k\geq 1}$, where $(n_k)_{k\geq 1}$ is lacunary and $f$ is a 1-periodic function of bounded variation. This result completes a long line of investigations starting with the classical paper of Kac [12]. If the Diophantine equations in (3) have "too many" solutions, the probabilistic behavior of $(f(n_k x))_{k\geq 1}$ and $(n_k x)_{k\geq 1}$ can show considerable differences from the behavior of i.i.d. random variables. Fukuyama's result shows that in this case the value of the lim sup can be different from 1/2, and in [1] we showed that the lim sup in (4) does not even have to be constant almost everywhere.

In contrast to the Hadamard lacunary case, relatively little is known in the sub-lacunary case. Berkes and Philipp [7] showed that for any sequence $\varepsilon_k \searrow 0$ there exists a sequence $(n_k)_{k\geq 1}$ of positive integers satisfying

$$n_{k+1}/n_k \geq 1 + \varepsilon_k, \quad k \geq 1,$$

such that

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} = +\infty \quad \text{a.e.}$$

This means that the LIL is generally false for the discrepancy of $(n_k x)_{k\ge1}$ for sub-lacunary sequences $(n_k)_{k\ge1}$. However, Aistleitner and Berkes [4] showed that an LIL-type result will hold for the discrepancy of $(n_k x)_{k\ge1}$ for sub-lacunary $(n_k)_{k\ge1}$, if the norming function $(2N\log\log N)^{1/2}$ in (2) is replaced by $(2B_N\log\log B_N)^{1/2}$, where $(B_N)_{N\ge1}$ depends on the "density" of the sequence $(n_k)_{k\ge1}$. Higher order Diophantine conditions for the CLT and LIL for sublacunary sequences were given in Berkes, Philipp and Tichy [8].

The purpose of the present paper is to give simple and nearly optimal sufficient conditions for the exact LIL (4) for the discrepancy of a class of sub-lacunary growing sequences of integers. As we will see, in addition to a bound for the number of solutions of the Diophantine equation (3), required already in the Hadamard lacunary case, for the LIL we need a bound for the density of $(n_k)_{k\ge1}$ among the integers. It is easy to see that our density condition on $(n_k)_{k\ge1}$ corresponds to a Kolmogorov type condition for the random variables

$$X_k = \sum_{2^k\le j\le 2^{k+1}} f(n_j x) \qquad k=1,2,\dots \tag{5}$$

and thus, as a comparison with the classical paper Kolmogorov [14] shows, the random variables $X_k$ in (5) behave like independent random variables.

Let $(n_k)_{k\ge1}$ be an increasing sequence of positive integers. Letting

$$A_j = \#\left\{k:\ 2^j \le n_k < 2^{j+1}\right\}, \qquad j\ge 0,$$

we say that $(n_k)_{k\ge1}$ satisfies

- the density condition $(\mathbf{K}_\alpha)$, $0\le\alpha\le1$, if there exists a constant $C_\alpha>0$ such that

$$A_N \le C_\alpha \left(\sum_{0\le j<N} A_j\right)^\alpha, \qquad N\ge1.$$

- the Diophantine condition $(\mathbf{D}_\delta)$, $0\le\delta<1$, if there exists a constant $C_\delta$ such that for every $N\ge1$ and for fixed integers $j_i$ with $0<|j_i|\le N^2$, $i=1,2$, the number of solutions $(k_1,k_2)$ of the Diophantine equation

$$j_1 n_{k_1} - j_2 n_{k_2} = b,$$

subject to

$$1\le k_i\le N, \qquad i=1,2,$$

does not exceed $C_\delta N^\delta$, uniformly for all $b\in\mathbb{Z}$, $b\ne0$.

- the Diophantine condition $(\mathbf{D}^{\mathbf{0}}_\gamma)$, $0\le\gamma<1$, if there exists a constant $C_\gamma$ such that for every $N\ge1$ and for fixed integers $j_i$ with $0<|j_i|\le N^2$, $i=1,2$, the number of solutions $(k_1,k_2)$ of the Diophantine equation

$$j_1 n_{k_1} - j_2 n_{k_2} = 0, \quad (j_1,k_1)\ne(j_2,k_2),$$

subject to

$$1 \le k_i \le N, \qquad i = 1, 2,$$

does not exceed $C_\gamma N^\gamma$.

Conditions ($\mathbf{K}_\alpha$) and ($\mathbf{D}_\delta$) guarantee that the system $(f(n_k x))_{k\ge1}$ has almost i.i.d. properties, while condition ($\mathbf{D}^{\mathbf{0}}_\gamma$) controls the value of the integral

$$\int_0^1 \left( \sum_{k=1}^N f(n_k x) \right)^2 dx. \tag{6}$$

Together they will imply (see Theorem 1) an exact LIL for the sequence $(f(n_k x))_{k\ge1}$ for fixed $f$, but they are not sufficient to obtain the LIL (4) for the discrepancy of $(n_k x)_{k\ge1}$. Using condition ($\mathbf{D}^{\mathbf{0}}_\gamma$), we can calculate the (asymptotic) value of the integral (6), and obtain an exact LIL for the discrepancy of $(n_k x)_{k\ge1}$, where the value of the lim sup is exactly the same as in the Chung-Smirnov LIL for i.i.d. random variables (Theorem 2).

We note that higher order Diophantine conditions were shown in Berkes, Philipp and Tichy [8] to imply asymptotic results for the discrepancy of $(n_k x)_{k\ge1}$. The substantial improvement of the present paper is to use the two-term Diophantine conditions ($\mathbf{D}_\delta$), ($\mathbf{D}^{\mathbf{0}}_\gamma$) and the density condition ($\mathbf{K}_\alpha$), which are essentially optimal for LIL type results.

We will prove the following results:

**Theorem 1** *Let $f$ be a real function satisfying*

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\ dx = 0, \qquad f \in BV([0,1]), \tag{7}$$

*and assume there exists a positive constant $C_1$ such that*

$$b_N = \int_0^1 \left( \sum_{k=1}^N f(n_k x) \right)^2 dx \ge C_1 N, \qquad N \ge 1.$$

*Let $(n_k)_{k\ge1}$ a sequence of positive integers satisfying conditions ($\mathbf{K}_\alpha$) and ($\mathbf{D}_\delta$), where*

$$\alpha + \delta < 1. \tag{8}$$

*Let $S_N = \sum_{k=1}^N f(n_k x)$. Then the sequence $(S_N)_{N\ge1}$ can be redefined on a new probability space (without changing its distribution) together with a Wiener process $\xi(t)$ such that*

$$S_N = \xi(b_N) + o\left(N^{1/2-\lambda}\right) \qquad \text{a.s.},$$

*where $\lambda > 0$ depends on $\alpha$ and $\delta$.*

**Corollary 1** *Let $f$ be a real function satisfying (7) and*

$$\|f\|_2 := \left( \int_0^1 f(x)^2\ dx \right)^{1/2} > 0,$$

*and let $(n_k)_{k\geq 1}$ a sequence of positive integers satisfying conditions* $(\mathbf{K}_\alpha)$ *and* $(\mathbf{D}_\delta)$*, where*

$$\alpha+\delta<1.$$

*Assume that $(n_k)_{k\geq 1}$ also satisfies condition* $(\mathbf{D}^{\mathbf{0}}_\gamma)$ *for some $\gamma<1$. Then, letting $S_N=\sum_{k=1}^N f(n_k x)$, the sequence $(S_N)_{N\geq 1}$ can be redefined on a new probability space (without changing its distribution) together with a Wiener process $\xi(t)$ such that*

$$S_N=\xi\left(\|f\|_2^2 N\right)+o\left(N^{1/2-\lambda}\right) \qquad \text{a.s.},$$

*where $\lambda>0$ depends on $\alpha,\delta$ and $\gamma$.*

**Corollary 2** *Let $f$ be a real function satisfying (7), and let $(n_k)_{k\geq 1}$ a sequence of positive integers satisfying conditions* $(\mathbf{K}_\alpha)$ *and* $(\mathbf{D}_\delta)$*, where*

$$\alpha+\delta<1.$$

*Then*

$$\sum_{k=1}^N f(n_k x)=\mathcal{O}\left(N^{1/2}(\log\log N)^{3/2}\right) \qquad \text{a.e.}$$

**Theorem 2** *Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers satisfying conditions* $(\mathbf{K}_\alpha)$ *and* $(\mathbf{D}_\delta)$*for $\alpha+\delta<1$, and condition* $(\mathbf{D}^{\mathbf{0}}_\gamma)$ *for $\gamma<1$. Then*

$$\limsup_{N\to\infty}\frac{ND_N(n_k x)}{\sqrt{2N\log\log N}}=\limsup_{N\to\infty}\frac{ND_N^*(n_k x)}{\sqrt{2N\log\log N}}=\frac{1}{2} \qquad \text{a.e.}$$

A simple example for a class of sequences satisfying conditions $(\mathbf{K}_\alpha)$ and $(\mathbf{D}_\delta)$ is the class of Hardy-Littlewood-Pólya sequences (cf. Berkes, Philipp, Tichy [8] and Philipp [16]), i.e. the class of sequences generated by a finite number of coprime integers, sorted in increasing order. A Hardy-Littlewood-Pólya sequence does not satisfy condition $(\mathbf{D}^{\mathbf{0}}_\gamma)$, since the equation

$$j_1 n_{k_1}-j_2 n_{k_2}=0$$

has "too many" solutions for certain values of $j_1,j_2$, and we can not expect that a Hardy-Littlewood-Pólya sequence will satisfy the LIL for the discrepancy (4) with the value of the $\limsup$ equal to 1/2. In fact, recently Fukuyama and Nakata [11] have been able to calculate the value of the $\limsup$ in the LIL for the discrepancy of $(n_k x)_{k\geq 1}$, where $(n_k)_{k\geq 1}$ is a Hardy-Littlewood-Pólya sequence, and in general the value is different from 1/2. On the other hand, using a random construction, it is easy to give examples of sequences satisfying our conditions $(\mathbf{K}_\alpha)$, $(\mathbf{D}_\delta)$, $(\mathbf{D}^{\mathbf{0}}_\gamma)$. In fact one can show that "almost all" sequences (with respect to a certain "natural" measure) satisfying condition $(\mathbf{K}_\alpha)$ will also satisfy the Diophantine conditions $(\mathbf{D}_\delta)$, $(\mathbf{D}^{\mathbf{0}}_\gamma)$ (for such an argument see [8, p. 117]). It is not clear whether (8) is really necessary, or it can be replaced by e.g.

$$\alpha<1, \quad \delta<1. \tag{9}$$

There is some reason to believe that for slowly growing sequences $(n_k)_{k\geq 1}$ the Diophantine (or number-theoretic) structure has to be "stronger", while for fast growing sequences the Diophantine structure can be weaker. This is in accordance with our results for lacunary sequences, which in some sense represent the case $\alpha=0,\ \delta=1$. In particular, we do not know if (8) can be improved, and it would be interesting to see if results similar to Theorem 1, Theorem 2 can be shown if (8) is replaced by (9).

## 2 Preliminaries

To prove our theorems we will need some auxiliary results. In the sequel we will always assume that $f$ satisfies (7), and, without loss of generality, that $\mathrm{Var}_{[0,1]} f \le 2$, i.e. the total variation of $f$ in the interval $[0,1]$ is at most 2. This implies that for the Fourier series of $f$, i.e.

$$f \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j x + b_j \sin 2\pi j x,$$

we have (see Zygmund [21, p. 48])

$$|a_j| \le j^{-1}, \qquad |b_j| \le j^{-1}, \qquad j \ge 1.$$

**Theorem 3 (Strassen [20])** *Let $(Y_i, \mathcal{F}_i, i \ge 1)$ be a martingale difference sequence with finite fourth moments, let $V_M = \sum_{i=1}^{M} \mathbb{E}(Y_i^2 | \mathcal{F}_{i-1})$ and suppose $V_M \sim s_M$ a.s. with some positive sequence $s_M$ and*

$$\sum_{M=1}^{\infty} \frac{\mathbb{E} Y_M^4}{s_M^{2\vartheta}} < +\infty \tag{10}$$

*with $0 < \vartheta < 1$. Then $Y_1, Y_2, \ldots$ can be redefined on a new probability space without changing its distribution together with a Wiener process $\xi(t)$ such that*

$$Y_1 + \cdots + Y_M = \xi(V_M) + o\left(V_M^{(1+\vartheta)/4} \log V_M\right) \quad \text{a.s.}$$

The Beppo Levi theorem and (10) imply the a.s. convergence of $\sum_{M=1}^{\infty} s_M^{-2\vartheta} \mathbb{E}(Y_M^4 | \mathcal{F}_{M-1})$ and hence by $V_M \sim s_M$ the series $\sum_{M=1}^{\infty} V_M^{-2\vartheta} \mathbb{E}(Y_M^4 | \mathcal{F}_{M-1})$ is also a.s. convergent. Thus

$$\begin{aligned} \sum_{M=1}^{\infty} \frac{1}{V_M^{\vartheta}} \int_{x^2 > V_M^{\vartheta}} x^2 \, dP(Y_M < x | \mathcal{F}_{M-1}) &\le \sum_{M=1}^{\infty} \frac{1}{V_M^{2\vartheta}} \int_{-\infty}^{+\infty} x^4 \, dP(Y_M < x | \mathcal{F}_{M-1}) \\ &\le \sum_{M=1}^{\infty} \frac{1}{V_M^{2\vartheta}} \mathbb{E}(Y_M^4 | \mathcal{F}_{M-1}) \; < \; +\infty \quad \text{a.s.} \end{aligned}$$

Thus Theorem 3 follows from Theorem (4.4) of [20] (this argument is copied from [6, Theorem B]).

**Lemma 1 (Strassen [20])** *Let $\varepsilon > 0$ be given. Then there exists an $\vartheta > 0$ such that*

$$\begin{aligned} \mathbb{P}\Big\{ & |\xi(V_n) - \xi(s_n)| = o\left(s_n^{1/2-\vartheta}\right) \text{ as } n \to \infty \text{ for any sequences } V_n, s_n, \\ & \text{such that } V_n \to \infty, s_n \to \infty, |V_n - s_n| = o\left(s_n^{1-\varepsilon}\right) \text{ as } n \to \infty \Big\} = 1. \end{aligned}$$

This is a special case of [20, Lemma 4.2].

**Theorem 4 ([19, p. 299])** *Let $(U_i, \mathcal{F}_i, i \ge 1)$ be a supermartingale with $\mathbb{E} U_1 = 0$. Put*

$$U_0 = 0 \quad \text{and} \quad Y_i = U_i - U_{i-1}, \quad i \ge 1.$$

*Suppose that*

$$Y_i \le c \quad \text{a.s.}$$

*for some constant* $c > 0$ *and for all* $i \ge 1$. *For* $\lambda > 0$ *define*

$$T_n = \exp\left(\lambda U_n - \frac{1}{2}\lambda^2\left(1 + \frac{1}{2}\lambda c\right)\sum_{i\le n}\mathbb{E}(Y_i^2|\mathcal{F}_{i-1})\right), \quad n \ge 1,$$

*and* $T_0 = 1$ *a.s. Then for each* $\lambda$ *with* $\lambda c \le 1$ *the sequence* $(T_n, \mathcal{F}_n, n \ge 0)$ *is a non-negative supermartingale satisfying*

$$\mathbb{P}\left(\sup_{n\ge 0} T_n > a\right) \le 1/a$$

*for each* $a > 0$.

**Lemma 2** *Assume that condition* $(\mathbf{K}_\alpha)$ *holds for the sequence* $(n_k)_{k\ge1}$ *for some* $\alpha < 1$, *and assume without loss of generality that* $C_\alpha > 1$. *Then*

$$\frac{n_{k+3C_\alpha^2k^\alpha}}{n_k} \ge 2, \quad k \ge 1.$$

*Remark:* Here and in the sequel, $n_k$ will stand for $n_{\lceil k\rceil}$ if $k$ is not an integer.

*Proof:* For given $k$, there exists an $j$ such that $n_k \in [2^j, 2^{j+1})$, and by condition $(\mathbf{K}_\alpha)$ there are at most $(C_\alpha k^\alpha - 1)$ other indices $k'$ for which $n_{k'}$ lies in this interval. Accordingly, the number of indices $k''$ for which $n_{k''} \in [2^{j+1}, 2^{j+2})$ is bounded by

$$C_\alpha\left(k + C_\alpha k^\alpha\right)^\alpha \le C_\alpha\left(k^\alpha + C_\alpha k^{(\alpha^2)}\right) \le 2C_\alpha^2 k^\alpha.$$

This implies

$$n_{k+(C_\alpha k^\alpha-1)+2C_\alpha^2k^\alpha+1} \in \left[2^{j+2}, \infty\right)$$

and

$$\frac{n_{k+3C_\alpha^2k^\alpha}}{n_k} \ge \frac{n_{k+(C_\alpha k^\alpha-1)+2C_\alpha^2k^\alpha+1}}{n_k} \ge 2. \qquad \square$$

**Lemma 3** *Assume that condition* $(\mathbf{K}_\alpha)$ *holds for the sequence* $(n_k)_{k\ge1}$ *for some* $\alpha < 1$, *and assume without loss of generality that* $C_\alpha > 1$. *Then for any integer* $m \ge 1$

$$\frac{n_{k(1+3C_\alpha^2k^{\alpha-1})^m}}{n_k} \ge 2^m, \quad k \ge 1.$$

*Proof*: We use induction on $m$. The case $m = 1$ was shown in Lemma 2. For $m \ge 1$ we have

$$\begin{aligned}\frac{n_{k(1+3C_\alpha^2k^{\alpha-1})^{m+1}}}{n_k} &\ge \frac{n_{k(1+3C_\alpha^2k^{\alpha-1})^m(1+3C_\alpha^2(k(1+3C_\alpha^2k^{\alpha-1})^m)^{\alpha-1})}}{n_{k(1+3C_\alpha^2k^{\alpha-1})^m}} \cdot \frac{n_{k(1+3C_\alpha^2k^{\alpha-1})^m}}{n_k}\\ &\ge 2\cdot 2^m = 2^{m+1}. \qquad \square\end{aligned}$$

**Lemma 4** *Assume that condition* $(\mathbf{K}_\alpha)$ *holds for the sequence* $(n_k)_{k\ge1}$ *for some* $\alpha < 1$, *and assume without loss of generality that* $C_\alpha > 1$. *Then there exists a number* $c > 0$, *depending only on* $\alpha$ *and* $C_\alpha$, *such that*

$$\frac{n_{k+28C_\alpha^2(\log k)k^\alpha}}{n_k} \ge ck^6, \quad k \ge 1. \tag{11}$$

*Proof:* For sufficiently large $k$,

$$\left(1+3C_\alpha^2 k^{\alpha-1}\right)^{9\log k} \leq 1+28C_\alpha^2(\log k)k^{\alpha-1}.$$

Thus for such $k$

$$\frac{n_{k+28C_\alpha^2(\log k)k^\alpha}}{n_k} \geq \frac{n_{k(1+3C_\alpha^2 k^{\alpha-1})^{(9\log k)}}}{n_k} \geq 2^{9\log k} \geq k^{9\log 2} \geq k^{-6}$$

by Lemma 3. By choosing $c$ sufficiently small (11) holds for all $k \geq 1$. $\square$

**Lemma 5** *For any function $f$ satisfying (7), we have*

$$\left|\int_a^b f(\lambda x)\ dx\right| \leq \frac{2}{\lambda}\int_0^1 |f(x)|\ dx \leq \frac{2}{\lambda}\|f\|_\infty$$

*for any real numbers $a<b$ and any $\lambda>0$. In particular,*

$$\left|\int_a^b \cos(2\pi\lambda x)\ dx\right| \leq \frac{2}{\lambda}.$$

*Proof.* The lemma follows from

$$\int_a^b f(\lambda x)\ dx = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} f(x)\ dx. \quad \square$$

## 3 Proof of Theorem 1

Assume the conditions of Theorem 1 are satisfied. We put

$$\eta = \frac{\alpha}{1-\alpha}+\nu$$

for a $\nu>0$ such that $\alpha+\delta+\nu(1-\alpha)\delta<1$ (since by assumption $\alpha+\delta<1$ it is possible to choose such a $\nu$). Then

$$(\eta+1)\delta = \left(\frac{1}{1-\alpha}+\nu\right)\delta = \frac{\delta+\nu(1-\alpha)\delta}{1-\alpha} < 1. \tag{12}$$

We choose an $\eta'$ such that

$$\eta' < \eta \quad \text{and} \quad \eta' > \frac{\alpha}{1-\alpha}+\nu\alpha,$$

and observe that

$$(\eta+1)\alpha = \left(\frac{1}{1-\alpha}+\nu\right)\alpha < \eta'. \tag{13}$$

We have

$$ci^{\eta+1} \leq \sum_{l=1}^{i}\lfloor l^\eta\rfloor \leq ci^{\eta+1}.$$

(Remark: Throughout this section $c$ will denote appropriate positive numbers, not always the same, that may only depend on the constants $\alpha, \delta, C_\alpha, C_\delta, C_1$ in the statement of Theorem 1 and on $\eta, \eta'$, but not on $f, N, k, i, M, p, \varphi$ or anything else. $\varepsilon$ will denote appropriate, "small" positive number, that may only depend on $\alpha$ and $\delta$.)

We divide the set of positive integers into consecutive blocks

$$\Delta_1, \Delta_1', \Delta_2, \Delta_2', \dots, \Delta_i, \Delta_i', \dots$$

of lengths $\lfloor i^\eta \rfloor$ and $\lceil i^{\eta'} \rceil$, respectively. Let $(i-1)^+$ denote the largest integer in $\Delta_i$. Then

$$c i^{\eta+1} \le (i-1)^+ \le c i^{\eta+1}, \quad i \ge 2. \tag{14}$$

Therefore by (13) for sufficiently large $i$

$$(i-1)^+ + 28 C_\alpha^2 (\log((i-1)^+))((i-1)^+)^\alpha \le (i-1)^+ + c(\log i) i^{(\eta+1)\alpha} \le (i-1)^+ + i^{\eta'} \le i^-,$$

where $i^-$ denotes the smallest integer in the block $\Delta_i$. Thus by condition $(\mathbf{K}_\alpha)$, Lemma 4 and (14) for sufficiently large $i$

$$\frac{n_{(i-1)^+}}{n_{i^-}} \le c\left((i-1)^+\right)^{-6} \le c\left(i^{\eta+1}\right)^{-6}, \tag{15}$$

and by changing $c$ this is valid for $i \ge 2$.

For simplicity of writing without loss of generality we assume that $f$ is an even function, i.e. the Fourier series of $f$ is of the form

$$f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j x.$$

The proof in the general case is exactly the same. We approximate $f$ by trigonometric polynomials

$$p_k(x) = \sum_{j=1}^{k^2} a_j \cos 2\pi j x, \quad k \ge 1.$$

Then

$$\|f - p_k\|_2 \le \left( \sum_{j=k^2+1}^{\infty} a_j^2 \right)^{1/2} \le \left( \sum_{j=k^2+1}^{\infty} j^{-2} \right)^{1/2} \le k^{-1}.$$

For every $k \in \bigcup_{i \ge 1} \Delta_i$, we set

$$m(k) = \lceil \log_2 n_k + 4 \log_2 k \rceil$$

and approximate $p_k(n_k x)$ by a discrete function $\varphi_k(x)$ such that the following properties are satisfied:

(P1) $\varphi_k(x)$ is constant for $\frac{v}{2^{m(k)}} \le x < \frac{v+1}{2^{m(k)}}, \quad v = 0, 1, \dots, 2^{m(k)} - 1$
(P2) $|\varphi_k(x) - p_k(n_k x)| \le c k^{-2}, \qquad x \in [0, 1)$
(P3) $\mathbb{E}(\varphi_k(x) | \mathcal{F}_{i-1}) = 0$

Here $\mathcal{F}_i$ denotes the $\sigma$-field generated by the intervals

$$\left[\frac{v}{2^{m(i^+)}}, \frac{v+1}{2^{m(i^+)}}\right), \quad 0 \le v < 2^{m(i^+)}.$$

We have

$$\begin{aligned} & |p_k(n_k x) - p_k(n_k \bar{x})| \le \sum_{j=1}^{k^2} |a_j| 2\pi j n_k 2^{-m(k)} \le ck^{-2} \\ & \text{for} \quad \frac{v}{2^{m(k)}} \le x, \bar{x} \le \frac{v+1}{2^{m(k)}}, \ 0 \le v < 2^{m(k)}. \end{aligned}$$

Thus it is possible to approximate $p_k(n_k x)$ by discrete functions $\hat{\varphi}_k(x)$ that satisfy (P1) and (P2) only. For $k \in \Delta_i$ and any interval $I$ of the form $\left[\frac{v}{2^{m((i-1)^+)}}, \frac{v+1}{2^{m((i-1)^+)}}\right), 0 \le v < 2^{m((i-1)^+)}$, letting $|I|$ denote the length of $I$,

$$\begin{aligned} \frac{1}{|I|}\left|\int_I \hat{\varphi}_k(x)\ dx\right| &\le \frac{1}{|I|}\left|\int_I p_k(n_k x)\ dx\right| + \frac{1}{|I|}\int_I ck^{-2}\ dx \\ &\le c2^{m((i-1)^+)} \sum_{j=1}^{k^2} \frac{|a_j|}{j n_{i^-}} + \frac{c}{k^2} \\ &\le \frac{c\left(i^{\eta+1}\right)^4 n_{(i-1)^+}}{n_{i^-}} + \frac{c}{k^2} \\ &\le ck^{-2}, \end{aligned}$$

by (15) and Lemma 5. Every $x \in [0,1)$ is contained in one interval of the form $I$ (for some $v$), so we put $\varphi_k(x) = \hat{\varphi}_k(x) - |I|^{-1}\int_I \hat{\varphi}_k(t)dt$ for $x \in I$ and get functions that satisfy (P1), (P2) and (P3).

We put $Y_0 = 0$, and, for $i \ge 1, M \ge 1$, we define

$$\begin{aligned} Y_i &= Y_i(x) = \sum_{k \in \Delta_i} \varphi_k(x), \\ T_i &= T_i(x) = \sum_{k \in \Delta_i} p_k(n_k x), \\ T_i' &= T_i'(x) = \sum_{k \in \Delta_i'} p_k(n_k x), \\ V_M &= \sum_{i=1}^{M} \mathbb{E}(Y_i^2 | \mathcal{F}_{i-1}), \end{aligned}$$

and

$$\sigma_i^2 = \int_0^1 \left(\sum_{k \in \Delta_i} p_k(n_k x)\right)^2 dx, \qquad s_M = \sum_{i=1}^{M} \sigma_i^2.$$

We want to estimate $\|V_M - s_M\|_2$. We have

$$\begin{aligned}
& T_i(x)^2 - \sigma_i^2 \\
= \quad & \left( \sum_{k \in \Delta_i} p(n_k x) \ dx \right)^2 - \int_0^1 \left( \sum_{k \in \Delta_i} p_k(n_k x) \right)^2 \\
= \quad & \sum_{\substack{k, k' \in \Delta_i, 1 \le j \le k^2, 1 \le j \le (k')^2 \\ 0 < |jn_k - j'n_{k'}| \le n_{(i-1)^+}}} \frac{1}{2} \ a_j a_{j'} \cos 2\pi (jn_k - j'n_{k'}) x \\
& + W_i(x) \\
= \quad & U_i(x) + W_i(x). \qquad (16)
\end{aligned}$$

Note that the trigonometric functions in $T_i^2$ with frequency zero cancel out with $\sigma_i^2$. $W_i(x)$ is a sum of trigonometric functions with frequencies at least $n_{(i-1)^+}$, and the sum of the coefficients of these trigonometric functions is at most

$$c \left( \sum_{k \in \Delta_i} \sum_{j=1}^{k^2} |a_j| \right)^2 \le c (\log i)^2 i^{2\eta}$$

Thus we can write

$$W_i(x) = \sum_{u \ge n_{(i-1)^+}} c_u \cos 2\pi u x,$$

where

$$\sum_{u \ge n_{(i-1)^+}} |c_u| \le c (\log i)^2 i^{2\eta}, \qquad (17)$$

and by condition $(\mathbf{D}_\delta)$

$$|c_u| \le c \left( i^{\eta+1} \right)^\delta. \qquad (18)$$

By (16), Minkowski's inequality and

$$\begin{aligned}
\left\| Y_i^2 - T_i^2 \right\|_\infty \quad & \le \quad \|Y_i - T_i\|_\infty \ \|Y_i + T_i\|_\infty \\
& \le \quad c |\Delta_i| \ i^{-2(\eta+1)} \sum_{k \in \Delta_i} \|p_k\|_\infty \\
& \le \quad c (\log i) i^{-2} \qquad (19)
\end{aligned}$$

we get

$$\begin{aligned}
& \|V_M - s_M\|_2 \\
\le \quad & \left\| \sum_{i=1}^M \left( \mathbb{E}(T_i^2 | \mathcal{F}_{i-1}) - \sigma_i^2 \right) \right\|_2 + \sum_{i=1}^M c (\log i) i^{-2} \\
\le \quad & \left\| \sum_{i=1}^M \mathbb{E}(U_i | \mathcal{F}_{i-1}) \right\|_2 + \left\| \sum_{i=1}^M \mathbb{E}(W_i | \mathcal{F}_{i-1}) \right\|_2 + c. \qquad (20)
\end{aligned}$$

To estimate $\left\|\sum_{i=1}^{M}\mathbb{E}(W_i|\mathcal{F}_{i-1})\right\|_2$, we observe

$$\left(\sum_{i=1}^{M}\mathbb{E}(W_i|\mathcal{F}_{i-1})\right)^2 \leq 2\sum_{1\leq i\leq i'\leq M}\mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i'-1}).$$

By (17), (18) and the Jensen inequality

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{M}(\mathbb{E}(W_i|\mathcal{F}_{i-1}))^2\right) &\leq \sum_{i=1}^{M}\mathbb{E}W_i^2\\
&\leq \sum_{i=1}^{M}c(\log i)^2 i^{2\eta}\left(i^{\eta+1}\right)^{\delta}\\
&\leq c(\log M)^2 M^{2\eta+1+(\eta+1)\delta}.
\end{aligned}$$

For $i<i'$, since $\mathbb{E}(W_i|\mathcal{F}_{i-1})$ is $\mathcal{F}_{i-1}$-measurable,

$$\begin{aligned}
\left|\mathbb{E}\Big(\mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i'-1})\Big|\mathcal{F}_{i-1}\Big)\right| &= |\mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})|\\
&\leq \|W_i\|_\infty\,|\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})|\\
&\leq c(\log i)^2 i^{2\eta}\,|\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})|\,.
\end{aligned}$$

Using (17) and Lemma 5, we get

$$\begin{aligned}
|\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})| &\leq c(\log i')^2(i')^{2\eta}\frac{i^{4\eta+4}n_{(i-1)^+}}{n_{(i'-1)^+}}\\
&\leq c\frac{(\log i')^2(i')^{2\eta}i^{4\eta+4}}{\prod_{j=i+1}^{i'}j^{6\eta+6}},
\end{aligned}$$

which finally yields

$$\begin{aligned}
&\left\|\sum_{i=1}^{M}\mathbb{E}(W_i|\mathcal{F}_{i-1})\right\|_2\\
&\leq c\left((\log M)^2M^{2\eta+1+(\eta+1)\delta}+\sum_{1\leq i<i'\leq M}\frac{(\log i)^2i^{2\eta}(\log i')^2(i')^{2\eta}i^{4\eta+4}}{\prod_{j=i+1}^{i'}j^{6\eta+6}}\right)^{1/2}\\
&\leq c(\log M)\sqrt{M^{2\eta+1+(\eta+1)\delta}}. \qquad (21)
\end{aligned}$$

Next we estimate $\left\|\sum_{i=1}^{M}\mathbb{E}(U_i|\mathcal{F}_{i-1})\right\|_2$. Writing

$$U_i=\sum_{u=1}^{n_{(i-1)^+}}c_u\cos 2\pi ux,$$

wee see that the fluctuation of $U_i$ on any atom of $\mathcal{F}_{i-1}$ is at most

$$\begin{aligned}
\sum_{u=1}^{n_{(i-1)^+}}|c_u|2\pi u2^{-m((i-1)^+)} &\leq cn_{(i-1)^+}2^{-m((i-1)^+)}\sum_{u=1}^{n_{(i-1)^+}}|c_u|\\
&\leq ci^{-4\eta-4}(\log i)^2i^{2\eta}\;\leq\; c(\log i)^2i^{-2\eta-4}.
\end{aligned}$$

Therefore

$$\left\| \sum_{i=1}^{M} \mathbb{E}(U_i | \mathcal{F}_{i-1}) \right\|_2 \leq \left\| \sum_{i=1}^{M} U_i \right\|_2 + c. \tag{22}$$

Writing

$$\sum_{i=1}^{M} U_i = \sum_{u=1}^{n_{(M-1)^+}} c_u \cos 2\pi u x,$$

we have

$$\sum_{u=1}^{n_{(M-1)^+}} |c_u| \leq \sum_{i=1}^{M} c(\log i)^2 i^{2\eta} \leq c(\log M)^2 M^{2\eta+1},$$

and, by condition $(\mathbf{D}_\delta)$,

$$|c_u| \leq c \left(M^{\eta+1}\right)^{\delta}, \qquad 1 \leq u \leq n_{(M-1)^+}.$$

This, together with (22), yields

$$\left\| \sum_{i=1}^{M} \mathbb{E}(U_i | \mathcal{F}_{i-1}) \right\|_2 \leq \left( \sum_{u=1}^{n_{(M-1)^+}} c_u^2 \right)^{1/2} + c \ \leq \ c(\log M) \sqrt{M^{2\eta+1+(\eta+1)\delta}}. \tag{23}$$

Combining (20), (21) and (23), we finally arrive at

$$\|V_M - s_M\|_2 \leq c(\log M) \sqrt{M^{2\eta+1+(\eta+1)\delta}} \leq c M^{\eta+1-\varepsilon},$$

since by (12) we have $(\eta+1)\delta < 1$.

By assumption

$$\int_0^1 \left( \sum_{k=1}^{N} f(n_k x) \right)^2 dx > C_1 N, \qquad N \geq 1. \tag{24}$$

We observe that

$$\begin{aligned}
& \left| \int_0^1 \left( \sum_{i=1}^{M} \sum_{k \in \Delta_i} p_k(n_k x) \right)^2 dx - \int_0^1 \sum_{i=1}^{M} \left( \sum_{k \in \Delta_i} p_k(n_k x) \right)^2 dx \right| \\
= & \left| 2 \int_0^1 \sum_{1 \leq i < i' \leq M} \left( \sum_{k \in \Delta_i} p_k(n_k x) \right) \left( \sum_{k' \in \Delta_{i'}} p_k(n_{k'} x) \right) dx \right| \\
\leq & \ c \sum_{1 \leq i < i' \leq M} \# \left\{ k \in \Delta_i, k' \in \Delta_{i'}, 1 \leq j \leq k^2, 1 \leq j' \leq (k')^2 : \ j n_k - j' n_{k'} = 0 \right\} \qquad (25) \\
\leq & \ c,
\end{aligned}$$

since for $k \in \Delta_i, k' \in \Delta_{i'}$, where $i' > i$, we have

$$\frac{n_{k'}}{n_k} \geq \frac{n_{(i')^-}}{n_{i^+}} \geq \frac{n_{(i+1)^-}}{n_{i^+}} \geq c i^{6(\eta+1)} \geq c k^6$$

and thus the equation in (25) has only finitely many solutions. Letting $r_k(x) = f(x) - p_k(x),\ k \geq 1$, Minkowski's inequality yields

$$\begin{aligned}
& \left|\sqrt{s_M} - \sqrt{b_{M^+}}\right| \\
\leq & \left| \left( \int_0^1 \left( \sum_{i=1}^{M} \sum_{k \in \Delta_i} p_k(n_k x) \right)^2 dx \right)^{1/2} - \left( \int_0^1 \left( \sum_{k=1}^{M^+} f(n_k x) \right)^2 dx \right)^{1/2} \right| + c \\
\leq & \left( \int_0^1 \left( \sum_{k=1}^{M^+} r_k(n_k x) \right)^2 dx \right)^{1/2} + \left( \int_0^1 \left( \sum_{1 \leq k \leq M^+, k \notin \bigcup_{i=1}^M \Delta_i} p_k(n_k x) \right)^2 dx \right)^{1/2} + c \\
\leq & \left( \sum_{k=1}^{M^+} \|r_k\|_2 \right) + \sum_{j=1}^{(M^+)^2} |a_j| \left( \int_0^1 \left( \sum_{1 \leq k \leq M^+, k \notin \bigcup_{i=1}^M \Delta_i, k \geq \sqrt{j}} \cos 2\pi j n_k x \right)^2 dx \right)^{1/2} + c \\
\leq & \left( \sum_{k=1}^{M^+} c k^{-1} \right) + \sum_{j=1}^{(M^+)^2} c j^{-1} \sqrt{\sum_{i=1}^{M} |\Delta_i'|} + c \\
\leq & \ c \left( (\log M) + (\log M) \sqrt{M^{\eta'+1}} \right) \ \leq \ c(M^+)^{1/2-\varepsilon}.
\end{aligned}$$

It is easy to see that

$$s_M \leq c(\log M)^2 M^+, \qquad b_{M^+} \leq c(\log M)^2 M^+,$$

and thus

$$|s_M - b_{M^+}| \leq \left|\sqrt{s_M} - \sqrt{b_{M^+}}\right| \left(\sqrt{s_M} + \sqrt{b_{M^+}}\right) \leq c(M^+)^{1-\varepsilon}. \tag{26}$$

Therefore by (24)

$$s_M \geq c M^{\eta+1}$$

for sufficiently large $M$, which implies, together with (26), that

$$\frac{b_{M^+}}{s_M} \to 1 \quad \text{and} \quad |s_M - b_{M^+}| \leq \mathcal{O}\left((b_{M^+})^{1-\varepsilon}\right) \quad \text{as} \quad M \to \infty. \tag{27}$$

Choose $\ell$ "large" (the exact value will be determined later), and define

$$N_j = j^\ell, \quad j \geq 1.$$

We have

$$\begin{aligned}
s_{N_j} - s_{N_{j-1}} &= \sum_{i=N_{j-1}}^{N_j} \int_0^1 \left( \sum_{k \in \Delta_i} p_k(n_k x) \right)^2 dx \\
&\leq c(\log j)^2 \sum_{i=N_{j-1}}^{N_j} i^\eta \\
&\leq c(\log j)^2 \left( \left(j^\ell\right)^{\eta+1} - \left((j-1)^\ell\right)^{\eta+1} \right) \\
&\leq c(\log j)^2 j^{\ell(\eta+1)-1} \\
&\leq c_1 (\log j)^2 (s_{N_{j-1}})^{\frac{\ell(\eta+1)-1}{\ell(\eta+1)}}
\end{aligned}$$

for some positive number $c_1$, which does not depend on $j$. Since $s_M$ and $V_M$ are both increasing in $M$

$$\begin{aligned}
& \mathbb{P}\left( \bigcup_{N_{j-1}<M\leq N_j} \left\{ |V_M - s_M| \geq 2c_1(\log j)^2 (s_M)^{\frac{\ell(\eta+1)-1}{\ell(\eta+1)}} \right\} \right) \\
\leq \; & \mathbb{P}\left( \left\{ \left|V_{N_{j-1}} - s_{N_j}\right| \geq 2c_1(\log j)^2 (s_{N_{j-1}})^{\frac{\ell(\eta+1)-1}{\ell(\eta+1)}} \right\} \right) \\
& + \mathbb{P}\left( \left\{ \left|V_{N_j} - s_{N_{j-1}}\right| \geq 2c_1(\log j)^2 (s_{N_{j-1}})^{\frac{\ell(\eta+1)-1}{\ell(\eta+1)}} \right\} \right) \\
\leq \; & \mathbb{P}\left( \left\{ |V_{N_{j-1}} - s_{N_{j-1}}| \geq c_1(\log j)^2 (s_{N_{j-1}})^{\frac{\ell(\eta+1)-1}{\ell(\eta+1)}} \right\} \right) \\
& + \mathbb{P}\left( \left\{ |V_{N_j} - s_{N_j}| \geq c_1(\log j)^2 (s_{N_{j-1}})^{\frac{\ell(\eta+1)-1}{\ell(\eta+1)}} \right\} \right) \\
\leq \; & c(s_{N_j})^{\frac{-2(\ell(\eta+1)-1)}{\ell(\eta+1)}} (N_j)^{2(\eta+1-\varepsilon)} \\
\leq \; & c\left((N_j)^{\eta+1}\right)^{\frac{-2(\ell(\eta+1)-1)}{\ell(\eta+1)}} (N_j)^{2(\eta+1-\varepsilon)} \\
\leq \; & c\left(j^{\ell(\eta+1)}\right)^{\frac{-2(\ell(\eta+1)-1)}{\ell(\eta+1)}} \left(j^{\ell}\right)^{2(\eta+1-\varepsilon)} \\
\leq \; & cj^{-2(\ell(\eta+1)-1)+2\ell(\eta+1-\varepsilon)} \\
\leq \; & cj^{-2}
\end{aligned}$$

for sufficiently large $\ell$ (depending only on $\varepsilon$). Therefore by the Borel-Cantelli lemma

$$\mathbb{P}\left\{ x \in [0,1): \; x \in \left\{ |V_M - s_M| \geq 2c_1(\log M)^2 (s_M)^{\frac{\ell(\eta+1)-1}{\ell(\eta+1)}} \right\} \text{ for finitely many } M \right\} = 1,$$

or, in other words,

$$|V_M - s_M| = o\left((s_M)^{1-\varepsilon}\right) \qquad \text{a.e.} \tag{28}$$

Next we estimate $\|Y_M\|_4$. We have

$$\begin{aligned}
\|Y_M\|_4 \; & \leq \; \left\| \sum_{k\in\Delta_M} p_k(n_k x) \right\|_4 + \sum_{k\in\Delta_M} \|p_k - \varphi_k\|_\infty \\
& \leq \; \left( \sum_{j=1}^{M^+} \frac{1}{j} \left\| \sum_{k\in\Delta_M, k\geq\sqrt{j}} \cos 2\pi n_k x \right\|_4 \right) + cM^\eta (M^{\eta+1})^{-2} \\
& \leq \; c(\log M)\left(M^{2\eta} M^{(\eta+1)\delta}\right)^{1/4} \\
& \leq \; c\left(M^{2\eta+1-\varepsilon}\right)^{1/4},
\end{aligned} \tag{29}$$

since

$$
\begin{aligned}
& \left\| \sum_{k \in \Delta_M} \cos 2\pi n_k x \right\|_4^4 && (30)\\
\leq \; & \# \left\{ (k_1,k_2,k_3,k_4), k_i \in \Delta_M, i=1,2,3,4: \; n_{k_1} \pm n_{k_2} + n_{k_3} \pm n_{k_4} = 0 \right\} \\
\leq \; & \sum_{b \in \mathbb{Z}} \left( \# \left\{ (k_1,k_2), k_i \in \Delta_M, i=1,2: \; n_{k_1} \pm n_{k_2} = b \right\} \right)^2 \\
\leq \; & \left( \# \left\{ (k_1,k_2), k_i \in \Delta_M, i=1,2: \; n_{k_1} - n_{k_2} = 0 \right\} \right)^2 \\
& + |\Delta_M|^2 \max_{b \in \mathbb{Z}, b \neq 0} \# \left\{ (k_1,k_2), k_i \in \Delta_M, i=1,2: \; n_{k_1} \pm n_{k_2} = b \right\} \\
\leq \; & |\Delta_M|^2 + |\Delta_M|^2 C_\delta (M^+)^\delta \\
\leq \; & c M^{2\eta + (\eta+1)\delta}
\end{aligned}
$$

(here $n_{k_1} \pm n_{k_2} = b$ means that either $n_{k_1} + n_{k_2} = b$ or $n_{k_1} - n_{k_2} = b$). Now we can apply Theorem 3: by (28) we have $V_M \sim s_M$ a.e., and by (29) we have

$$
\sum_{M=1}^{\infty} \frac{\mathbb{E} Y_M^4}{s_M^{2-\varepsilon_1}} \leq c \sum_{M=1}^{\infty} \frac{M^{2\eta+1-\varepsilon}}{\left(M^{\eta+1}\right)^{2-\varepsilon_1}} \leq c \sum_{M=1}^{\infty} \frac{M^{(\eta+1)\varepsilon_1}}{M^{1+\varepsilon}} < +\infty
$$

for a sufficiently small $\varepsilon_1 > 0$ (depending on the value of $\varepsilon$ in (29)). This implies that there exists a Wiener process $\xi$ such that

$$
Y_1 + \dots + Y_M = \xi(V_M) + o\left( V_M^{1/2-\varepsilon} \right) \quad \text{a.e.}
$$

Since by (27) and (28)

$$
|V_M - b_{M^+}| = o\left( (b_{M^+})^{1-\varepsilon} \right) \quad \text{a.e.},
$$

by Lemma 1 and since $|Y_i - T_i| \leq c|\Delta_i| i^{-2\eta-2} \leq c i^{-2}$

$$
T_1 + \dots + T_M = \xi(b_{M^+}) + o\left( b_{M^+}^{1/2-\varepsilon} \right) \quad \text{a.e.}
$$

To prove Theorem 1 it remains to replace the functions $p_k$ by $f$, add the remaining function values in $T_i'$ and break into the blocks of integers $\Delta_i$ and $\Delta_i'$. First we observe that

$$
\begin{aligned}
\left\| \max_{N \geq 1} \left| \sum_{k \in (\Delta_i \cup \Delta_i'),\; k \leq N} f(n_k x) - p_k(n_k x) \right| \right\|_2 \; & \leq \; \sum_{k \in (\Delta_i \cup \Delta_i')} \|r_k\|_2 \\
& \leq \; c(|\Delta_i| + |\Delta_i'|) i^{-\eta-1} \\
& \leq \; c i^{-1}.
\end{aligned}
$$

On the other hand, using the Carleson-Hunt inequality (for a survey see e.g. [5] or [15])

$$
\begin{aligned}
\left\| \max_{N \geq 1} \left| \sum_{k \in (\Delta_i \cup \Delta_i'),\; k \leq N} p_k(n_k x) \right| \right\|_4 \; & \leq \; c \sum_{j=1}^{(i^+)^2} \frac{1}{j} \left\| \max_{N \geq 1} \left| \sum_{k \in (\Delta_i \cup \Delta_i'),\; k \leq N} \cos 2\pi j n_k x \right| \right\|_4 \\
& \leq \; c (\log i) \left\| \sum_{k \in (\Delta_i \cup \Delta_i')} \cos 2\pi n_k x \right\|_4 \\
& \leq \; c (\log i) \left( i^{2\eta} i^{(\eta+1)\delta} \right)^{1/4} \; \leq \; c i^{\frac{2\eta+1-\varepsilon}{4}}, && (31)
\end{aligned}
$$

where the last inequality follows from an argument similar to the one in (30). Thus for a sufficiently small $\varepsilon_2 > 0$ (depending on the value of $\varepsilon$ in (31))

$$\begin{aligned} & \mathbb{P}\left\{x \in (0,1): \max_{N \geq 1} \left| \sum_{k \in (\Delta_i \cup \Delta_i'),\ k \leq N} f(n_k x) \right| > i^{(\eta+1)/2-\varepsilon_2} \right\} \\ \leq\ & c \frac{i^{2\eta+1-\varepsilon}}{i^{2\eta+2-4\varepsilon_2}} \\ \leq\ & c i^{-1-\varepsilon}. \end{aligned}$$

By the Borel-Cantelli lemma

$$\sum_{k=1}^{N} f(n_k x) = \xi(b_{M^+(N)}) + o\left( (b_{M^+(N)})^{1/2-\varepsilon} \right) \quad \text{a.e.},$$

where $M^+(N)$ denotes the value of $M^+$, where $M$ is defined by $N \in \Delta_M \cup \Delta_M'$. Since

$$b_N \leq cN (\log N)^2$$

and

$$\left| b_{M^+(N)} - b_N \right| = o\left( (b_N)^{1-\varepsilon} \right),$$

by Lemma 1 we finally arrive at

$$\sum_{k=1}^{N} f(n_k x) = \xi(b_N) + o\left( N^{1/2-\varepsilon} \right) \quad \text{a.e.},$$

which is Theorem 1.

# 4 Proof of Corollary 1 and Corollary 2

Corollary 2 is a consequence of the proof of Theorem 1 and the fact that for a function of bounded variation $g(x)$, satisfying (7), and any increasing sequence $(n_k)_{k \geq 1}$ of positive integers,

$$\int_0^1 \left( \sum_{k=1}^{N} g(n_k x) \right)^2 dx \leq cN (\log\log N)^2 \tag{32}$$

(the proof of (32) is due to Koksma [13], who used a deep result of Gál [9]). In fact, let a function of bounded variation $f$, satisfying (7), be given. Then, again without loss of generality assuming $\mathrm{Var}_{[0,1]} f \leq 2$ and $f$ is even,

$$f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j n_k x,$$

where

$$|a_j| \leq j^{-1}, \quad j \geq 1. \tag{33}$$

We can decompose

$$f = f_1 + f_2, \quad \text{where} \quad f_1(x) \sim \sum_{j=1}^{\infty} \max\{0, a_j\} \cos 2\pi j n_k x, \quad f_2 \sim \sum_{j=1}^{\infty} \min\{0, a_j\} \cos 2\pi j n_k x,$$

i.e. the Fourier coefficients of $f_1$ are all greater or equal zero, those of $f_2$ are all less or equal zero. Thus it is clear that

$$\int_0^1 \left( \sum_{k=1}^N f_1(n_k x) \right)^2 dx \geq cN, \qquad \int_0^1 \left( \sum_{k=1}^N f_2(n_k x) \right)^2 dx \geq cN,$$

except for the trivial case $\|f_1\|_2 = 0$ and $\|f_2\|_2 = 0$, respectively. In the proof of Theorem 1 we did not need the fact that $f$ is of bounded variation, but only the estimate (33) for the Fourier coefficients of $f$. Of course, (33) holds for the Fourier coefficients of $f_1$ and $f_2$ as well. Thus we can apply Theorem 1 and get, using (32),

$$\left| \sum_{k=1}^N f(n_k x) \right| \leq \left| \sum_{k=1}^N f_1(n_k x) \right| + \left| \sum_{k=1}^N f_2(n_k x) \right| \leq \mathcal{O}\left( N^{1/2} (\log \log N)^{3/2} \right) \quad \text{a.e.} \quad \square$$

Corollary 1 is a direct consequence of Theorem 1 and the fact that under condition $(\mathbf{D}_\gamma^\mathbf{0})$, again writing

$$b_N = \int_0^1 \left( \sum_{k=1}^N f(n_k x) \right)^2 dx, \quad N \geq 1,$$

the value of $b_N$ is almost $\|f\|_2^2 N$. More precisely we have to show that under condition $(\mathbf{D}_\gamma^\mathbf{0})$

$$\left| b_N - \|f\|_2^2 N \right| = o\left( N^{1-\varepsilon} \right) \qquad \text{as} \qquad N \to \infty \tag{34}$$

for a small $\varepsilon > 0$ (depending only on $\alpha, \delta, \gamma$). In fact, for given $N \geq 1$, writing

$$p(x) = \sum_{j=1}^{N^2} c_j \cos 2\pi j x$$

for the $N^2$-th partial sum of the Fourier series of $f$ (again assuming without loss of generality that $f$ is even), we get, using Hölder's inequality,

$$\begin{aligned}
& \left| \left( b_N - \|f\|_2^2 N \right) - \left( \int_0^1 \left( \sum_{k=1}^N p(n_k x) \right)^2 dx - \|p\|_2^2 N \right) \right| \\
\leq \; & \left| b_N - \int_0^1 \left( \sum_{k=1}^N p(n_k x) \right)^2 dx \right| + N(\|f\|_2^2 - \|p\|_2^2) \\
\leq \; & \left\| \sum_{k=1}^N (f(n_k x) - p(n_k x)) \right\|_2 \left\| \sum_{k=1}^N (f(n_k x) + p(n_k x)) \right\|_2 + N \sum_{j=N^2+1}^{\infty} j^{-2} \\
\leq \; & cN \; N^{-1} (\log N) \sqrt{N} + cN \; N^{-2} \; \leq \; c\sqrt{N} \log N.
\end{aligned}$$

Additionally, we have

$$\begin{aligned}
&\left| \int_0^1 \left( \sum_{k=1}^{N^2} p(n_k x) \right)^2 dx - \|p\|_2^2 N \right| \\
\leq\ & c \sum_{j_1,j_2=1}^{N^2} \frac{1}{j_1 j_2} \# \left\{ (k_1,k_2),\ 1 \leq k_1,k_2 \leq N, k_1 \neq k_2 :\ j_1 n_{k_1} - j_2 n_{k_2} = 0 \right\} \\
\leq\ & c (\log N)^2 N^{1-\gamma}
\end{aligned}$$

by condition $(\mathbf{D}_\gamma^\mathbf{0})$, which implies (34). $\square$

## 5 The LIL for the discrepancy

For $r \geq 0$, $N \geq 1$ and $(x_1, \dots, x_N) \in \mathbb{R}^N$ we define

$$D_N^{(\leq 2^{-r})}(x_1, \dots, x_N) = \sup_{0 \leq a < b < 1,\ b-a < 2^{-r}} \left| \frac{\sum_{k=1}^N \mathbf{I}_{[a,b)}(x_k)}{N} \right|$$

and

$$\begin{aligned}
D_N^{(\geq 2^{-r})}(x_1, \dots, x_N) &= \max_{0 \leq a_1 < a_2 \leq 2^r} \left| \frac{\sum_{k=1}^N \mathbf{I}_{[a_1 2^{-r}, a_2 2^{-r})}(x_k)}{N} \right|, \\
D_N^{*\,(\geq 2^{-r})}(x_1, \dots, x_N) &= \max_{0 < a_1 < 2^r} \left| \frac{\sum_{k=1}^N \mathbf{I}_{[0, a_1 2^{-r})}(x_k)}{N} \right|
\end{aligned}$$

Here and in the sequel $\mathbf{I}_{[a,b)}$ denotes the indicator function of the interval $[a,b)$, extended with period 1 and centered at expectation zero, i.e.

$$\mathbf{I}_{[a,b)}(x) = \mathbb{1}_{[a,b)}(\langle x \rangle) - (b-a), \qquad x \in \mathbb{R}.$$

It is easy to see that always

$$D_N^{(\geq 2^{-r})} \leq D_N \leq D_N^{(\geq 2^{-r})} + 2 D_N^{(\leq 2^{-r})} \tag{35}$$

and

$$D_N^{*\,(\geq 2^{-r})} \leq D_N^* \leq D_N^{*\,(\geq 2^{-r})} + 2 D_N^{(\leq 2^{-r})}. \tag{36}$$

The idea to split the discrepancies $D_N$ and $D_N^*$ into a discrepancy $D_N^{(\geq 2^{-r})}$ for finitely many "large" intervals, and a discrepancy $D_N^{(\leq 2^{-r})}$ for "small" intervals to obtain an exact LIL is due to Fukuyama [10]. This method is also used in [1], [2].

**Lemma 6** *Let $(n_k)_{k \geq 1}$ be a sequence of positive integers satisfying conditions $(\mathbf{K}_\alpha)$, $(\mathbf{D}_\delta)$ and $(\mathbf{D}_\gamma^\mathbf{0})$, where $\gamma < 1$ and $\alpha + \delta < 1$. Then*

$$\limsup_{N \to \infty} \frac{N D_N^{(\leq 2^{-r})}(n_k x)}{\sqrt{2N \log \log N}} \leq \frac{K}{r} \qquad \text{a.e.},$$

*where $K$ is a positive number that may only depend on $\alpha, \delta, \gamma, C_\alpha, C_\delta, C_\gamma$.*

**Lemma 7** *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying conditions $(\mathbf{K}_\alpha)$, $(\mathbf{D}_\delta)$ and $(\mathbf{D}^{\mathbf{0}}_\gamma)$, where $\gamma < 1$ and $\alpha + \delta < 1$. Then*

$$\limsup_{N\to\infty} \frac{N D_N^{(\geq 2^{-r})}(n_k x)}{\sqrt{2N \log\log N}} = \limsup_{N\to\infty} \frac{N D_N^{*\,(\geq 2^{-r})}(n_k x)}{\sqrt{2N \log\log N}} = \frac{1}{2} \qquad \text{a.e.}$$

Lemma 7 is a direct consequence of Corollary 2, which implies that under the assumptions of Lemma 7

$$\frac{\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)}{\sqrt{2N \log\log N}} = \left\| \mathbf{I}_{[a,b)} \right\|_2 \qquad \text{a.e.,}$$

and the fact that for $r \geq 1$

$$\max_{0\leq a_1 < a_2 \leq 2^r} \left\| \mathbf{I}_{[a_1 2^{-r}, a_2 2^{-r})} \right\|_2 = \frac{1}{2}, \qquad \max_{0< a_1 < 2^r} \left\| \mathbf{I}_{[0, a_1 2^{-r})} \right\|_2 = \frac{1}{2}.$$

It is easy to see that Theorem 2 follows from Lemma 6, Lemma 7, (35) and (36).

It remains to prove Lemma 6. The proof of this lemma is similar to the proof of Theorem 1. The main difference is that we now have to consider a class of functions instead of one single function. The notation in this section will be the same as in Section 3.

Throughout this section we will assume that $r \geq 1$ is fixed. We define $\eta$ and $\eta'$ like in Section 3, and again we divide the set of positive integers into consecutive blocks

$$\Delta_1, \Delta_1', \Delta_2, \Delta_2', \dots, \Delta_i, \Delta_i', \dots$$

of lengths $\lfloor i^\eta \rfloor$ and $\lceil i^{\eta'} \rceil$, respectively.

Assume that $M \geq 1$ is given. We put

$$H = \lfloor ((\eta + 1)/2) \log_2 M \rfloor \tag{37}$$

and define a class of functions

$$F_M = \left\{ \mathbf{I}_{[m 2^{-h}, (m+1)2^{-h})}(x), 1 \leq h \leq H, 0 \leq m < 2^h \right\}$$

For all functions $f \in F_M$ we have

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\, dx = 0, \quad \mathrm{Var}_{[0,1]} f \leq 2, \quad \|f\|_\infty \leq 1.$$

For every $f \in F_M$ and every $k, 1 \leq k \leq M^+$ we write

$$p_k(f,x) = \sum_{j=1}^{k^2} a_j(f) \cos 2\pi j x$$

for the $k^2$-th partial sum of the Fourier series of the even part

$$\sum_{j=1}^{\infty} a_j(f) \cos 2\pi j x$$

of $f$ (without loss of generality we consider only the even parts; the proof in the general case is exactly the same), and approximate $p_k(f, n_k x)$ by discrete functions $\varphi_k(f, x)$ having properties (P1)-(P3). Note, that all the functions $Y_i = Y_i(f, x),\ 1 \leq i \leq M$ are $\mathcal{F}_i$-measurable, if the $\sigma$-fields $(\mathcal{F}_i)_{i \geq 0}$ are defined like in Section 3. Similar to (30) we get

$$\begin{aligned}
& \left\| \max_{f \in F_M} \left| \sum_{k \in \Delta_i} \sum_{j=1}^{k^2} a_j(f) \cos 2\pi j n_k x \right| \right\|_4 \\
\leq \quad & \sum_{j=1}^{(i^+)^2} \frac{1}{j} \left\| \sum_{k \in \Delta_i} \cos 2\pi j n_k x \right\|_4 \\
\leq \quad & c (\log i) \left( i^{2\eta + (\eta+1)\delta} \right)^{1/4}.
\end{aligned}$$

(Remark: In this section, the numbers $c$ and $\varepsilon$ may depend on $C_\alpha, \alpha, C_\delta, \delta, \eta, \eta'$, and additionally on $C_\gamma$ and $\gamma$). We put

$$A_i = \left\{ \max_{f \in F_M} |Y_i(f)| > \frac{\sqrt{M^{\eta+1}}}{(\log_2 M)^4} \right\}, \quad 1 \leq i \leq M$$

($\log_2 i$ is meant as the maximum of $\log_2 i$ and 1) and get

$$\mathbb{P}(A_i) \leq c i^{2\eta + (\eta+1)\delta} (\log_2 i)^{20} M^{-2\eta - 2} \leq c M^{-1-\varepsilon}, \qquad 1 \leq i \leq M,$$

and

$$\sum_{i=1}^{M} \mathbb{P}(A_i) \leq c M^{-\varepsilon}.$$

We define $Z_0 = 0$ and $Z_1 = Y_1$, which yields $\mathbb{E} Z_1 = 0$. Each set $A_i, i \geq 1$ can be written as a union of intervals of the form

$$\left[ v 2^{-m(i^+)}, (v+1) 2^{-m(i^+)} \right), 0 \leq v < 2^{m(i^+)}. \tag{38}$$

For $1 \leq i \leq M$ we define

$$Z_i = Y_i \cdot \mathbb{1}_{Y_i \leq \frac{\sqrt{M^{\eta+1}}}{(\log_2 M)^4}}.$$

Then the $Z_i$'s are also $\mathcal{F}_i$-measurable, $1 \leq i \leq M$. We put

$$X_n(f) = \begin{cases} 0 & \text{if n=0} \\ \sum_{i=1}^n Z_i(f), & \text{if } 1 \leq n \leq M \\ \sum_{i=1}^M Z_i(f), & \text{if } n > M \end{cases}$$

Then $(X_n, \mathcal{F}_n, n \geq 1)$ is a supermartingale such that

$$X_n - X_{n-1} = Z_n \leq \frac{\sqrt{M^{\eta+1}}}{(\log_2 M)^4}, \quad 1 \leq n \leq M,$$

and trivially $X_n - X_{n-1} = 0$ for $n > M$. That means the system $(X_n, \mathcal{F}_n, n \geq 1)$ satisfies the conditions of Theorem 4.

All functions $f \in F_M$ are indicator functions of certain intervals, centered at expectation. Let $|I(f)|$ denote the length of the interval $f$ corresponds to. For all $f \in F_M$ we put $L(f) = -\log_2 |I(f)|$. Note that $L(f)$ always is a positive integer, and by (37) $L(f) \leq H \leq (\eta + 1)/2\ (\log_2 M)$. Now we want to calculate

$$\mathbb{P}\left( \bigcup_{f \in F_M} \left\{ \sum_{i=1}^{M} \mathbb{E}(Z_i(f)^2 | \mathcal{F}_{i-1}) > L(f)^{-5} M^{\eta+1} \right\} \right).$$

We define $T_i = T_i(f,x), V_M(f,x)$ and $s_M(f)$ like in Section 3, and, using the abbreviations

$$\overline{\cos}^+(x) = \cos 2\pi \left(j_1 n_{k_1} + j_2 n_{k_2}\right) x, \quad \overline{\cos}^-(x) = \cos 2\pi \left(j_1 n_{k_1} - j_2 n_{k_2}\right) x,$$

we get

$$\begin{aligned}
& \left\| \max_{f \in F_M} \left| \left( \sum_{i=1}^{M} \mathbb{E}\left(T_i^2 | \mathcal{F}_{i-1}\right) \right) - s_M \right| \right\|_2 \\
= & \left\| \max_{f \in F_M} \left| \left( \sum_{i=1}^{M} \mathbb{E}\left( \left( \sum_{k \in \Delta_i} \sum_{j=1}^{k^2} a_j(f) \cos 2\pi j n_k x \right)^2 \middle| \mathcal{F}_{i-1} \right) \right) - s_M \right| \right\|_2 \\
\leq & \ 2 \left\| \max_{f \in F_M} \left| \sum_{i=1}^{M} \mathbb{E}\left( \left( \underbrace{\sum_{k_1,k_2 \in \Delta_i} \sum_{j_1=1}^{k_1^2} \sum_{j_2=1}^{k_2^2}}_{j_1 n_{k_1} \neq j_2 n_{k_2}} \frac{a_{j_1}(f) a_{j_2}(f)}{2}\ \overline{\cos}^-(x) \right)^2 \middle| \mathcal{F}_{i-1} \right) \right| \right\|_2 \\
& +2 \left\| \max_{f \in F_M} \left| \sum_{i=1}^{M} \mathbb{E}\left( \left( \sum_{k_1,k_2 \in \Delta_i} \sum_{j_1=1}^{k_1^2} \sum_{j_2=1}^{k_2^2} \frac{a_{j_1}(f) a_{j_2}(f)}{2}\ \overline{\cos}^+(x) \right)^2 \middle| \mathcal{F}_{i-1} \right) \right| \right\|_2 \\
\leq & \sum_{j_1,j_2=1}^{(M^+)^2} \frac{1}{j_1 j_2} \left\| \sum_{i=1}^{M} \mathbb{E}\left( \left( \sum_{\substack{k_1, k_2 \in \Delta_i,\ j_1 k_1 \neq j_2 k_2 \\ k_1 \geq \sqrt{j_1}, k_2 \geq \sqrt{j_2}}} \overline{\cos}^-(x) \right)^2 \middle| \mathcal{F}_{i-1} \right) \right\|_2 \\
& + \sum_{j_1,j_2=1}^{(M^+)^2} \frac{1}{j_1 j_2} \left\| \sum_{i=1}^{M} \mathbb{E}\left( \left( \sum_{\substack{k_1, k_2 \in \Delta_i \\ k_1 \geq \sqrt{j_1}, k_2 \geq \sqrt{j_2}}} \overline{\cos}^+(x) \right)^2 \middle| \mathcal{F}_{i-1} \right) \right\|_2 .
\end{aligned}$$

Splitting into sums of trigonometric functions with "small" and "large" frequencies, respectively, and using the same methods as in Section 3 it is no problem to show

$$\left\| \max_{f \in F_M} \left| \left( \sum_{i=1}^{M} \mathbb{E}\left(T_i^2 | \mathcal{F}_{i-1}\right) \right) - s_M \right| \right\|_2 \leq c M^{\eta+1-\varepsilon},$$

which in view of (19), (26) and (34) yields

$$\left\| \max_{f \in F_M} \left| V_M - \|f\|_2^2 M^+ \right| \right\|_2 \leq c M^{\eta+1-\varepsilon}.$$

This implies

$$\mathbb{P}\left\{\max_{f\in F_M}\left|\sum_{i=1}^{M}\mathbb{E}(Y_i(f)^2|\mathcal{F}_{i-1})-\|f\|_2^2M^+\right|>L(f)^{-5}M^{\eta+1}\right\}\le cM^{-\varepsilon},$$

and, since $Z_i^2\le Y_i^2$ and $\|f\|_2^2\le cL(f)^{-5}$,

$$\mathbb{P}\left(\bigcup_{f\in F_M}\left\{\left|\sum_{i=1}^{M}\mathbb{E}\left(Z_i(f)^2|\mathcal{F}_{i-1}\right)\right|>cL(f)^{-5}M^{\eta+1}\right\}\right)\le cM^{-\varepsilon}. \tag{39}$$

Now we use Theorem 4. The supermartingale $(X_n(f),\mathcal{F}_n,n\ge 1)$ satisfies all conditions of the theorem. We know that

$$X_n-X_{n-1}\le\frac{\sqrt{M^{\eta+1}}}{(\log_2 M)^4},\quad n\ge 1,$$

so we put

$$\lambda=\frac{L(f)^3\sqrt{\log\log M}}{\sqrt{M^{\eta+1}}},$$

and get $\lambda\frac{\sqrt{M^{\eta+1}}}{(\log_2 M)^4}<1$ (without loss of generality we assume that $M$ is large enough). Thus

$$\lambda^2\ge\frac{1}{2}\lambda^2\left(1+\frac{1}{2}\right)\ge\frac{1}{2}\lambda^2\left(1+\frac{1}{2}\lambda\frac{\sqrt{M^{\eta+1}}}{(\log_2 M)^4}\right),$$

and

$$\begin{aligned}
&\mathbb{P}\left\{\max_{1\le n\le M}X_n(f)>2L(f)^{-2}\sqrt{M^{\eta+1}\log\log M}+\lambda\sum_{i=1}^{M}\mathbb{E}\left(Z_i^2|\mathcal{F}_{i-1}\right)\right\}\\
\le\;&\mathbb{P}\left\{\max_{1\le n\le M}\left(\lambda X_n(f)-\lambda^2\sum_{i=1}^{n}\mathbb{E}\left(Z_i^2|\mathcal{F}_{i-1}\right)\right)>2L(f)\log\log M\right\}\\
\le\;&\mathbb{P}\left\{\sup_{n\ge 0}\ \exp\left(\lambda X_n(f)-\frac{1}{2}\lambda^2\left(1+\frac{1}{2}\lambda\frac{\sqrt{M^{\eta+1}}}{(\log_2 M)^4}\right)\sum_{i=1}^{n}\mathbb{E}\left(Z_i^2|\mathcal{F}_{i-1}\right)\right)>(\log M)^{2L(f)}\right\}\\
\le\;&\frac{1}{(\log M)^{2L(f)}},
\end{aligned}$$

where the last inequality follows from Theorem 4.

In $F_M$ there are two functions that correspond to intervals of length $1/2$; for these functions $L(f)=1$. There are four functions that correspond to intervals of length $1/4$; for these functions $L(f)=2$. There are 8 functions with $L(f)=3$, etc., and $2^H$ functions with

$L(f)=H$. Thus by (39)

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{f\in F_M}\left\{\max_{1\le n\le M} X_n(f) > cL(f)^{-2}\sqrt{M^{\eta+1}\log\log M}\right\}\right)\\
\le\ & \sum_{f\in F_M}\mathbb{P}\left\{\max_{1\le n\le M} X_n(f) > 2L(f)^{-2}\sqrt{M^{\eta+1}\log\log M}+\lambda\sum_{i=1}^{M}\mathbb{E}\left(Z_i(f)^2|\mathcal{F}_{i-1}\right)\right\}\\
&+\mathbb{P}\left(\bigcup_{f\in F_M}\left\{\sum_{i=1}^{M}\mathbb{E}\left(Z_i(f)^2|\mathcal{F}_{i-1}\right) > cL(f)^{-5}M^{\eta+1}\right\}\right)\\
\le\ & \left(\sum_{h=1}^{H}\frac{2^h}{(\log M)^{2h}}\right)+cM^{-\varepsilon}\ \le\ c\frac{1}{(\log M)^2}.
\end{aligned}$$

We know that $X_n(f,x)=\sum_{i=1}^{n}Z_i(f,x)=\sum_{i=1}^{n}Y_i(f,x)$ for all $x$ and all functions $f\in F_M$, except for those $x$ in $\bigcup_{i=1}^{M}A_i$. Now

$$\|\varphi_k-p_k\|_\infty\le ck^{-2}$$

implies

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{f\in F_M}\left\{\max_{1\le n\le M}\sum_{k\in\Delta_n}p_k(f,n_kx) > cL(f)^{-2}\sqrt{M^{\eta+1}\log\log M}\right\}\right)\\
\le\ & \mathbb{P}\left(\bigcup_{f\in F_M}\left\{\max_{1\le n\le M} X_n(f) > cL(f)^{-2}\sqrt{M^{\eta+1}\log\log M}\right\}\right)+\sum_{i=1}^{M}\mathbb{P}(A_i)\\
\le\ & c\frac{1}{(\log M)^2}
\end{aligned}$$

In a similar way we get a corresponding result for $\sum -p_k(f,n_kx)$, so overall we have a result for $|\sum p_k(f,n_kx)|$:

$$\mathbb{P}\left(\bigcup_{f\in F_M}\left\{\max_{1\le n\le M}\left|\sum_{k\in\Delta_i}p_k(f,n_kx)\right| > cL(f)^{-2}\sqrt{M^{\eta+1}\log\log M}\right\}\right)\le c\frac{1}{(\log M)^2}$$

Now, like in Section 3, we break into the blocks $\Delta_i$, add the remainder terms $r_k=f-p_k$, and add the remaining values $n_k, k\in\Delta_i', 1\le i\le M$. Similar to (31), the Carleson-Hunt theorem yields

$$\begin{aligned}
&\left\| \max_{f \in F_M} \max_{N \geq 1} \left| \sum_{k \in \Delta_i \cup \Delta_i', k \leq N} \ \sum_{j=1}^{k^2} a_j(f) \cos 2\pi j n_k x \right| \right\|_4 \\
\leq \ & \sum_{j=1}^{(i^+)^2} \frac{1}{j} \left\| \max_{N \geq 1} \left| \sum_{k \in \Delta_i \cup \Delta_i', k \leq N} \cos 2\pi j n_k x \right| \right\|_4 \\
\leq \ & c (\log i) \left\| \sum_{k \in \Delta_i} \cos 2\pi n_k x \right\|_4 \\
\leq \ & c (\log i) \left( i^{2\eta + (\eta+1)\delta} \right)^{1/4},
\end{aligned}$$

and so

$$\begin{aligned}
&\mathbb{P} \left( \bigcup_{f \in F_M} \left\{ \max_{N \geq 1} \left| \sum_{k \in \Delta_i \cup \Delta_i', k \leq N} p_k(f, n_k x) \right| > L(f)^{-2} \sqrt{M^{\eta+1} \log \log M} \right\} \right) \\
\leq \ & c \frac{(\log M)^8 i^{2\eta+1-\varepsilon}}{M^{2\eta+2}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{P} \left( \bigcup_{f \in F_M} \left\{ \max_{1 \leq N \leq M^+} \left| \sum_{k=1}^{N} p_k(f, n_k x) \right| > c L(f)^{-2} \sqrt{M^{\eta+1} \log \log M} \right\} \right) \qquad (40) \\
\leq \ & \mathbb{P} \left( \bigcup_{f \in F_M} \left\{ \max_{1 \leq n \leq M} \left| \sum_{i=1}^{n} \sum_{k \in \Delta_i} p_k(f, n_k x) \right| > c L(f)^{-2} \sqrt{M^{\eta+1} \log \log M} \right\} \right) \\
&+ \sum_{i=1}^{M} \mathbb{P} \left( \bigcup_{f \in F_M} \left\{ \max_{N \geq 1} \left| \sum_{k \in \Delta_i \cup \Delta_i', k \leq N} p_k(f, n_k x) \right| > L(f)^{-2} \sqrt{M^{\eta+1} \log \log M} \right\} \right) \\
\leq \ & c \frac{1}{(\log M)^2} + \sum_{i=1}^{M} c \frac{(\log M)^8 i^{2\eta+1-\varepsilon}}{M^{2\eta+2}} \ \leq \ c \frac{1}{(\log M)^2}.
\end{aligned}$$

Now we need an estimate for the remainder terms $r_k(f, n_k x) = f(n_k x) - p_k(f, n_k x)$. We have

$$\left\| \max_{N \geq 1} \left| \sum_{k \in \Delta_i \cup \Delta_i', k \leq N} r_k(f, n_k x) \right| \right\|_2 \leq c \sum_{k \in \Delta_i \cup \Delta_i'} \|r_k\|_2 \leq c i^{-1},$$

and so

$$\left\| \max_{1 \leq N \leq M^+} \left| \sum_{k=1}^{N} r_k(f, n_k x) \right| \right\|_2 \leq \sum_{i=1}^{M^+} c i^{-1} \leq c \log M$$

and

$$\mathbb{P} \left( \bigcup_{f \in F_M} \left\{ \max_{1 \leq N \leq M^+} \left| \sum_{k=1}^{N} r_k(f, n_k x) \right| > L(f)^{-2} \sqrt{M^{\eta+1}} \right\} \right) \leq \sum_{f \in F_M} \frac{c (\log M)^6}{M^{\eta+1}} \leq c \frac{(\log M)^6}{\sqrt{M^{\eta+1}}}.$$

Combining this with (40) we get

$$\mathbb{P}\left(\bigcup_{f\in F_M}\left\{\max_{1\le N\le M^+}\left|\sum_{k=1}^N f(n_kx)\right| > cL(f)^{-2}\sqrt{M^{\eta+1}\log\log M}\right\}\right)\le c\frac{1}{(\log M)^2}. \tag{41}$$

Now we apply an argument similar to the one in [17, Section 3]. First we observe, that for any sequence $(x_k)_{k\ge1}$ of reals

$$D_N^{(\le 2^{-r})}(x_k)\le 2\max_{0\le m<2^r}\sup_{a<2^{-r}}\left|\sum_{k=1}^N \mathbf{I}_{[m2^{-r},m2^{-r}+a)}(x_k)\right|$$

Since there are only finitely many possible values of $m$, it suffices to prove

$$\limsup_{N\to\infty}\sup_{a<2^{-r}}\left|\frac{\sum_{k=1}^N \mathbf{I}_{[m2^{-r},m2^{-r}+a)}(n_kx)}{\sqrt{2N\log\log N}}\right|\le\frac{K}{r}\quad\text{a.e.}$$

for all possible values of $m$. We give a detailed proof for $m=0$, all other cases can be treated similarly. Let $a>0$ be given, and write $(.a_1a_2a_3\dots)_2$ for the dyadic expansion of $a$. We put $a^{(0)}=0$ and, for $n\ge1$, $a^{(n)}=(.a_1a_2\dots a_n000\dots)_2$, i.e. we cut off the dyadic expansion of $a$ after the first $n$ digits. Since $a<2^{-r}$, the first $r$ digits of the binary representation of $a$ are zero (We assume without loss of generality that $M$ is large enough such that $H>r$). Thus

$$\mathbf{I}_{[0,a)}(x)=\sum_{h=r}^{\infty}\mathbf{I}_{[a^{(r)},a^{(r+1)})}(x),\quad x\in(0,1),$$

and

$$\begin{aligned}
&\left(\sum_{h=r}^{H-1}\mathbf{I}_{[a^{(h)},a^{(h+1)})}(x)\right)-\left(a-a^{(H)}\right)\\
\le\ &\mathbf{I}_{[0,a)}(x)\\
\le\ &\left(\sum_{h=r}^{H-1}\mathbf{I}_{[a^{(h)},a^{(h+1)})}(x)\right)+\mathbf{I}_{[a^{(H)},a^{(H)}+2^{-H})}(x)+\left(a^{(H)}+2^{(-H)}-a\right).
\end{aligned}$$

We note that by (37)

$$a-a^{(H)}\le 2^{-H}\le c\frac{1}{\sqrt{M^+}},\qquad \left(a^{(H)}+2^{(-H)}-a\right)\le c\frac{1}{\sqrt{M^+}}.$$

All the indicators

$$\mathbf{I}_{[a^{(h)},a^{(h+1)})},\ r\le h\le H-1,\quad\text{and}\quad \mathbf{I}_{[a^{(H-1)},a^{(H-1)}+2^{-H})}$$

are contained in $F_M$. Thus, for those $x\in(0,1)$, which are not contained in the set

$$\bigcup_{f\in F_M}\left\{\max_{1\le N\le M^+}\left|\sum_{k=1}^N f(n_kx)\right| > cL(f)^{-2}\sqrt{M^{\eta+1}\log\log M}\right\}, \tag{42}$$

we have

$$\begin{aligned}
& \max_{1\le N\le M^+} \left| \sum_{k=1}^N \mathbf{I}_{[0,a)}(n_k x) \right| \\
\le\ & \max_{1\le N\le M^+} \sum_{r=0}^{H-1} \left| \sum_{k=1}^N \mathbf{I}_{[a^{(h)},a(h+1))}(n_k x) \right| \\
& + \max_{1\le N\le M^+} \left| \sum_{k=1}^N \mathbf{I}_{[a^{(H)},a^{(H)}+2^{-H})}(n_k x) \right| \\
& + \sum_{k=1}^{M^+} c \frac{1}{\sqrt{M^+}} \\
\le\ & c\sqrt{M^{\eta+1}\log\log M} \left( \sum_{h=r}^{H-1} L\left(\mathbf{I}_{[a^{(h)},a^{(h+1)})}\right)^{-2} \right) \\
& + c\sqrt{M^{\eta+1}\log\log M}\ L\left(\mathbf{I}_{[a^{(H)},a^{(H)}+2^{-H})}\right)^{-2} \\
& + c\sqrt{M^+} \\
\le\ & c\sqrt{M^{\eta+1}\log\log M} \left( \left( \sum_{h=r}^{\infty} h^{-2} \right) + cH^{-2} + c\left(\log\log M^+\right)^{-1/2} \right) \\
\le\ & cr^{-1}\sqrt{M^{\eta+1}\log\log M}.
\end{aligned}$$

Note that this holds for any $a \in (0, 2^{-r})$, and all $x \in (0,1)$ which are not contained in the sets in (42).

We write $M^{(1)}, M^{(2)}, \ldots$ for the values of $M^+$ for $M = 2^1, 2^2, \ldots$. Since $M^{(i+1)}/M^{(i)} \le c$, for all $N$ between $M^{(i)}$ and $M^{(i+1)}$

$$\begin{aligned}
\sup_{a<2^{-r}} \left| \sum_{k=1}^N \mathbf{I}_{[0,a)}(n_k x) \right| &\le \frac{c}{r}\sqrt{M^{(i+1)}\log\log M^{(i+1)}} \\
&\le \frac{c}{r}\sqrt{M^{(i)}\log\log M^{(i)}} \\
&\le \frac{c}{r}\sqrt{N\log\log N}
\end{aligned}$$

for all $x \in (0,1)$, except for those $x$ which are contained in the set in (42). By (41) the sum of the measures of the exceptional sets converges, since

$$\sum_{i=1}^{\infty} c\frac{1}{(\log 2^i)^2} \le c\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

Therefore the Borel-Cantelli lemma implies

$$\limsup_{N\to\infty} \sup_{a<2^{-r}} \left| \frac{\sum_{k=1}^N \mathbf{I}_{[0,a)}(n_k x)}{\sqrt{2N\log\log N}} \right| \le \frac{c}{r} \quad \text{a.e.}$$

Repeating the same argument for the other possible values of $m$ proves Lemma 6.

## References


[1] C. Aistleitner. Irregular discrepancy behavior of lacunary series. *Monatshefte Math.*, to appear.

[2] C. Aistleitner. On the law of the iterated logarithm for the discrepancy of lacunary sequences. *Trans. Amer. Math. Soc.*, to appear.

[3] C. Aistleitner and I. Berkes. On the central limit theorem for $f(n_k x)$. *Probab. Theory Related Fields*, to appear.

[4] C. Aistleitner and I. Berkes. On the law of the iterated logarithm for the discrepancy of $\langle n_k x \rangle$. *Monatshefte Math.*, to appear.

[5] J. Arias de Reyna. *Pointwise convergence of Fourier series.* Springer-Verlag, Berlin, 2002.

[6] I. Berkes. On the central limit theorem for lacunary trigonometric series. *Anal. Math.* **4**:159-180, 1978.

[7] I. Berkes and W. Philipp. The size of trigonometric and Walsh series and uniform distribution mod 1. *J. London Math. Soc. (2)* **50**:454-464, 1994.

[8] I. Berkes, W. Philipp and R. Tichy. Empirical processes in probabilistic number theory: the LIL for the discrepancy of $(n_k \omega)$ mod 1. *Illinois J. Math.* **50**:107-145, 2006

[9] I.S. Gál. A theorem concerning Diophantine approximations. *Nieuw. Arch. Wiskunde (2)* **23**:13-38, 1949.

[10] K. Fukuyama. The law of the iterated logarithm for discrepancies of $\{\theta^n x\}$. *Acta Math. Hung.* **118**:155-170, 2008.

[11] K. Fukuyama and K. Nakata. A metric discrepancy result for the Hardy-Littlewood-Pólya sequences. *Monatshefte Math.*, to appear.

[12] M. Kac, On the distribution of values of sums of the type $\sum f(2^k t)$. *Ann. of Math.* **47**, 33–49, 1946.

[13] J.F. Koksma. On a certain integral in the theory of uniform distribution. *Nederl. Akad. Wetensch., Proc. Ser. A.* **54** = *Indagationes Math.* **13**:285-287, 1951.

[14] A.N. Kolmogorov, Über das Gesetz des iterierten Logarithmus. *Math. Ann.* **101**:126-135, 1929.

[15] C.J. Mozzochi. *On the pointwise convergence of Fourier series.* Springer-Verlag, Berlin-New York 1971.

[16] W. Philipp. Empirical distribution functions and strong approximation theorems for dependent random variables. A problem of Baker in probabilistic number theory. *Trans. Amer. Math. Soc.* **345**:705-727, 1994.

[17] W. Philipp. Limit theorems for lacunary series and uniform distribution mod 1. *Acta Arith.* **26**:241-251, 1975.

[18] R. Shorack and J. Wellner, *Empirical Processes with Applications to Statistics*, Wiley, New York, 1986.

[19] W.F. Stout. *Almost sure convergence.* Academic Press, New York - London, 1974.

[20] V. Strassen. Almost sure behavior of sums of independent random variables and martingales. *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66) Vol. II: Contributions to Probability Theory*:315-343, 1967.

[21] A. Zygmund. *Trigonometric series. Vol. I, II.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2002.