Irregular discrepancy behavior of lacunary series II

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Abstract

In 1975 Philipp proved the following law of the iterated logarithm (LIL) for the discrepancy of lacunary series: Let $(n_k)_{k\geq 1}$ be a lacunary sequence of positive integers, i.e. a sequence satisfying the Hadamard gap condition $n_{k+1}/n_k > q > 1$. Then $1/(4\sqrt{2}) \leq \limsup_{N\to\infty} ND_N(n_k x)(2N \log\log N)^{-1/2} \leq C_q$ for almost all $x \in (0,1)$ in the sense of Lebesgue measure. The same result holds, if the "extremal discrepancy" D_N is replaced by the "star discrepancy" D_N^*. It has been a long standing open problem whether the value of the limsup in the LIL has to be a constant almost everywhere or not. In a preceding paper we constructed a lacunary sequence of integers, for which the value of the limsup in the LIL for the star discrepancy is not a constant a.e. Now, using a refined version of our methods from this preceding paper, we finally construct a sequence for which also the value of the limsup in the LIL for the extremal discrepancy is not a constant a.e.

1 Introduction

In 1975 Philipp [7] solved the Erdős-Gál conjecture and proved a law of the iterated logarithm for the discrepancy of $(n_k x)_{k\geq 1}$, where $(n_k)_{k\geq 1}$ is a sequence of positive integers satisfying the Hadamard gap condition

$$\frac{n_{k+1}}{n_k} > q > 1, \quad k \geq 1,$$

and $x \in (0,1)$. In fact, he showed that

$$\frac{1}{4\sqrt{2}} \leq \limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N \log\log N}} \leq C_q \quad \text{a.e.,} \tag{1}$$

where C_q is a constant depending on q (this classical result is discussed e.g. in [4],[6]). Exactly the same conclusion follows if the (extremal) discrepancy D_N is replaced by the "star discrepancy" D_N^*. The exact value of the $\limsup$ in (1) is very difficult to calculate, since it depends on number-theoretic properties of $(n_k)_{k\geq 1}$ in an extremely involved way. Fukuyama [5] calculated the exact value of the $\limsup$ for sequences of the type $\left(\theta^k\right)_{k\geq 1}$ ($\theta > 1$, not necessarily an integer), and in all cases the $\limsup$ is a constant a.e. Of particular interest is

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the case $\theta^r \notin \mathbb{Q},\ r \geq 1$, where the lim sup equals 1/2 a.e. This is the same constant as in the Chung-Smirnov-LIL for i.i.d. random variables (see e.g. [8], p. 504).

In [2] we showed that for a large class of lacunary sequences, characterized in terms of the number of solutions of Diophantine equations of the type

$$j_1 n_{k_1} \pm j_2 n_{k_2} = \nu, \quad j_1, j_2, \nu \in \mathbb{Z},\ j_1, j_2 \geq 1,\ \nu \geq 0 \tag{2}$$

the value of the lim sup in the LIL for the discrepancy also equals 1/2 a.e. The same phenomenon, namely a relation between the "regularity" of the probabilistic behavior and the number of solutions of certain Diophantine equations, appears also in the theory of the central limit theorem for $f(n_k x)$ (see Aistleitner and Berkes [3]).

It is a long standing open problem, whether the lim sup in (1) has to be a constant a.e. or not for lacunary $(n_k)_{k \geq 1}$. In [1] we showed that this is not the case if the extremal discrepancy D_N is replaced by the star discrepancy D_N^*: for the sequence

$$n_{2k-1} = 2^{k^2}, \quad n_{2k} = 2^{k^2+1} - 1 \quad k \geq 1 \tag{3}$$

we have

$$\limsup_{N \to \infty} \frac{N D_N^*(n_k x)}{\sqrt{2N \log\log N}} \quad = \quad \Psi^*(x) \quad \text{a.e.},$$

where

$$\Psi^*(x) = \begin{cases} \dfrac{3}{4\sqrt{2}}, & 0 \leq x \leq 3/8,\ 5/8 \leq x \leq 1 \\ \dfrac{\sqrt{(4x(1-x)-x)}}{\sqrt{2}}, & 3/8 \leq x \leq 1/2 \\ \dfrac{\sqrt{(4x(1-x)-(1-x))}}{\sqrt{2}}, & 1/2 \leq x \leq 5/8. \end{cases}$$

Regrettably, for this sequence

$$\limsup_{N \to \infty} \frac{N D_N(n_k x)}{\sqrt{2N \log\log N}} \quad = \quad \frac{3}{4\sqrt{2}} \quad \text{a.e.},$$

so the problem remained partially unsolved. It turned out, that the structure of the sequence defined in (3) is "too simple": the only Diophantine equations of type (2), that have "many" solutions, are the equations

$$\begin{aligned} 2n_{k_1} - n_{k_2} &= 1, \\ 4n_{k_1} - 2n_{k_2} &= 2, \\ 6n_{k_1} - 3n_{k_2} &= 3, \\ &\vdots \end{aligned}$$

In this paper we slightly modify our construction from (3), and obtain a sequence, for which also the lim sup in the LIL for the extremal discrepancy D_N is not a constant a.e.

Theorem 1 *Let* $(n_k)_{k\geq 1}$ *be defined by*

$$n_k = \begin{cases} 2^{k^2} & \text{for} \quad k \equiv 1 \mod 4 \\ 2^{(k-1)^2+1} - 1 & \text{for} \quad k \equiv 2 \mod 4 \\ 2^{k^2+k} & \text{for} \quad k \equiv 3 \mod 4 \\ 2^{(k-1)^2+(k-1)+1} - 2 & \text{for} \quad k \equiv 0 \mod 4 \end{cases} \tag{4}$$

Then

$$\limsup_{N\to\infty} \frac{N D_N(n_k x)}{\sqrt{2N \log\log N}} = \Psi(x),$$

where

$$\Psi(x) = \begin{cases} \dfrac{3}{4\sqrt{2}} & \text{for} \quad 0 \leq x \leq 3/8 \\ \sqrt{2(1-x)x - x/2} & \text{for} \quad 3/8 \leq x \leq 7/16 \\ \sqrt{\dfrac{49}{128} - \dfrac{x}{4}} & \text{for} \quad 7/16 \leq x \leq 1/2 \\ \Psi(1-x) & \text{for} \quad 1/2 < x \leq 1. \end{cases}$$

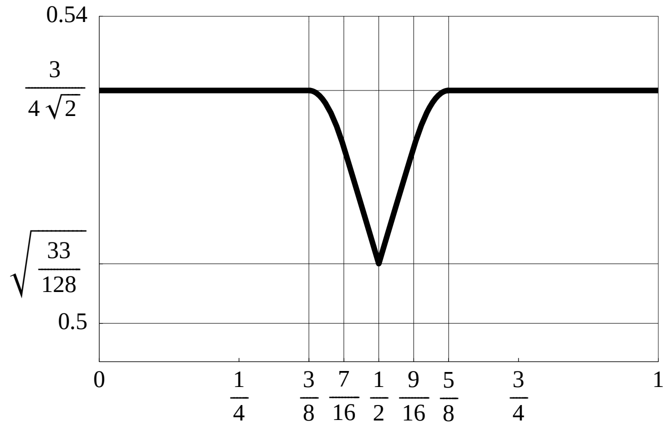

Figure 1: $\Psi(x)$

Many interesting problems remain unsolved. We mention the following:

Open problem: What are the largest numbers $\bar{C} > 0, \bar{C}^* > 0$, such that for any lacunary sequence $(n_k)_{k\geq 1}$ of positive integers

$$\begin{aligned} \limsup_{N\to\infty} \frac{N D_N(n_k x)}{\sqrt{2N \log\log N}} &\geq \bar{C} \quad \text{a.e.} \\ \limsup_{N\to\infty} \frac{N D_N^*(n_k x)}{\sqrt{2N \log\log N}} &\geq \bar{C}^* \quad \text{a.e.} \end{aligned}$$

Philipp's result gives $\bar{C} \geq 1/(4\sqrt{2})$, and trivially $\bar{C} \geq \bar{C}^*$. Interestingly, there is some reason to believe that actually the sharp inequality $\bar{C} > \bar{C}^*$ holds. In particular, we have been intensely trying to find a sequence $(n_k)_{k\geq 1}$, for which the lim sup in the LIL for the discrepancy D_N is not $\geq 1/2$ a.e., but have not been able to find a single sequence for which this is the case. On the other hand, we have been able to construct some sequences for which the value of the lim sup in the LIL for the star discrepancy D_N^* is strictly smaller than 1/2 on a set of positive measure (no proof given here).

2 Outline

The proof of Theorem 1 is structured in three lemmas, which are presented here. The proofs of the lemmas are given in Sections 3-5.

It is easy to see that the sequence

$$(n_k)_{k\geq 1} = (2, 3, 4096, 8190, 33554432, 67108863, \dots)$$

defined in (4) is a lacunary sequence. We have

$$\begin{aligned} \frac{n_{k+1}}{n_k} &\to 2 \quad \text{for} \quad k \equiv 1,3 \mod 4 \\ \frac{n_{k+1}}{n_k} &\to \infty \quad \text{for} \quad k \equiv 2,0 \mod 4, \end{aligned}$$

and

$$\frac{n_{k+1}}{n_k} \geq \frac{3}{2}, \quad k \geq 1.$$

The Diophantine structure of the sequence is described in the following lemma:

Lemma 1 *For $j_1, j_2, \nu \in \mathbb{Z}$, $j_1, j_2 \geq 1$, $\nu \geq 0$ let*

$$L(j_1, j_2, \nu, N) = \# \{(k_1, k_2),\ k_1, k_2 \leq N : (j_1, k_1) \neq (j_2, k_2),\ j_1 n_{k_1} - j_2 n_{k_2} = \nu\}.$$

Then

$$L(j_1, j_2, \nu, N) = \begin{cases} \frac{N}{4} + \mathcal{O}(1) & \text{as } N \to \infty \text{ if} \qquad j_1 = 2j_2 \quad \text{and} \quad \nu = j_2 \\ \frac{N}{4} + \mathcal{O}(1) & \text{as } N \to \infty \text{ if} \qquad j_1 = 2j_2 \quad \text{and} \quad \nu = 2j_2 \\ \mathcal{O}(1) & \text{as } N \to \infty \text{ otherwise,} \end{cases}$$

where the implied constant may depend on j_1 and j_2, but not on ν.

As usual, for $0 \leq a \leq b \leq 1$ we write $[a, b)$ for the half-open interval $\{x \in [0, 1) :\ a \leq x < b\}$. If $0 \leq b \leq a \leq 1$, then $[a, b)$ shall denote the set $\{x \in [0, 1) : a \leq x\} \cup \{x \in [0, 1) : x \leq b\}$. In this case, we will still consider the set $[a, b)$ as *one* interval.

For $0 \leq a \leq b \leq 1$ we define $\mathbf{I}_{[a,b)}(x)$ as the indicator of the interval $[a, b)$, centered at expectation and extended with period 1, i.e.

$$\mathbf{I}_{[a,b)}(x) = \mathbf{1}_{[a,b)}(\langle x \rangle) - (b - a),$$

where $\mathbf{1}$ is the ordinary indicator function, and $\langle\cdot\rangle$ denotes the fractional part. For convenience we will also allow $0 \le b \le a \le 1$, and define

$$\mathbf{I}_{[a,b)}(x) = \mathbf{1}_{[a,b)}(\langle x\rangle) - (1+b-a) = -\mathbf{I}_{[b,a)}(x).$$

Thus we can write

$$ND_N(n_k x) = \sup_{0\le a<b\le 1}\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|.$$

In the same way as in [1] we can show

$$\limsup_{N\to\infty}\frac{\sup_{0\le a<b\le 1}\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|}{\sqrt{2N\log\log N}} = \sup_{0\le a<b\le 1}\limsup_{N\to\infty}\frac{\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|}{\sqrt{2N\log\log N}} \quad \text{a.e.}$$

Thus we will first calculate the value of

$$\limsup_{N\to\infty}\frac{\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|}{\sqrt{2N\log\log N}}$$

for fixed $0 \le a < b \le 1$. In a second step we will calculate the supremum of all this lim sup's as a, b run over all possible values $0 \le a < b \le 1$.

Lemma 2 *Let a, b be given, such that $0 \le a < b \le 1$. Then*

$$\limsup_{N\to\infty}\frac{\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|}{\sqrt{2N\log\log N}} = \sqrt{\sigma_{[a,b)}(x)} \quad \text{a.e.,}$$

where

$$\begin{aligned}\sigma_{[a,b)}(x) &= (1-(b-a))(b-a)\\ &+\frac{1}{4}\int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(x-t)\ dt\\ &+\frac{1}{4}\int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(\langle 2x\rangle-t)\ dt\end{aligned}$$

Lemma 3

$$\sup_{0\le a<b\le 1}\sigma_{[a,b)}(x) = \begin{cases}\dfrac{9}{32} & \text{for} \quad 0\le x\le 3/8\\ 2(1-x)x-\dfrac{x}{2} & \text{for} \quad 3/8<x\le 7/16\\ \dfrac{49}{128}-\dfrac{x}{4} & \text{for} \quad 7/16<x\le 1/2\\ \sup\limits_{0\le a<b\le 1}\sigma_{[a,b)}(1-x) & \text{for} \quad 1/2<x<1.\end{cases}$$

3 Proof of Lemma 1

This lemma is similar to [1, Lemma 2]. We subdivide the proof into three parts (Lemmas 4-6), which together yield the desired result.

Lemma 4 *For $j_1 \geq 1, j_2 \geq 1$ and all $N \geq 1$*

$$L(j_1, j_2, 0, N) = \mathcal{O}(1) \qquad \text{as} \quad N \to \infty.$$

Proof: W.l.o.g. $j_1 \leq j_2$. For sufficiently large $k_0 = k_0(j_1, j_2)$

$$j_1 n_{k_1} - j_2 n_{k_2} = 0, \qquad k_1, k_2 \geq k_0$$

is only possible if k_1, k_2 are of the form $k+1, k$ for some k which satisfies $k \equiv 1 \mod 2$. Since

$$j_1 n_{k+1} - j_2 n_k = 0 \tag{5}$$

implies

$$\frac{j_1}{j_2} = \frac{n_k}{n_{k+1}},$$

and since n_k and n_{k+1} are coprime (except possibly a common factor 2) there can exist only finitely many solutions of (5). $\square$

Lemma 5 *Assume $\nu \neq 0$, and either*

$$j_1 \neq 2j_2$$

or

$$\nu \notin \{j_2, 2j_2\}.$$

Then

$$L(j_1, j_2, \nu, N) = \mathcal{O}(1) \qquad \text{as} \quad N \to \infty, \tag{6}$$

where the implied constant may only depend on j_1, j_2, but not on ν.

Proof: Let us first assume that $j_1 \neq 2j_2$. Then the sequence $(m_k)_{k\geq 1}$, which consists of all numbers

$$\left(\bigcup_{k\geq 1}\{j_1 n_k\}\right) \cup \left(\bigcup_{k\geq 1}\{j_2 n_k\}\right),$$

sorted in increasing order, is a lacunary sequence, which means there exists some $q > 1$ such that $m_{k+1}/m_k > q,\ k \geq 1$. By a well-known property of lacunary sequences (cf. Zygmund [9, p. 203])

$$\#\{(k_1, k_2):\ m_{k_1} \pm m_{k_2} = \nu\} = \mathcal{O}(1) \qquad \text{as} \quad N \to \infty,$$

where the implied constant does not depend on ν. By the construction of the sequence $(m_k)_{k\geq 1}$ this implies that $(n_k)_{k\geq 1}$ has property (6).

Now let us assume that $j_1 = 2j_2$, and $\nu \notin \{j_2, 2j_2\}$. The sequence $(m_k)_{k\geq 1}$, which consits of the elements

$$\left(\bigcup_{k\geq 1}\{j_1 n_k\}\right) \cup \left(\bigcup_{k\geq 1}\{j_2 n_k\}\right),$$

sorted in increasing order, is of the form

$$(j_2n_1, j_2n_2, j_2n_2+1, 2j_2n_2, j_2n_3, j_2n_4, j_2n_4+2, 2j_2n_4, \dots)$$

E.g., if $j_1 = 2, j_2 = 1$, we have $(m_k)_{k\geq 1} = (2, 3, 4, 6, 4096, 8190, 8192, 16380, \dots)$. The number of solutions (k_1, k_2) of

$$m_{k_1} - m_{k_2} = \nu, \qquad \text{where} \quad (k_1, k_2) \not\in \{(4k+3, 4k+2), k \geq 0\},$$

is bounded by $\mathcal{O}(1)$ uniformly for $\nu \geq 1$, since $(j_2n_k)_{k\geq 1}$ is lacunary. On the other hand, for (k_1, k_2) of the form $(4k+3, 4k+2)$ for some $k \geq 0$,

$$m_{k_1} - m_{k_2}$$

is always either j_2 or $2j_2$ by the construction of $(m_k)_{k\geq 1}$. Thus $m_{k_1} - m_{k_2} = \nu$ has only finitely many solutions, and the same holds for $j_1n_{k_1} - j_2n_{k_2} = \nu$. $\Box$

Lemma 6 *Assume that*

$$j_1 = 2j_2$$

and

$$\nu \in \{j_2, 2j_2\}.$$

Then

$$\left| L(j_1, j_2, \nu, N) - \frac{N}{4} \right| = \mathcal{O}(1).$$

Proof: We show only the case $\nu = j_2$. The case $\nu = 2j_2$ can be treated similarly. Obviously

$$2j_2n_{k_1} - j_2n_{k_2} = j_2$$

implies

$$2n_{k_1} - n_{k_2} = 1. \tag{7}$$

Since $n_{k+2}/n_k \to \infty$ there are only finitely many solutions (k_1, k_2) of (7), for which $k_2 > k_1+1$. For $k_2 = k_1+1$, the number of solutions $\{(k_1, k_2) : \ k_1, k_2 \leq N\}$ of (7) is exactly $\lfloor (N+2)/4 \rfloor$, as can be easily seen. $\Box$

4 Proof of Lemma 2

For fixed a, b, the indicator function $\mathbf{I}_{[a,b)}(x)$ is approximated by a partial sum of its Fourier series. This means, writing

$$\mathbf{I}_{[a,b)}(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi jx + b_j \sin 2\pi jx,$$

where

$$\begin{aligned} a_j = a_j(a,b) &= \frac{\sin 2\pi jb - \sin 2\pi ja}{\pi j}, \qquad j \geq 1, \\ b_j = b_j(a,b) &= \frac{-\cos 2\pi jb + \cos 2\pi ja}{\pi j}, \qquad j \geq 1, \end{aligned}$$

we choose $d \geq 1$ and set

$$\begin{aligned}
p(x) = p_{[a,b),d}(x) &= \sum_{j=1}^{d} a_j \cos 2\pi jx + b_j \sin 2\pi jx, \\
r(x) = r_{[a,b),d}(x) &= \sum_{j=d+1}^{\infty} a_j \cos 2\pi jx + b_j \sin 2\pi jx.
\end{aligned}$$

Then by [2, Lemma 3.1]

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} r(n_k x)}{\sqrt{2N \log\log N}} \leq c\ d^{-1/4} \quad \text{a.e.}$$

(*Remark:* Throughout this section c stands for appropriate positive numbers, not always the same, which must not depend on d). Using methods from [1, Section 4] we can easily show

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} p(n_k x)}{\sqrt{2N \log\log N}} = \sqrt{\sigma_{[a,b),d}(x)} \quad \text{a.e.,}$$

where

$$\begin{aligned}
\sigma_{[a,b),d}(x) &= \frac{1}{2}\sum_{j=1}^{d} \left(a_j^2 + b_j^2\right) \\
&\quad + \frac{1}{4}\sum_{j=1}^{\lfloor d/2 \rfloor} (a_j a_{2j} + b_j b_{2j}) \cos 2\pi jx + (a_j b_{2j} - a_{2j} b_j) \sin 2\pi jx \\
&\quad + \frac{1}{4}\sum_{j=1}^{\lfloor d/2 \rfloor} (a_j a_{2j} + b_j b_{2j}) \cos 4\pi jx + (a_j b_{2j} - a_{2j} b_j) \sin 4\pi jx
\end{aligned} \tag{8}$$

(the proof of this result can be easily modeled after the proof of [1, Corollary 1]. The proof remains literally the same, except that the function $\sigma_{[a,b),d}$ in [1] has to be replaced by the function in (8)).
We observe that for $j \geq 1$

$$\begin{aligned}
a_j(a,b) &= -a_j(1-a, 1-b) && (9) \\
b_j(a,b) &= b_j(1-a, 1-b) && (10) \\
a_{2j}(a,b) &= \frac{a_j(\langle 2a \rangle, \langle 2b \rangle)}{2} \\
b_{2j}(a,b) &= \frac{b_j(\langle 2a \rangle, \langle 2b \rangle)}{2}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{4}\Bigg(\sum_{j=1}^{\lfloor d/2\rfloor}\big(a_j(a,b)a_{2j}(a,b)+b_j(a,b)b_{2j}(a,b)\big)\cos 2\pi jx \\
&\quad +\big(a_j(a,b)b_{2j}(a,b)-a_{2j}(a,b)b_j(a,b)\big)\sin 2\pi jx\Bigg) \\
= \;& -\frac{1}{8}\Bigg(\sum_{j=1}^{\lfloor d/2\rfloor}\big(a_j(1-a,1-b)a_j(\langle 2a\rangle,\langle 2b\rangle)-b_j(1-a,1-b)b_j(\langle 2a\rangle,\langle 2b\rangle)\big)\cos 2\pi jx \\
&\quad +\big(a_j(1-a,1-b)b_j(\langle 2a\rangle,\langle 2b\rangle)+a_j(\langle 2a\rangle,\langle 2b\rangle)b_j(1-a,1-b)\big)\sin 2\pi jx\Bigg)
\end{aligned}$$

is the $\lfloor d/2\rfloor$-th partial sum of the Fourier series of

$$\begin{aligned}
&-\frac{1}{4}\int_0^1 \mathbf{I}_{[1-a,1-b)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(x-t)\ dt \\
=\;& \frac{1}{4}\int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(x-t)\ dt \qquad (11)
\end{aligned}$$

(this is a consequence of the "convolution theorem", cf. the proof of Lemma 4 in [1]). Thus $\sigma_{[a,b),d}(x)-\sum_{j=1}^{d}(a_j^2+b_j^2)$ is the $\lfloor d/2\rfloor$-th partial sum of the Fourier series of

$$\frac{1}{4}\int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(x-t)\ dt+\frac{1}{4}\int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(\langle 2x\rangle-t)\ dt.$$

For

$$\begin{aligned}
\sigma_{[a,b)}(x) \;=\;& (1-(b-a))(b-a) \\
&+\frac{1}{4}\int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(x-t)\ dt \\
&+\frac{1}{4}\int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(\langle 2x\rangle-t)\ dt
\end{aligned}$$

we have

$$\begin{aligned}
\|\sigma_{[a,b),d}-\sigma_{[a,b)}\|_\infty &\le \big((1-(b-a))(b-a)-\|p_{[a,b),d}\|_2^2\big)+cd^{-1} \\
&\le cd^{-1},
\end{aligned}$$

since the function $\sigma_{[a,b)}$ is Lipschitz-continuous and

$$(1-(b-a))(b-a)-\|p_{[a,b),d}\|_2^2=\frac{1}{2}\sum_{j=d+1}^{\infty}\big(a_j^2+b_j^2\big)\le\frac{1}{2}\sum_{j=d+1}^{\infty}\frac{4}{\pi j}\le d^{-1}.$$

Thus

$$\begin{aligned}
\limsup_{N\to\infty}\frac{\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_kx)\right|}{\sqrt{2N\log\log N}} &\le \limsup_{N\to\infty}\frac{\left|\sum_{k=1}^N p_{[a,b),d}(n_kx)\right|}{\sqrt{2N\log\log N}}+\limsup_{N\to\infty}\frac{\left|\sum_{k=1}^N r_{[a,b),d}(n_kx)\right|}{\sqrt{2N\log\log N}} \\
&\le \sqrt{\sigma_{[a,b),d}(x)}+cd^{-1/4} \quad \text{a.e.} \\
&\le \sqrt{\sigma_{[a,b)}(x)}+cd^{-1/4} \quad \text{a.e.}
\end{aligned}$$

and similarly

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|}{\sqrt{2N\log\log N}} \geq \sqrt{\sigma_{[a,b)}(x)} - cd^{-1/4} \quad \text{a.e.}$$

Since d can be chosen arbitrarily, this yields

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \mathbf{I}_{[a,b)}(n_k x)\right|}{\sqrt{2N\log\log N}} = \sqrt{\sigma_{[a,b)}(x)} \quad \text{a.e.}$$

5 Proof of Lemma 3

In this section we want to calculate

$$\sup_{0\leq a<b\leq 1} \sigma_{[a,b)}(x), \tag{12}$$

which is a very elaborate problem.

We have

$$\mathbf{I}_{[a,b)}(x) = \mathbf{I}_{[1-b,1-a)}(1-x),$$

except for $x=a$, and therefore, since $\langle -2b\rangle = 1-\langle 2b\rangle$,

$$\begin{aligned}
& \int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(x-t)\ dt \\
= & \int_0^1 \mathbf{I}_{[a,b)}(1-t)\mathbf{I}_{[1-\langle 2b\rangle,1-\langle 2a\rangle)}(1-x+t)\ dt \\
= & \int_0^1 \mathbf{I}_{[a,b)}(1-t)\mathbf{I}_{[\langle 2(1-b)\rangle,\langle 2(1-a)\rangle)}(1-x-(1-t))\ dt \\
= & \int_0^1 \mathbf{I}_{[a,b)}(s)\mathbf{I}_{[\langle 2(1-b)\rangle,\langle 2(1-a)\rangle)}(1-x-s)\ ds.
\end{aligned}$$

The second integral in the definition of $\sigma_{[a,b)}(x)$ can be transformed in a similar way, which yields

$$\sigma_{[a,b)}(x) = \sigma_{[1-b,1-a)}(1-x).$$

Thus it suffices to calculate (12) for $x\leq 1/2$.
For $0\leq b<a\leq 1$ we define

$$\sigma_{[a,b)}(x) = \sigma_{[b,a)}(x).$$

Then

$$\sup_{0\leq a<b\leq 1} \sigma_{[a,b)}(x) = \sup_{0\leq a,b,\leq 1,\ \langle b-a\rangle\leq 1/2} \sigma_{[a,b)}(x), \tag{13}$$

and we will calculate the expression on the right-hand side of (13) instead of (12), where again we can assume w.l.o.g. $x\leq 1/2$.

We set

$$\begin{aligned}
\sigma^{(1)}_{[a,b)}(x) &= (1-(\langle b-a\rangle))(\langle b-a\rangle) \\
\sigma^{(2)}_{[a,b)}(x) &= \frac{1}{4}\int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(x-t)\ dt \\
\sigma^{(3)}_{[a,b)}(x) &= \frac{1}{4}\int_0^1 \mathbf{I}_{[1-b,1-a)}(t)\mathbf{I}_{[\langle 2a\rangle,\langle 2b\rangle)}(\langle 2x\rangle-t)\ dt.
\end{aligned}$$

Then

$$\sigma_{[a,b)}(x) = \sigma^{(1)}_{[a,b)}(x) + \sigma^{(2)}_{[a,b)}(x) + \sigma^{(3)}_{[a,b)}(x).$$

The function $\sigma^{(1)}_{[a,b)}(x)$ is constant for fixed a, b. For the functions $\sigma^{(2)}_{[a,b)}(x)$ and $\sigma^{(3)}_{[a,b)}(x)$ we have (cf. [1, Lemma 4])

$$\begin{aligned}
\sigma^{(2)}_{[a,b)}(x) &= \sigma^{(2)}_{[0,\langle b-a\rangle)}(x-a) \\
\sigma^{(3)}_{[a,b)}(x) &= \sigma^{(2)}_{[a,b)}(\langle 2x\rangle) = \sigma^{(2)}_{[0,\langle b-a\rangle)}(\langle 2x\rangle - a),
\end{aligned}$$

and trivially $\sigma^{(1)}_{[a,b)}(x) = \sigma^{(1)}_{[0,b-a)}(x)$. Writing $\beta = \langle b-a\rangle$ we have by assumption $\beta \le 1/2$, which implies $\langle 2\beta\rangle = 2\beta$. Observing that $x \le 1/2$ implies $\langle 2x\rangle = 2x$ we have

$$\begin{aligned}
\sigma_{[a,b)}(x) &= \sigma^{(1)}_{[0,\langle b-a\rangle)}(x) + \sigma^{(2)}_{[0,\langle b-a\rangle)}(x-a) + \sigma^{(2)}_{[0,\langle b-a\rangle)}(\langle 2x\rangle - a) \\
&= (1-\beta)\beta \\
&\quad -\frac{1}{4}\int_0^1 \mathbf{I}_{[0,1-\beta)}(t)\cdot \mathbf{I}_{[0,\langle 2\beta\rangle)}(x-a-t)\ dt \\
&\quad -\frac{1}{4}\int_0^1 \mathbf{I}_{[0,1-\beta)}(t)\cdot \mathbf{I}_{[0,\langle 2\beta\rangle)}(\langle 2x\rangle-a-t)\ dt \\
&= (1-\beta)\beta \\
&\quad -\frac{1}{4}\int_0^1 \big(\mathbf{1}_{[0,1-\beta)}(t)-(1-\beta)\big)\cdot\big(\mathbf{1}_{[0,2\beta)}(\langle x-a-t\rangle)-2\beta\big)\ dt \\
&\quad -\frac{1}{4}\int_0^1 \big(\mathbf{1}_{[0,1-\beta)}(t)-(1-\beta)\big)\cdot\big(\mathbf{1}_{[0,2\beta)}(\langle 2x-a-t\rangle)-2\beta\big)\ dt \\
&= 2(1-\beta)\beta-\beta \qquad (14) \\
&\quad +\frac{1}{4}\int_0^1 \mathbf{1}_{[0,\beta)}(t)\cdot\mathbf{1}_{[0,2\beta)}(\langle x-a+t\rangle)\ dt \\
&\quad +\frac{1}{4}\int_0^1 \mathbf{1}_{[0,\beta)}(t)\cdot\mathbf{1}_{[0,2\beta)}(\langle 2x-a+t\rangle)\ dt.
\end{aligned}$$

Since for any $y \in [0,1]$

$$\int_0^1 \mathbf{1}_{[0,\beta)}(t)\cdot\mathbf{1}_{[0,2\beta)}(\langle y+t\rangle) \le \beta$$

we have

$$\sigma_{[a,b)}(x) \le \max_{\beta\in[0,1/2]} 2(1-\beta)\beta-\beta+\frac{1}{2}\beta \le \frac{9}{32}, \qquad 0 \le a,b, \le 1,\ \langle b-a\rangle \le 1/2,\ x\in[0,1/2]. \quad (15)$$

Let $x \in [0, 3/8)$. Then choosing $a = x$, using formula (14) we can calculate

$$\sigma_{[a,a+3/8)}(x) = \frac{3}{32} + \frac{3}{32} + \frac{3}{32} = \frac{9}{32}.$$

This proves

$$\sup_{0 \le a,b \le 1,\ \langle b-a \rangle \le 1/2} \sigma_{[a,b)}(x) = \frac{9}{32}, \qquad 0 \le x \le 3/8.$$

It remains to calculate (12) for $x \in [3/8, 1/2]$, which is particularly involved. We have to distinguish between several cases. First we observe that, using (14), we have

$$\begin{aligned} \sigma_{[a,b)}(x) &= 2(1-\beta)\beta - \beta \\ &\quad + \frac{1}{4}\int_0^1 \mathbf{1}_{[x-a,x-a+\beta)}(t)\mathbf{1}_{[0,2\beta)}(t)\ dt \\ &\quad + \frac{1}{4}\int_0^1 \mathbf{1}_{[2x-a,2x-a+\beta)}(t)\mathbf{1}_{[0,2\beta)}(t)\ dt \\ &= 2(1-\beta)\beta - \beta \\ &\quad + \frac{1}{4}\mathbb{P}\left\{[x-a, x-a+\beta) \cap [0, 2\beta)\right\} \\ &\quad + \frac{1}{4}\mathbb{P}\left\{[2x-a, 2x-a+\beta) \cap [0, 2\beta)\right\}. \end{aligned} \tag{16}$$

Here and in the sequel $\mathbb{P}$ denotes the Lebesgue measure on $[0, 1)$.

We will assume that $x \in [3/8, 1/2]$ is fixed.

Case 1: $x \le \beta$
Like in (15) we have

$$\sigma_{[a,b)}(x) \quad \le \quad 2(1-\beta)\beta - \frac{\beta}{2}. \tag{17}$$

Since the function in (17) is decreasing for $\beta \in [3/8, 1/2]$, and since by assumption $x \le \beta$, we have

$$\sigma_{[a,b)}(x) \le 2(1-x)x - \frac{x}{2}. \tag{18}$$

Case 2: $x > \beta$
In this case $A = [x-a, x-a+\beta)$ and $B = [2x-a, 2x-a+\beta)$ are two disjoint intervals. Between these two intervals there are two gaps of lengths $x - \beta$ and $1 - (x + \beta)$, respectively. Since by assumption $x \le 1/2$, we have

$$x - \beta \le (1 - 2x) + x - \beta = 1 - (x + \beta).$$

If $\beta \le 1/4$, then by (15)

$$\sigma_{[a,b)}(x) \le 2\left(1 - \frac{1}{4}\right)\frac{1}{4} - \frac{1}{8} = \frac{1}{4}.$$

On the other hand, if $\beta > 1/4$, then $x - \beta \le \beta$, and therefore

$$\mathbb{P}\left([0, 2\beta) \cap (A \cup B)\right) \le 2\beta - (x - \beta) = 3\beta - x.$$

Therefore

$$\sigma_{[a,b)}(x) \leq \max \left\{ \frac{1}{4},\ 2(1-\beta)\beta - \frac{\beta}{4} - \frac{x}{4} \right\}.$$

For fixed x this function is increasing for $\beta \in [3/8, 7/16]$, and decreasing for $\beta \in [7/16, 1/2]$. Since by assumption $x > \beta$, this yields

$$\sigma_{[a,b)}(x) < 2(1-x)x - \frac{x}{2}, \qquad 3/8 \leq x \leq 7/16 \tag{19}$$

and

$$\sigma_{[a,b)}(x) \leq 2\left(1 - \frac{7}{16}\right)\frac{7}{16} - \frac{7}{64} - \frac{x}{4} = \frac{49}{128} - \frac{x}{4}, \qquad 7/16 < x \leq 1/2. \tag{20}$$

Combining (18) (19) and (20) finally yields

$$\sigma_{[a,b)}(x) \leq \begin{cases} 2(1-x)x - \frac{x}{2} & 3/8 \leq x \leq 7/16 \\ \frac{49}{128} - \frac{x}{4} & 7/16 \leq x \leq 1/2. \end{cases} \tag{21}$$

It remains to show that the upper bounds in (21) can really be attained for certain values $a, b = a(x), b(x)$. For given x, we choose $a = x$ and $\beta = \min(x, 7/16)$, which implies $b = a + \min(x, 7/16)$. Then by (16)

$$\begin{aligned} \sigma_{[a,b)}(x) &= 2(1-\beta)\beta - \beta + \frac{\mathbb{P}\{[0,\beta)\}}{4} + \frac{\mathbb{P}\{[x, x+\beta) \cap [0, 2\beta)\}}{4} \\ &= 2(1-\beta)\beta - \beta + \frac{\beta}{4} + \frac{\beta - (x-\beta)}{4} \\ &= \begin{cases} 2(1-x)x - \frac{x}{2} & \text{for} \quad 3/8 \leq x \leq 7/16 \\ \frac{49}{128} - \frac{x}{4} & \text{for} \quad 7/16 \leq x \leq 1/2 \end{cases} \qquad \square \end{aligned}$$

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