On the law of the iterated logarithm for the discrepancy of $\langle n_k x \rangle$

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Dedicated to Professor Robert F. Tichy on the occasion of his 50th birthday

Abstract

By a well known result of Philipp (1975), the discrepancy $D_N(\omega)$ of the sequence $(n_k\omega)_{k\geq 1}$ mod 1 satisfies the law of the iterated logarithm

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under the Hadamard gap condition $n_{k+1}/n_k \geq q > 1$ $(k = 1, 2, \ldots)$. Recently Berkes, Philipp and Tichy (2006) showed that this result remains valid, under Diophantine conditions on (n_k), for subexpenentially growing (n_k), but in general the behavior of $(n_k\omega)$ becomes very complicated in the subexponential case. Using a different norming factor depending on the density properties of the sequence (n_k), in this paper we prove a law of the iterated logarithm for the discrepancy $D_N(\omega)$ for subexponentially growing (n_k) without number theoretic assumptions.

1 Introduction

Given a sequence $(x_1, \ldots, x_N)$ of real numbers, the value

$$D_N = D_N(x_1, \ldots, x_N) := \sup_{0 \le a < b \le 1} \left| \frac{A(N, a, b)}{N} - (b - a) \right|$$

is called the discrepancy of this sequence. Here $A(N, a, b)$ denotes the number of indices $k \le N$ for which the fractional part $\langle x_k \rangle$ of x_k belongs to the interval $[a, b)$. An infinite sequence $(x_k)_{k \ge 1}$ is called uniformly distributed mod 1 if $D_N(x_1, \ldots x_N) \to 0$ as $N \to \infty$. By a classical result of analysis (Bohl [5], Sierpinski [18], Weyl [20]), the sequence $(kx)_{k \ge 1}$ is uniformly distributed mod 1 for any irrational x. This result extends to $(n_k x)_{k \ge 1}$ for a large class of increasing sequences (n_k) such as $n_k = k^r$ $(r = 1, 2, \ldots)$, $n_k = (\log k)^\alpha$ (where $\alpha > 1$), $n_k = p_k$, where p_k denotes the k-th prime, etc. On the other hand, it is easy to see that $(k!x)$ is not uniformly distributed for $x = e$. However, Weyl [21] proved that given any increasing sequence $(n_k)_{k \ge 1}$ of positive integers, $(n_k x)_{k \ge 1}$ is uniformly distributed mod 1 for all x except for a set with Lebesgue measure 0. Determining the discrepancy of this sequence is a difficult problem and precise results are known only for a few special (n_k). R.C. Baker [1] proved, improving earlier results of Erdős and Koksma [8] and Cassels [6], that for any increasing sequence

(n_k) of integers the discrepancy $D_N(x)$ of $(n_k x)_{k\geq 1}$ satisfies

$$ND_N(x) = O(N^{\frac{1}{2}}(\log N)^{\frac{3}{2}+\varepsilon}) \qquad \text{a.e.}$$

for every $\varepsilon > 0$. Except the power of the logarithm, this result is sharp: Berkes and Philipp [3] constructed an increasing sequence (n_k) for which

$$ND_N(x) \geq c(N\log N)^{\frac{1}{2}} \qquad \text{for infinitely many } N$$

for some constant $c > 0$ and almost every x. Kesten [11] proved that for $n_k = k$ we have

$$\frac{ND_N(x)}{\log N \log\log N} \to \frac{2}{\pi^2} \qquad \text{in measure.}$$

For the remainder term, see Schoissengeier [16]. On the other hand, Philipp [14] proved that if $(n_k)_{k\geq 1}$ is a lacunary sequence of integers, i.e. a sequence of integers satisfying

$$n_{k+1}/n_k \geq q > 1 \qquad k = 1, 2\ldots, \tag{1}$$

then the discrepancy $D_N(x)$ of the sequence $(n_k x)_{k\geq 1}$ satisfies the law of the iterated logarithm (LIL), i.e.

$$\frac{1}{4} \leq \limsup_{N\to\infty} \frac{ND_N(x)}{\sqrt{N\log\log N}} \leq C \qquad \text{a.e.} \tag{2}$$

where $C \leq 166 + 664(q^{1/2}-1)^{-1}$. Recall that by the Chung-Smirnov LIL (see [17], p. 504), for an i.i.d. uniform sequence (ξ_k) in $[0, 1)$ we have

$$\limsup_{N\to\infty} \frac{ND_N(\xi_1,\ldots,\xi_N)}{\sqrt{N\log\log N}} = 1/2 \qquad \text{a.s.}$$

Thus the result of Philipp means that, in some sense, the sequence $\langle n_k x\rangle$ behaves like a sequence of independent random variables. However, the analogy is not perfect: for any $K>0$ one can construct a sequence (n_k) of integers satisfying the Hadamard gap condition (1) with some $q>1$ such that the lower bound $1/4$ in (2) can be replaced by K (see Berkes and Philipp [3]). In general, it is unknown if the limsup in (2) is a constant almost everywhere. Very recently, Fukuyama [9] managed to compute the limsup in the special case $n_k=\theta^k$, $\theta>1$. He showed that if θ^r is irrational for $r=1,2,\ldots$ then the limsup is $1/2$ a.e. (that is, exactly the same as in the Chung-Smirnov LIL). Further, if $\theta>1$ is integer, then the limsup is $\sqrt{42}/9$ if $\theta=2$, it is $1/2\ \sqrt{\theta(\theta+1)(\theta-2)/(\theta-1)^3}$ if $\theta\geq 4$ is even and it is $1/2\ \sqrt{(\theta+1)/(\theta-1)}$ if θ is odd. These results show that the limsup in (2) depends on θ very sensitively and this is an indication that computing the limsup (possibly depending on x) for general Hadamard lacunary (n_k) will be a very difficult problem.

If we weaken the Hadamard gap condition (1), the LIL (2) becomes generally false: as Berkes and Philipp [3] proved, for any positive sequence $\varepsilon_k \to 0$ there exists a sequence (n_k) of integers satisfying

$$n_{k+1}/n_k \geq 1 + \varepsilon_k \qquad k = 1, 2, \ldots \tag{3}$$

such that

$$\limsup_{N\to\infty} \frac{ND_N(x)}{\sqrt{N \log\log N}} = +\infty \qquad \text{a.e.}$$

For subexponential sequences, i.e. sequences (n_k) with $n_{k+1}/n_k \to 1$, the behavior of the discrepancy of $\langle n_k x\rangle_{k\geq 1}$ depends strongly on the number-theoretic properties of (n_k). For example, Berkes, Philipp and Tichy [4] proved that if the number of solutions of certain Diophantine equations

$$a_1 n_{k_1} + a_2 n_{k_2} + \ldots + a_p n_{k_p} = b$$

is "not too large" for $p = 2, 4$, then the law of the iterated logarithm (2) holds. For example, this is the case if (n_k) is the sequence generated by finitely many primes $p_1, \ldots, p_r$ (see Philipp [15]).

The purpose of the present paper is to prove a law of the iterated logarithm for the discrepancy of $\langle n_k x\rangle$ for subexponentially growing (n_k) without assuming any arithmetical conditions on n_k. We define

$$a_{N,r} = \#\{k \leq N :\ n_k \in [2^r, 2^{r+1})\}, \qquad r \geq 0,\ N \geq 1$$

and

$$B_N = \left(\sum_{r=0}^{\infty} a_{N,r}^2 \right)^{1/2}, \qquad N \geq 1. \tag{4}$$

Note that for each $N \geq 1$ the sum in (4) contains only finitely many nonzero terms. It is easy to see that $\sqrt{N} \leq B_N \leq N$. We will show that an LIL holds for $D_N(x)$ provided $a_{N,r}$ is small compared with B_N, uniformly in r. More precisely, we shall prove the following

Theorem 1 *Let $(n_k)_{k \geq 1}$ be a nondecreasing sequence of positive integers satisfying*

$$a_{N,r} = \mathcal{O}\left(\frac{B_N}{(\log N)^{\alpha}} \right) \tag{5}$$

for some constant $\alpha > 3$, uniformly for $r \in \mathbb{N}$. Then

$$\limsup_{N \to \infty} \frac{N D_N(x)}{\sqrt{B_N^2 \log \log N}} \leq C \quad a.e., \tag{6}$$

where C is a positive constant.

If n_k satisfies the Hadamard gap condition (1) then $a_{N,r} \leq K$ for $N \geq 1, r \geq 1$ with some constant $K > 0$ and consequently $B_N^2 = O(N)$. Hence in this case Theorem 1 reduces to the upper bound in the theorem of Philipp formulated above. In the subexponential case $n_{k+1}/n_k \to 1$ we have $B_N^2/N \to \infty$, and thus the bound

$$N D_N(x) = O\left(B_N (\log \log N)^{1/2} \right) \qquad \text{a.e.}$$

given by (6) is weaker than the LIL in (2). Of course, this must be the case, since we already noted that in the subexponential case the LIL for the discrepancy of $\langle n_k x \rangle$ is generally false. If n_k "almost" satisfies the Hadamard gap condition, i.e. n_{k+1}/n_k tends to 1 very slowly, then B_N^2/N grows very slowly and thus in this case the discrepancy of $\langle n_k x \rangle$ "almost" satisfies the LIL. Specifically, if (n_k) satisfies the subexponential gap condition

$$n_{k+1}/n_k \geq 1 + 1/\psi(k) \qquad k = 1, 2, \ldots, \tag{7}$$

where ψ is a nondecreasing, slowly varying function with $\lim_{k\to\infty} \psi(k) = +\infty$, then a simple calculation yields

$$a_{N,r} \leq c\psi(N) \quad \text{and} \quad B_N^2 = O(N\psi(N)).$$

Thus in this case Theorem 1 yields

$$ND_N(x) = O\left(\sqrt{N\psi(N)\log\log N}\right) \qquad \text{a.e.} \tag{8}$$

Conversely, Berkes and Philipp [3] showed that given a function ψ with the above properties, there is a sequence (n_k) of integers satisfying (7) such that for almost all x we have

$$ND_N(x) \geq c\sqrt{N\psi^*(N)\log\log N} \qquad \text{for infinitely many N} \tag{9}$$

where $\psi^*(N) = (\log\log\psi(N))^2$. There is a gap between (8) and (9) and the precise order of magnitude of $D_N(x)$ remains open.

Note that conclusion (6) of Theorem 1 implies

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N f(\langle n_k x\rangle)}{\sqrt{B_N^2 \log\log N}} < +\infty \qquad \text{a.e.} \tag{10}$$

for any f of the form $f = \mathbf{1}_{[a,b)}(x) - (b-a)$, $0 \le a < b \le 1$. Actually, by Koksma's inequality (10) remains valid for any function f on $(0,1)$ with bounded variation, satisfying $\int_0^1 f(x)dx = 0$.

Condition (5) of Theorem 1 has a simple probabilistic meaning. Let $X_1, X_2, \ldots$ be independent random variables with $EX_n = 0$, $EX_n^2 < +\infty$ $(n = 1, 2, \ldots)$ and let $S_n = \sum_{k=1}^n X_k$, $B_n^2 = \sum_{k=1}^n EX_k^2$. By a standard version of the central limit theorem, under the assumption

$$|X_n| = o(B_n) \tag{11}$$

we have

$$S_n/B_n \to_d \mathcal{N}(0,1).$$

Assuming the slightly stronger assumption

$$|X_n| = o\left(\frac{B_n}{(\log\log B_n)^{1/2}}\right) \tag{12}$$

("Kolmogorov condition"), we have the law of the iterated logarithm (see [12]):

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2B_n^2 \log\log B_n^2}} = 1 \qquad \text{a.s.} \tag{13}$$

Both (11) and (12) are "uniform asymptotic negligibility" conditions, requiring that the individual random variables X_n be negligible compared with the norming factor in the CLT and LIL. This condition plays a crucial role in several other limit theorems for sums of independent random variables as well. Note that condition (5) expresses the same effect for the random variables

$$X_k = \sum_{n_j \in [2^k, 2^{k+1})} \mathbf{1}_{[a,b)}(\langle n_j x \rangle) \tag{14}$$

in connection with the LIL (10). Thus Theorem 1 means that, roughly speaking, the dyadic block sums X_k in (14) behave like independent random variables. This heuristics generalizes the classical heuristics (see e.g. Kac [10]) that for Hadamard lacunary (n_k), $\langle n_k x \rangle$ are nearly independent random variables.

As the proof of Theorem 1 will show, assumption (5) can be slightly weakened. In fact, we shall prove the following stronger result:

Theorem 2 *Let $(n_k)_{k \geq 1}$ be a nondecreasing sequence of positive integers satisfying*

$$a_{N,r} \leq \frac{C_1 \ B_N}{(\log N)^3 (\log \log N)^{5/2}} \tag{15}$$

for a constant $C_1 \geq 1$, uniformly for $r \in \mathbb{N}$, $N \geq N_0$. Then

$$\limsup_{N\to\infty} \frac{ND_N(x)}{\sqrt{B_N^2 \log\log N}} \leq C_2 \tag{16}$$

for almost all x, where $C_2 \leq 10^4\ C_1$.

To prove Theorem 2 we use techniques developed by Takahashi [19], Berkes [2] and Philipp [14].

Whether Theorem 2 holds under the assumption

$$a_{N,r} = o\left(\frac{B_N}{(\log\log N)^{1/2}}\right)$$

remains open.

2 Truncation error

In this section we give an estimate for the error we make when in the sum $\sum_{k=M}^{N} f(n_k x)$ we replace $f(x)$ by a partial sum $g(x)$ of its Fourier series. We use a maximal inequality by Móricz, Serfling und Stout [13].

In the following let a positive integer N be given, let $f(x)$ denote an even function satisfying

$$f(x+1) = f(x), \quad \text{Var } f \leq 2, \quad \|f\|_\infty \leq 1, \quad \int_0^1 f(x)\ dx = 0, \tag{17}$$

and let

$$f \sim \sum_{j=1}^{\infty} c_j \cos 2\pi j x$$

be the Fourier series of f. Here Var f stands for the variation of f in the interval $[0,1)$. We will additionally assume that

$$2^{-h-2} \leq \int_0^1 f(x)^2 \, dx \leq 2^{-h-1}, \tag{18}$$

where h is a positive integer with $h \leq \lceil (\log_2 N)/2 \rceil$. This condition will play a crucial role in the chaining argument in Section 5. Let

$$g(x) = \sum_{j=1}^{N} c_j \cos 2\pi j x,$$

$$a_{M_1,M_2,r} = \#\{M_1 \leq k \leq M_2 : \ n_k \in [2^r, 2^{r+1})\}$$

for $1 \leq M_1 \leq M_2 \leq N, \ r \geq 1$, and

$$B_{M_1,M_2} = \left(\sum_{r=1}^{\infty} a_{M_1,M_2,r}^2 \right)^{1/2}, \quad 1 \leq M_1 \leq M_2 \leq N.$$

Note that $a_{M_1,M_2,r}$ and B_{M_1,M_2} depend also on N, but to lighten the notations, we will not mark the dependence on N. According to (17) we have (see Zygmund [22, p. 48])

$$|c_j| \leq \frac{\text{Var } f}{2j} \leq \frac{1}{j}, \qquad j \geq 1,$$

and thus for $J \geq 0$

$$\sum_{j=J+1}^{\infty} c_j^2 \leq \int_J^{\infty} \frac{1}{j^2} \, dj = \frac{1}{J}. \tag{19}$$

Lemma 1 *We have*

$$\int_0^1 \left(\sum_{k=M_1}^{M_2} (f(n_k x) - g(n_k x)) \right)^2 dx \leq \frac{3B_{M_1,M_2}^2}{N}.$$

Proof:

$$\begin{aligned}
& \int_0^1 \left(\sum_{k=M_1}^{M_2} (f(n_k x) - g(n_k x)) \right)^2 dx \\
= & \int_0^1 \left(\sum_{k=M_1}^{M_2} \sum_{j=N+1}^{\infty} c_j \cos 2\pi n_k j x \right)^2 dx \\
\leq & \sum_{k=M_1}^{M_2} \sum_{k'=k}^{M_2} \sum_{j,j'=N+1}^{\infty} |c_j c_{j'}| \{j n_k, j' n_{k'}\} \\
\leq & \sum_{k=M_1}^{M_2} \sum_{k'=k}^{M_2} \sum_{j,j'=N+1}^{\infty} \frac{1}{jj'} \{j n_k, j' n_{k'}\}, \qquad (20)
\end{aligned}$$

where $\{x,y\}$ denotes the Kronecker symbol. For fixed k, k' and j' there is at most one j for which $\{j n_k, j' n_{k'}\} = 1$ is true, namely

$$j = \frac{j' n_{k'}}{n_k}.$$

If $n_k \in [2^r, 2^{r+1})$ and $n_{k'} \in [2^{r+i}, 2^{r+i+1})$ for some r and $i \geq 1$, then $n_k/n_{k'} \leq 2^{-i+1}$. Hence by the Cauchy-Schwarz inequality the last expression in (20) is at most

$$\begin{aligned}
&\sum_{k=M_1}^{M_2} \sum_{k'=k}^{M_2} \sum_{j'=N+1}^{\infty} \frac{n_k}{j'^2 n_{k'}} \\
\leq\ & \frac{1}{N} \sum_{k=M_1}^{M_2} \sum_{k'=k}^{M_2} \frac{n_k}{n_{k'}} \\
\leq\ & \frac{1}{N} \left(\sum_{r=0}^{\infty} a_{M_1,M_2,r}^2 + \sum_{i=1}^{\infty} \sum_{r=0}^{\infty} a_{M_1,M_2,r}\ a_{M_1,M_2,r+i}\ 2^{-i+1} \right) \\
\leq\ & \frac{3B_{M_1,M_2}^2}{N} \qquad \Box
\end{aligned}$$

Note that B_{M_1,M_2}^2 is superadditive as a function of M_1 and M_2. Consequently we can use [13, Corollary 3.1] to get

$$\int_0^1 \max_{1 \leq K \leq N} \left(\sum_{k=1}^{K} \left(f(n_k x) - g(n_k x) \right) \right)^2 dx \leq \frac{3B_N^2 (1 + \log_2 N)^2}{N},$$

and thus for $h \geq 1$ we obtain (here and in the sequel $\log h$ will mean $\max\{1, \log h\}$)

Lemma 2

$$\mathbb{P} \left\{ \max_{1 \leq K \leq N} \left| \sum_{k=1}^{K} \left(f(n_k x) - g(n_k x) \right) \right| > h^{-1} (\log h)^{-2} B_N (\log \log N)^{1/2} \right\}$$
$$\leq \frac{3h^2 (\log h)^4 (1 + \log_2 N)^2}{N} \leq \frac{(\log N)^5}{N}$$

for sufficiently large N.

Here $\mathbb{P}$ denotes the Lebesgue measure on the interval $(0,1)$.

3 Exponential bounds

The following lemma is an extension of [19, Lemma 1] and [14, Lemma 3].

Lemma 3 *If λ is a positive number satisfying*

$$\lambda \leq \frac{(\log N)^2 (\log\log N)^2}{9\ C_1} \sqrt{\frac{\log\log N}{B_N^2}}, \tag{21}$$

then for sufficiently large N

$$\int_0^1 \exp\left(\lambda \sum_{k=1}^{N} g(n_k x)\right)\ dx \leq e^{24\|f\|_2 \lambda^2 B_N^2}.$$

Proof: Let $R = \lfloor \log_2 N \rfloor + 3$, then $3N < 2^R$. We define $L = \lfloor (\log_2 n_N)/R \rfloor$, $m_l = \min\{1 \leq k \leq N : n_k \geq 2^{Rl}\}$ for $l = 0, 1, \ldots, L$, further $m_{L+1} = N+1$. Clearly the numbers m_l are well defined and $1 \leq m_1 \leq \cdots \leq m_{L+1}$. Finally for $l = 0, 1, \ldots, L$ we set

$$U_l(x) = \begin{cases} \sum_{k=m_l}^{m_{l+1}-1} g(n_k x) & \text{if} \qquad m_{l+1} > m_l \\ 0 & \text{otherwise.} \end{cases}$$

Let $G_{\text{even}} = \{0 \leq l \leq L : l \text{ even and } m_{l+1} - m_l > 0\}$ and $G_{\text{odd}} = \{0 \leq l \leq L : l \text{ odd and } m_{l+1} - m_l > 0\}$. Then $\sum_{k=1}^N g(n_k x) = \sum_{l \in G_{\text{even}}} U_l(x) + \sum_{l \in G_{\text{odd}}} U_l(x)$. We define

$$I_1(\lambda) = \int_0^1 \exp\left(2\lambda \sum_{l \in G_{\text{even}}} U_l(x)\right)\ dx$$

and

$$I_2(\lambda) = \int_0^1 \exp\left(2\lambda \sum_{l \in G_{\mathrm{odd}}} U_l(x)\right)\ dx.$$

For $|z| \le 1$

$$e^z \le 1 + z + z^2,$$

and by (17) we get

$$\|g\|_\infty \le \|f\|_\infty + \mathrm{Var}\ f \le 3,$$

(see Philipp [14]). For sufficiently large N by (15) and (21)

$$\begin{aligned} \|2\lambda U_l\|_\infty &\le 2\lambda(m_{l+1} - m_l)\|g\|_\infty \le 6\lambda R \max_{r \ge 0} a_{N,r} \\ &\le \frac{6(\lfloor \log_2 N \rfloor + 3)}{9 \log N} \le 1 \end{aligned}$$

and so

$$I_1(\lambda) \le \int_0^1 \prod_{l \in G_{\mathrm{even}}} \left(1 + 2\lambda U_l(x) + 4\lambda^2 U_l^2(x)\right)\ dx. \tag{22}$$

For $l \in G_{\mathrm{even}}$

$$U_l^2(x) \le 2 \sum_{k=m_l}^{m_{l+1}-1} \sum_{k'=k}^{m_{l+1}-1} g(n_k x) g(n_{k'} x)$$

and for $m_l \le k \le k' \le m_{l+1} - 1$

$$g(n_k x) g(n_{k'} x) - \frac{1}{2} \sum_{\substack{1 \le j, j' \le N \\ |n_k j - n_{k'} j'| < n_{m_l}}} c_j c_{j'} \cos\left(2\pi(n_k j - n_{k'} j')x\right)$$

is a sum of trigonometric functions whose frequencies lie between n_{m_l} and $2Nn_{m_{l+1}-1}$ and consequently between 2^{Rl} and $2N\ 2^{R(l+1)}$. Thus

$$\begin{aligned}
& 2\sum_{k=m_l}^{m_{l+1}-1}\sum_{k'=k}^{m_{l+1}-1} g(n_k x)g(n_{k'}x) \\
= \quad & W_l(x) + \sum_{k=m_l}^{m_{l+1}-1}\sum_{k'=k}^{m_{l+1}-1} \sum_{\substack{1\le j,j'\le N \\ |n_k j - n_{k'}j'| < n_{m_l}}} c_j c_{j'} \cos\left(2\pi(n_k j - n_{k'}j')x\right),
\end{aligned}$$

where $W_l(x)$ is a sum of trigonometric functions whose frequencies lie between n_{m_l} and $2Nn_{m_{l+1}-1}$, and consequently between 2^{Rl} and $2N\ 2^{R(l+1)}$. If $V_l(x)$ denotes the triple sum on the right-hand side of the last equation, then

$$\begin{aligned}
|V_l(x)| \quad & \le \quad \sum_{k=m_l}^{m_{l+1}-1}\sum_{k'=k}^{m_{l+1}-1} \sum_{\substack{1\le j,j'\le N \\ |n_k j - n_{k'}j'| < n_{m_l}}} |c_j c_{j'}| \\
& \le \quad \sum_{k=m_l}^{m_{l+1}-1}\sum_{k'=k}^{m_{l+1}-1} \sum_{|j'n_{k'}/n_k - j|<1} |c_j c_{j'}| \\
& \le \quad 2\sum_{k=m_l}^{m_{l+1}-1}\sum_{k'=k}^{m_{l+1}-1} \left(\sum_{j'=1}^{\infty} c_{j'}^2\right)^{1/2} \left(\sum_{j>n_{k'}/n_k-1} c_j^2\right)^{1/2} \\
& \le \quad 2\sum_{k=m_l}^{m_{l+1}-1}\sum_{k'=k}^{m_{l+1}-1} \sqrt{2}\|f\|_2 \left(\sum_{j>n_{k'}/n_k-1} c_j^2\right)^{1/2}.
\end{aligned}$$

If $n_k \in [2^r, 2^{r+1})$ and $n_{k'} \in [2^{r+i}, 2^{r+i+1})$ for some r and $i \ge 0$, then $n_k/n_{k'} \le 2^{-i+1}$. Hence by Minkowski's inequality and since

$\|f\|_2 \le 1/2$ by (18),

$$
\begin{aligned}
& |V_l(x)| \\
\le \quad & 2\sqrt{2}\|f\|_2 \sum_{i=0}^{\infty} \sum_{r=Rl}^{R(l+1)-1-i} a_{N,r} a_{N,r+i} \sqrt{\min\left\{2\|f\|_2^2, \sum_{j=2^{i-1}}^{\infty} j^{-2}\right\}} \\
\le \quad & 2\sqrt{2}\|f\|_2 \sum_{r=Rl}^{R(l+1)-1} a_{N,r}^2 \sum_{i=0}^{\infty} \sqrt{\min\left\{\frac{1}{2}, \sum_{j=2^{i-1}}^{\infty} j^{-2}\right\}} \\
\le \quad & 12\|f\|_2 \sum_{r=Rl}^{R(l+1)-1} a_{N,r}^2. \qquad (23)
\end{aligned}
$$

Therefore

$$
U_l^2(x) \quad \le \quad W_l(x) + 12 \ \|f\|_2 \sum_{r=Rl}^{R(l+1)-1} a_{N,r}^2,
$$

and by (22) we see that $I_1(\lambda)$ is bounded by

$$
\int_0^1 \prod_{l \in G_{\text{even}}} \left(1 + 2\lambda U_l(x) + 4\lambda^2 W_l(x) + 48\lambda^2 \|f\|_2 \sum_{r=Rl}^{R(l+1)-1} a_{N,r}^2 \right) dx.
$$

If $d_l \cos 2\pi u_l x$ for $l \in G_{\text{even}},\ l > 0$ is any term of the trigonometric polynomial $2\lambda U_l(x) + 4\lambda^2 W_l(x)$, then $2^{Rl} \le n_{m_l} \le u_l \le 2N n_{m_{l+1}-1} \le 2N 2^{R(l+1)}$ and therefore

$$
\begin{aligned}
& u_l - \sum_{i \in G_{\text{even}}, i<l} u_i \\
\ge \quad & 2^{Rl} - 2N \sum_{j=0}^{(l-2)/2} 2^{(2j+1)R} = 2^{Rl} \left(1 - 2N \sum_{j=0}^{(l-2)/2} \frac{2^{(2j+1)R}}{2^{Rl}} \right) \\
\ge \quad & 1 - 2N \frac{2^{R-Rl}(2^{Rl}-1)}{4^R - 1} \ge 1 - \frac{3N}{2^R} > 0.
\end{aligned}
$$

Hence for any $(l_0, l_1, \dots, l_j)$ such that $0 \leq l_0 < \dots < l_j$ and $l_i \in G_{\text{even}}$ for $i = 0, \dots, j$ we have

$$\int_0^1 \prod_{i=0}^{j} \cos 2\pi u_{l_i} x \; dx = 0,$$

and thus

$$I_1(\lambda) \leq \prod_{l \in G_{\text{even}}} \left(1 + 48 \|f\|_2 \; \lambda^2 \sum_{r=Rl}^{R(l+1)-1} a_{N,r}^2 \right).$$

In the same way we can show a corresponding inequality for $I_2(\lambda)$, and thus using $1 + u \leq e^u, \; u \in \mathbb{R}$ we get

$$\begin{aligned}
& \int_0^1 \exp\left(\lambda \sum_{k=1}^{N} g(n_k x) \right) \; dx \\
= \; & \int_0^1 \exp\left(\lambda \sum_{l \in G_{\text{even}} \cup G_{\text{odd}}} U_l(x) \right) \; dx \\
\leq \; & \left(\int_0^1 \exp\left(2\lambda \sum_{l \in G_{\text{even}}} U_l(x) \right) dx \int_0^1 \exp\left(2\lambda \sum_{l \in G_{\text{odd}}} U_l(x) \right) dx \right)^{1/2} \\
\leq \; & \left(\prod_{l \in G_{\text{even}}} \left(1 + 48 \|f\|_2 \; \lambda^2 \sum_{r=Rl}^{R(l+1)-1} a_{N,r}^2 \right) \times \right. \\
& \left. \prod_{l \in G_{\text{odd}}} \left(1 + 48 \|f\|_2 \lambda^2 \sum_{r=Rl}^{R(l+1)-1} a_{N,r}^2 \right) \right)^{1/2} \\
\leq \; & \left(\exp\left(48 \|f\|_2 \; \lambda^2 \sum_{l=0}^{L} \sum_{r=Rl}^{R(l+1)-1} a_{N,r}^2 \right) \right)^{1/2} \\
= \; & e^{24 \|f\|_2 \lambda^2 B_N^2} \qquad \square
\end{aligned}$$

Lemma 4 *For sufficiently large N we have*

$$\mathbb{P}\left\{\sum_{k=1}^{N} g(n_k x) > 378\ C_1 h^{-1}(\log h)^{-2} B_N (\log\log N)^{1/2}\right\} \le e^{-2h\log\log N}$$

Proof: If we put

$$\lambda = \frac{h^2(\log h)^2}{94\ C_1}\sqrt{\frac{\log\log N}{B_N^2}},$$

then, since $1 \le h \le \lceil(\log_2 N)/2\rceil$, for sufficiently large N relation (21) is satisfied. Simple calculations show that for sufficiently large N

$$24\ (94)^{-2}\ 2^{(-h-1)/2}\ h^4(\log h)^4 - 378\ (94)^{-1}h \le -2h \qquad \text{for } h \ge 1,$$

and therefore by (18), Lemma 3 (recall that $C_1 \ge 1$) and the Markov inequality

$$\begin{aligned}
& \mathbb{P}\left\{\sum_{k=1}^{N} g(n_k x) > 378 C_1\ h^{-1}(\log h)^{-2} B_N (\log\log N)^{1/2}\right\} \\
\le\ & \exp\left(24\|f\|_2\lambda^2 B_N^2 - 378\ C_1\lambda h^{-1}(\log h)^{-2} B_N(\log\log N)^{1/2}\right) \\
=\ & \exp\left(24\ (94C_1)^{-2}\|f\|_2 h^4(\log h)^4\log\log N - 378\ (94)^{-1}h\log\log N\right) \\
\le\ & \exp\left(\left(24\ (94)^{-2}\ 2^{(-h-1)/2}h^4(\log h)^4 - 378\ (94)^{-1}h\right)\log\log N\right) \\
\le\ & e^{-2h\log\log N}
\end{aligned}$$

for sufficiently large N. $\square$

If $f(x)$ satisfies the conditions in (17), the same is valid for $-f(x)$. So we get the following

Corollary 1 *For sufficiently large N we have*

$$\mathbb{P}\left\{\left|\sum_{k=1}^{N} g(n_k x)\right| > 378 C_1 h^{-1}(\log h)^{-2} B_N (\log\log N)^{1/2}\right\} \leq 2e^{-2h\log\log N}.$$

4 A maximal inequality

For $2^r \leq n_k < 2^{r+1}$ we put $m = \lfloor r + 3\log_2 N\rfloor$ and approximate $g(n_k x) = \sum_{j=1}^{N} c_j \cos 2\pi j n_k x$ by

$$\varphi_k(x) = \sum_{j=1}^{N} c_j \cos 2\pi j n_k \frac{i}{2^m} \quad \text{for} \quad \frac{i}{2^m} \leq x < \frac{i+1}{2^m},\ i = 0, 1, \ldots, 2^m - 1.$$

Then for all k and all x we have by $|c_j| \leq 1/j$

$$|\varphi_k(x) - g(n_k x)| \leq \sum_{j=1}^{N} 2\pi |c_j| j n_k 2^{-m} \leq \sum_{j=1}^{N} 8\pi 2^{-3\log_2 N} \leq 8\pi N^{-2},$$

and for $1 \leq K \leq N$

$$\sum_{k=1}^{K} |\varphi_k(x) - g(n_k x)| \leq \sum_{k=1}^{K} 8\pi N^{-2} \leq 8\pi N^{-1} \quad \text{for all} \quad x \in [0,1]. \quad (24)$$

Further we put $Z_K = \sum_{k=1}^{K} \varphi_k(x)$.

Lemma 5 *For sufficiently large N*

$$\begin{aligned} &\mathbb{P}\left\{\max_{1\leq K\leq N} |Z_K| > 423\ C_1 h^{-1}(\log h)^{-2} B_N (\log\log N)^{1/2}\right\} \\ \leq\ & 2\mathbb{P}\left\{|Z_N| > 379\ C_1 h^{-1}(\log h)^{-2} B_N (\log\log N)^{1/2}\right\}. \qquad (25)\end{aligned}$$

Proof: We use an idea going back to Kolmogorov [12]. Let

$$\lambda = C_1 h^{-1} (\log h)^{-2} B_N (\log\log N)^{1/2}$$

and define the sets

$$\begin{aligned}
E &= \left\{ \max_{1\le K\le N} Z_K > 423\ \lambda \right\}, \\
F &= \{ Z_N > 379\ \lambda \}, \\
E_1 &= \{ Z_1 > 423\ \lambda \}, \\
E_K &= \{ Z_1 \le 423\ \lambda, \dots,\ Z_{K-1} \le 423\ \lambda,\ Z_K > 423\ \lambda \},\ 2 \le K \le N, \\
G_K &= \{ Z_N - Z_K > -44\ \lambda \},\ 1 \le K \le N.
\end{aligned}$$

Then the sets $E_K G_K$ are pairwise disjoint and $\bigcup_{K=1}^N E_K G_K \subset F$, hence

$$\sum_{K=1}^N \mathbb{P}(E_K G_K) \le \mathbb{P}(F). \tag{26}$$

On the set $\overline{G}_K$ ($\overline{G}_K$ denotes the complement of G_K)

$$(Z_N - Z_K)^2 \ge (44\ \lambda)^2,$$

and so

$$\begin{aligned}
\mathbb{P}(E_K \overline{G}_K) &\le \frac{1}{1936\ \lambda^2} \int_{E_K} (Z_N - Z_K)^2\ dx \\
&= \frac{1}{1936\ \lambda^2} \int_{E_K} \left(\sum_{k=K+1}^N \varphi_k(x) \right)^2 dx. \qquad (27)
\end{aligned}$$

Let s be the integer for which $2^s \leq n_K < 2^{s+1}$, and put $w = \lfloor s + 3\log_2 N \rfloor$. Every $\varphi_k,\ 1 \leq k \leq K$ is constant on intervals of the form

$$I = [i2^{-w}, (i+1)2^{-w}) \quad \text{for} \quad i = 0, 1, \ldots, 2^w - 1, \tag{28}$$

and thus the set E_K can be written as a union of such intervals. We now want to show that

$$\frac{1}{1936\ \lambda^2} \int_I \left(\sum_{k=K+1}^{N} \varphi_k(x) \right)^2 dx \leq \frac{1}{2}\ \mathbb{P}(I) \tag{29}$$

for intervals of the form (28). Then by (29) the right side of (27) is at most $\frac{1}{2}\mathbb{P}(E_K)$, and so $\mathbb{P}(E_K \overline{G}_K) \leq \frac{1}{2}\mathbb{P}(E_K)$ for $1 \leq K \leq N$. This together with (26) implies

$$\begin{aligned} \frac{1}{2}\mathbb{P}(E) \quad &= \quad \sum_{K=1}^{N} \frac{1}{2}\mathbb{P}(E_K)\ \leq\ \sum_{K=1}^{N} \left(\mathbb{P}(E_K) - \mathbb{P}(E_K\overline{G}_K)\right) \\ &= \sum_{K=1}^{N} \mathbb{P}(E_K G_K)\ \leq\ \mathbb{P}(F), \end{aligned}$$

and the same argument, applied to $(-\varphi_k)_{k\geq 1}$ instead of $(\varphi_k)_{k\geq 1}$, yields (25).

It remains to prove (29). By $C_1 \geq 1$, the Minkowski inequality and (24) it is enough to show

$$\int_I \left(\sum_{k=K+1}^{N} g(n_k x) \right)^2 dx \leq \frac{927 B_N^2}{h^2 (\log h)^4}\ \mathbb{P}(I)$$

for intervals I of the form (28) and sufficiently large N, which will follow from

$$\int_I \left(\sum_{k=K+1}^{K+\lfloor B_N h^{-1} (\log h)^{-2} \rfloor} g(n_k x) \right)^2 dx \leq \frac{9 B_N^2}{h^2 (\log h)^4} \, \mathbb{P}(I) \tag{30}$$

and

$$\int_I \left(\sum_{k=K+\lfloor B_N h^{-1} (\log h)^{-2} \rfloor + 1}^{N} g(n_k x) \right)^2 dx \leq \frac{753 B_N^2}{h^2 (\log h)^4} \, \mathbb{P}(I). \tag{31}$$

Here (30) is trivial since the integrand is bounded by

$$(\|g\|_\infty B_N h^{-1} (\log h)^{-2})^2 \leq 9 B_N^2 h^{-2} (\log h)^{-4}.$$

In the case $K + \lfloor B_N h^{-1} (\log h)^{-2} \rfloor + 1 > N$ inequality (31) is trivial as well since the sum is empty. It remains to show (31) in the case $K + \lfloor B_N h^{-1} (\log h)^{-2}) \rfloor + 1 \leq N$. Using the substitution $t = 2^w x$, (31) is equivalent to

$$\int_i^{i+1} \left(\sum_{k=K+\lfloor B_N h^{-1} (\log h)^{-2} \rfloor + 1}^{N} g(m_k t) \right)^2 dt \leq \frac{753 B_N^2}{h^2 (\log h)^4}$$

for some integer i, where $m_k = 2^{-w} n_k$. We put

$$\theta = K + \lfloor B_N h^{-1} (\log h)^{-2} \rfloor + 1.$$

By (15) there are at most

$$\frac{C_1 B_N}{(\log N)^3 (\log \log N)^{5/2}}$$

elements of the sequence (n_k) in any interval $[2^r, 2^{r+1}), r \geq 0$, and thus, since $\theta - K = \lfloor B_N h^{-1} (\log h)^{-2} \rfloor + 1$ and $1 \leq h \leq \lceil (\log_2 N)/2 \rceil$, we get that

$$n_K/n_\theta \leq 2^{-\frac{\theta-K}{C_1 B_N (\log N)^{-3} (\log\log N)^{-5/2}}+1} \leq 2 \cdot 2^{-C_1^{-1} (\log N)^2 (\log\log N)^{1/2}}$$

for sufficiently large N. Thus

$$\frac{1}{m_\theta} = \frac{2^w}{n_\theta} \leq \frac{N^3 n_K}{n_\theta} \leq 2N^3 2^{-C_1^{-1} (\log N)^2 (\log\log N)^{1/2}} \tag{32}$$

for sufficiently large N. Now

$$\begin{aligned} & \int_i^{i+1} \left(\sum_{k=\theta}^{N} g(m_k t) \right)^2 dt \\ \leq & \left| \int_i^{i+1} \sum_{k=\theta}^{N} \sum_{k'=k}^{N} 2 \sum_{j,j'=1}^{N} c_j c_{j'} \cos 2\pi j m_k t \cos 2\pi j' m_{k'} t \ dt \right| \end{aligned} \tag{33}$$

and from

$$\begin{aligned} & \cos 2\pi j m_k t \ \cos 2\pi j' m_{k'} t \\ = & \frac{1}{2} \big(\cos 2\pi (j m_k + j' m_{k'}) t + \cos 2\pi (j m_k - j' m_{k'}) t \big) \end{aligned}$$

follows, in case

$$|j m_k - j' m_{k'}| \geq m_k, \tag{34}$$

that

$$\begin{aligned} & 2 \int_i^{i+1} \cos 2\pi j m_k t \ \cos 2\pi j' m_{k'} t \ dt \\ \leq & \frac{2}{2\pi (j m_k + j' m_{k'})} + \frac{2}{2\pi |j m_k - j' m_{k'}|} \leq \frac{1}{m_k}. \end{aligned} \tag{35}$$

The contribution of those summands in (33) for which (34) is not satisfied, is at most

$$\begin{aligned}
&\sum_{k=\theta}^{N}\sum_{k'=k}^{N}\sum_{j=1}^{N} \sum_{\substack{j'=1 \\ |j-j'm_{k'}/m_k|<1}}^{N} c_j c_{j'} \\
\leq\ & 2\sum_{k=1}^{N}\sum_{k'=k}^{N}\left(\sum_{j'=1}^{\infty} c_{j'}^2\right)^{1/2}\left(\sum_{j>(m_{k'}/m_k)-1}^{\infty} c_j^2\right)^{1/2} \\
\leq\ & 12\|f\|_2 B_N^2, \qquad (36)
\end{aligned}$$

which can be calculated similarly to (23). Since $12 \cdot 2^{(-h-1)/2} \leq 752\ (h^2(\log h)^4)^{-1}$ for integer $h \geq 1$ by a simple calculation, the last expression in (36) is at most

$$752\frac{B_N^2}{h^2(\log h)^4}. \qquad (37)$$

Using (32), (35) and (37) we have by (33) that

$$\begin{aligned}
&\left|\int_i^{i+1}\left(\sum_{k=\theta}^{N} g(m_k t)\right)^2 dt\right| \\
\leq\ & \frac{752B_N^2}{h^2(\log h)^4} + \sum_{k=\theta}^{N}\sum_{k'=k}^{N}\sum_{j,j'=1}^{N} c_j c_{j'}\frac{1}{m_k} \\
\leq\ & \frac{752B_N^2}{h^2(\log h)^4} + (1+\log N)^2\sum_{k=\theta}^{N}\sum_{k'=\theta}^{N}\frac{1}{m_\theta} \\
\leq\ & \frac{752B_N^2}{h^2(\log h)^4} + (1+\log N)^2 N^2\ 2N^3 2^{-C_1^{-1}(\log N)^2(\log\log N)^{1/2}} \\
\leq\ & \frac{753B_N^2}{h^2(\log h)^4}
\end{aligned}$$

for sufficiently large N, and thus the lemma is proved. $\square$

By (24) and (25)

$$\begin{aligned}
&\mathbb{P}\left\{\max_{1\leq K\leq N}\left|\sum_{k=1}^{K} g(n_k x)\right| > 424\ C_1 h^{-1}(\log h)^{-2}\sqrt{B_N^2 \log\log N}\right\} \\
\leq\ & 2\mathbb{P}\left\{\left|\sum_{k=1}^{N} g(n_k x)\right| > 378\ C_1 h^{-1}(\log h)^{-2}\sqrt{B_N^2 \log\log N}\right\}
\end{aligned}$$

for sufficiently large N, and thus by Lemma 2 and Corollary 1 we get

Corollary 2 *For sufficiently large N*

$$\begin{aligned}
&\mathbb{P}\left\{\max_{1\leq K\leq N}\left|\sum_{k=1}^{K} f(n_k x)\right| > 425\ C_1 h^{-1}(\log h)^{-2} B_N (\log\log N)^{1/2}\right\} \\
\leq\ & 4e^{-2h\log\log N} + N^{-1}(\log N)^5.
\end{aligned}$$

A similar result can be established for odd functions satisfying the conditions in (17) and (18). Since any periodic function f satisfying (17) can be decomposed into an even and an odd part (put $f_{even} = (f(x)+f(-x))/2$ and $f_{odd} = (f(x)-f(-x))/2$), which satisfy (17) as well, we get the following

Corollary 3 *Let $f(x)$ be a function satisfying the conditions in (17) and assume f can be decomposed into an even and an odd part both* *of which satisfy (18). Then*

$$\mathbb{P}\left\{\max_{1\le K\le N}\left|\sum_{k=1}^{K} f(n_k x)\right| > 850\ C_1 h^{-1}(\log h)^{-2}\sqrt{B_N^2 \log\log N}\right\}$$
$$\le\ 8e^{-2h\log\log N} + 2N^{-1}(\log N)^5.$$

5 Proof of Theorem 2

Using the exponential bounds and maximal inequalities obtained in Sections 3 and 4 we can complete the proof of Theorem 2 by using a chaining argument, a technique going back to Cassels. We will actually use the version of this method employed in Erdős and Gál [7] and Philipp [14].

Let N be given and put $H = \lceil (\log_2 N)/2 \rceil$. Every $a \in [0,1)$ can be written in dyadic expansion

$$a = \sum_{j=1}^{\infty} 2^{-j} a_j, \qquad a_j \in \{0,1\},$$

and obviously

$$\sum_{j=1}^{H} 2^{-j} a_j \le a \le \sum_{j=1}^{H} 2^{-j} a_j + 2^{-H}.$$

We define the functions

$$\varrho_h^{(j)}(x) = \mathbf{1}_{[(j-1)2^{-h}, j2^{-h})}(x), \qquad 1 \le j \le 2^h,\ 1 \le h \le H,$$

where $\mathbf{1}_{[a,b)}$ denotes the indicator of the interval $[a,b)$, extended with period 1, and we set

$$\varphi_h^{(j)}(x) = \varrho_h^{(j)}(x) - \int_0^1 \varrho_h^{(j)}(x)\ dx, \qquad 1 \leq j \leq 2^h,\ 1 \leq h \leq H.$$

Then for any a there exist coefficients $\varepsilon_h = \varepsilon_h(a) \in \{0,1\}$ and indices $j_h = j_h(a), 1 \leq h \leq H$, and an additional index $\bar{j_H} = \bar{j_H}(a)$ such that

$$\sum_{h=1}^{H} \varepsilon_h \varrho_h^{(j_h)}(x) \leq \mathbf{1}_{[0,a)}(x) \leq \sum_{h=1}^{H} \varepsilon_h \varrho_h^{(j_h)}(x) + \varrho_H^{(\bar{j_H})}(x).$$

For $1 \leq h \leq H,\ 1 \leq j \leq 2^h,\ N \geq 1$ we write

$$F(N,h,j) = \left| \sum_{k=1}^{N} \varphi_h^{(j)}(n_k x) \right|$$

and define for $n \geq 1$

$$G_n = \bigcup_{N:\ B_N \in (2^{n-1}, 2^n]} \bigcup_{h \leq H} \bigcup_{j \leq 2^h} \left\{ F(N,h,j) > \frac{1701 C_1}{h(\log h)^2} B_N (\log\log N)^{1/2} \right\}$$

We put $N_n = \max\{k : B_k \leq 2^n\}, n \geq 0$. Let now $B_N \in (2^{n-1}, 2^n]$, i.e. $N_{n-1} < N \leq N_n$. If

$$F(N,h,j) > 1701\ C_1 h^{-1} (\log h)^{-2} B_N (\log\log N)^{1/2}$$

is satisfied, then for any fixed $\varepsilon > 0$ and sufficiently large $N \geq N_0(\varepsilon)$ the inequality

$$F(N,h,j) > \frac{1701}{2+\varepsilon} C_1 h^{-1} (\log h)^{-2} B_{N_n} (\log\log N_n)^{1/2}$$

must be satisfied as well, since by $\sqrt{k} \leq B_k \leq k$ for$k \geq 1$

$$\frac{B_N(\log\log N)^{1/2}}{B_{N_n}(\log\log N_n)^{1/2}} \geq \frac{2^{n-1}}{2^n}\left(\frac{\log\log 2^{n-1}}{\log\log 2^{2n}}\right)^{1/2} \geq \frac{1}{2+\varepsilon}$$

for sufficiently large N and n. In particular for $\varepsilon = 850^{-1}$ and sufficiently large N and n

$$\mathbb{P}(G_n) \leq \sum_{h=1}^{H}\sum_{j=1}^{2^h}\mathbb{P}\left\{\max_{N_{n-1}<N\leq N_n} F(N,h,j) > \frac{850C_1}{h(\log h)^2}B_{N_n}(\log\log N_n)^{1/2}\right\}.$$

The functions $\varphi_h^{(j)}(x)$ satisfy the conditions of Corollary 3. Thus

$$\begin{aligned}
\mathbb{P}(G_n) &\leq \sum_{h=1}^{H}\sum_{j=1}^{2^h}\left(8e^{-2h\log\log N_n} + \frac{2(\log N_n)^5}{N_n}\right) \\
&\ll \sum_{h=1}^{\lceil(\log_2 N_n)/2\rceil} 2^h e^{-2h\log\log N_n} + \sum_{h=1}^{\lceil(\log_2 N_n)/2\rceil} 2^h\frac{(\log N_n)^5}{N_n} \\
&\ll \sum_{h=1}^{\infty}\left(\frac{\sqrt{2}}{\log N_n}\right)^{2h} + \frac{(\log N_n)^5}{N_n}\sum_{h=1}^{\lceil(\log_2 N_n)/2\rceil} 2^h \\
&\ll \frac{1}{(\log N_n)^2} + \frac{\sqrt{N_n}(\log N_n)^5}{N_n} \\
&\ll \frac{1}{n^2}
\end{aligned}$$

for sufficiently large n, since $N_n + 1 \geq 2^n$. Hence for arbitrary $\eta > 0$ there exists an n_0 such that

$$\sum_{n\geq n_0}\mathbb{P}(G_n) < \eta.$$

Now let $0 \le a < 1$ be abitrary. Then

$$\begin{aligned}
& \left| \sum_{k=1}^{N} \mathbf{1}_{[0,a)}(n_k x) - Na \right| \\
\le \ & \sum_{h=1}^{H} \left| \sum_{k=1}^{N} \varphi_h^{(j_h)}(n_k x) \right| + \left| \sum_{k=1}^{N} \varphi_H^{(\bar{j_H})}(n_k x) \right| + 2^{-H} N \\
\le \ & 2\sqrt{N} + \sum_{h \le H} F(N, h, j_h) + F(N, H, \bar{j_H}) \\
\le \ & 1703 \ C_1 B_N (\log\log N)^{1/2} \left(\sum_{h=1}^{H} \frac{1}{h(\log h)^2} + \frac{1}{H(\log H)^2} \right) \\
\le \ & 4376 \ C_1 B_N (\log\log N)^{1/2}
\end{aligned}$$

for all x and $N \ge N_0(\eta)$, except a set of measure less than η, no matter how a was chosen. For those x

$$\left| \sum_{k=1}^{N} \mathbf{1}_{[a,b)}(n_k x) - N(b-a) \right| \le 8752 \ C_1 B_N (\log\log N)^{1/2}$$

for all $0 \le a < b \le 1$. We divide by $B_N(\log\log N)^{1/2}$, take the supremum over $0 \le a < b \le 1$, take the lim sup and let $\eta \to 0$, all in that order. This proves Theorem 2.

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