The law of the iterated logarithm for $\sum c_k f(n_k x)$

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Abstract

By a classical heuristics, systems of the form $(\cos 2\pi n_k x)_{k\geq 1}$ and $(f(n_k x))_{k\geq 1}$, where $(n_k)_{k\geq 1}$ is a rapidly growing sequence of integers, show probabilistic properties similar to those of independent and identically distributed (i.i.d.) random variables. For example, Erdős and Gál proved the law of the iterated logarithm (LIL) in the form $\limsup_{N\to\infty}(2N \log\log N)^{-1/2} \sum_{k=1}^N \cos 2\pi n_k x = 1/\sqrt{2}$ a.e., valid for $(n_k)_{k\geq 1}$ satisfying the lacunary growth condition $n_{k+1}/n_k > q > 1$, $k \geq 1$. Weiss extended this to $\limsup_{N\to\infty}(2B_N^2 \log\log B_N)^{-1/2} \sum_{k=1}^N \cos 2\pi n_k x = 1$ a.e., again for lacunary $(n_k)_{k\geq 1}$, where $B_N^2 = \sum_{k=1}^N c_k^2$, under the additional assumption $c_N = o(B_N/(\log\log B_N)^{1/2})$ as $N \to \infty$. This directly corresponds to a general LIL for i.i.d. random variables due to Kolmogoroff. In this paper we generalize Weiss's result to systems $(f(n_k x))_{k\geq 1}$, where f is a function of bounded variation, under an almost best possible growth condition for the coefficients $(c_k)_{k\geq 1}$, thus partially solving a problem posed by Walter Philipp in his famous paper of 1975.

1 Introduction

A increasing sequence (n_k) of positive integers is called a lacunary sequence if it satisfies the Hadamard gap condition

$$\frac{n_{k+1}}{n_k} > q > 1, \quad k \geq 1.$$

A classical heuristics states that the systems

$$(\cos 2\pi n_k x)_{k\geq 1} \qquad \text{or} \qquad (f(n_k x))_{k\geq 1}, \tag{1}$$

where $(n_k)_{k\geq 1}$ is a lacunary sequence of integers and f is a "nice" function, exhibit properties similar to those of systems of independent and identically distributed (i.i.d.) random variables. For example, Erdős and Gál [6] proved in 1955 that for a lacunary sequence $(n_k)_{k\geq 1}$

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos 2\pi n_k x}{\sqrt{2N \log\log N}} = \frac{1}{\sqrt{2}} \quad \text{a.e.},$$

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which is similar to the law of the iterated logarithm for i.i.d. random variables, stating that for an i.i.d. sequence $X_1, X_2, \dots$ satisfying $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = \sigma^2 < \infty$,

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N X_k}{\sqrt{2N \log\log N}} = \sigma \quad \text{a.s.} \tag{2}$$

If the function $\cos 2\pi x$ is replaced by another function $f(x)$ satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\ dx = 0,$$

and if f is additionally Lipschitz continuous (Takahashi [12], 1962) or of bounded variation on $[0,1]$ (Philipp [10], 1975), then

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N f(n_k x)}{\sqrt{2N \log\log N}} \le C \quad \text{a.e.} \tag{3}$$

for some constant C. On the other hand, there are examples showing that an exact law of the iterated logarithm (LIL) like (2) will not necessarily hold in the case of general functions $f(x)$ instead of $\cos 2\pi x$: choose e.g. $f(x) = \cos 2\pi x + \cos 4\pi x$, and $n_k = 2^k + 1$, $k \ge 1$. Then

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N f(n_k x)}{\sqrt{2N \log\log N}} = \sqrt{2}|\cos \pi x| \quad \text{a.e.,} \tag{4}$$

as was pointed out by Erdős and Fortet.

There exists an important generalization of the LIL (2) to the case of independent, not identically distributed random variables, proved by Kolmogoroff [7] in 1929. Let $X_1, X_2, \dots$ be independent random variables satisfying $\mathbb{E}X_k = 0$, $\mathbb{E}X_k^2 = \sigma_k^2 < \infty$ $(k \ge 1)$, and define

$$B_N = \left(\sum_{k=1}^N \sigma_k^2 \right)^{1/2}, \quad N \ge 1.$$

Then

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N X_k}{\sqrt{2B_N^2 \log\log B_N^2}} = 1 \quad \text{a.s.,} \tag{5}$$

provided $B_N \to \infty$ and there exists a sequence m_k such that

$$|X_k| \le m_k,\ k \ge 1, \quad \text{and} \quad m_N = o\left(\frac{B_N}{\sqrt{\log\log B_N}} \right) \quad \text{as} \quad N \to \infty.$$

There are several possibilities to modify (5) to our situation of systems of the form (1). One way is to bound the number of elements n_k lying in dyadic intervals of the form $[2^r, 2^{r+1})$, $r = 1, 2, \dots$. The author proved, together with I. Berkes [4], the following result: Let $f(x)$ be a function satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\ dx = 0, \qquad \mathrm{Var}_{[0,1]}\ f \le 2. \tag{6}$$

Define

$$a_{N,r} = \#\{k \leq N : \ n_k \in [2^r, 2^{r+1})\}, \qquad r \geq 0, \ N \geq 1,$$

and

$$B_N = \left(\sum_{r=0}^{\infty} a_{N,r}^2 \right)^{1/2}, \qquad N \geq 1.$$

Then

$$\limsup_{N \to \infty} \frac{\sum_{k=1}^N f(n_k x)}{\sqrt{2B_N^2 \log\log B_N^2}} \leq C \quad \text{a.e.}$$

for some constant C, provided

$$a_{N,r} = \mathcal{O}\left(B_N (\log N)^{-\alpha}\right)$$

for some constant $\alpha > 3$, uniformly for $r \in \mathbb{N}$.

Another possibility is to introduce coefficients $(c_k)_{k \geq 1}$ and consider

$$(c_k \cos 2\pi n_k x)_{k \geq 1} \quad \text{or} \quad (c_k f(n_k x))_{k \geq 1}$$

instead of (1). In 1959 Weiss [13] proved the following: Let

$$B_N = \left(\frac{1}{2} \sum_{k=1}^{N} c_k^2 \right)^{1/2}, \qquad N \geq 1,$$

and assume $B_N \to \infty$ as $N \to \infty$. Then

$$\limsup_{N \to \infty} \frac{\sum_{k=1}^N c_k \cos 2\pi n_k x}{\sqrt{2B_N^2 \log\log B_N^2}} = 1 \quad \text{a.e.,} \tag{7}$$

provided

$$c_k = \mathcal{O}\left(\frac{B_N}{\sqrt{\log\log B_N}} \right) \quad \text{as} \quad N \to \infty,$$

in perfect analogy with Kolmogoroff's result (the upper bound in (7) was already shown in 1950 by Salem and Zygmund [11]).

It is reasonable to assume that a result similar to (7) should also hold if the function $\cos 2\pi x$ is replaced by $f(x)$ for f satisfying (6). In his famous paper [10] from 1975 entitled "Limit theorems for lacunary series and uniform distribution mod 1" Walter Philipp stated the problem in the following form:

Give a detailed proof that

$$\limsup_{N \to \infty} \frac{\sum_{k=1}^N c_k f(n_k x)}{\sqrt{2B_N^2 \log\log B_N^2}} \ll 1 \quad \text{a.e.,} \tag{8}$$

where

$$B_N = \left(\sum_{k=1}^{N} c_k^2 \right)^{1/2} \to \infty,$$
$$c_N = o\left(\frac{B_N}{\sqrt{\log\log B_N}} \right). \tag{9}$$

The purpose of this paper is to give a partial solution of the problem, and to verify (8) under a condition slightly stronger than (9):

Theorem 1 *Let $f(x)$ be a function satisfying (6), and let $(n_k)_{k\geq 1}$ be a lacunary sequence of integers. Then*

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} c_k f(n_k x)}{\sqrt{2B_N^2 \log\log B_N^2}} \leq C \quad \text{a.e.},$$

for some constant C, provided

$$B_N = \left(\sum_{k=1}^{N} c_k^2 \right)^{1/2} \to \infty \tag{10}$$

and

$$c_N = \mathcal{O}\left(\frac{B_N}{(\log\log B_N)^{3/2}} \right). \tag{11}$$

In fact, we are not sure if the theorem would really remain true with (11) replaced by (9). The problem to find the best possible upper bound for c_N in (11) remains unsolved.
In view of (3) and (4) it is clear that no stronger result than (8), i.e. no exact LIL like in (2) and (7) can be expected in our case. Nevertheless, we know that the exact law of the iterated logarithm for $(f(n_k x))_{k\geq 1}$ for lacunary $(n_k)_{k\geq 1}$ and f satisfying (6) is valid in the form

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} f(n_k x)}{\sqrt{2N \log\log N}} = \|f\| \quad \text{a.e.},$$

provided the number of solutions of Diophantine equations of the type

$$a n_k \pm b n_l = c, \qquad a, b, c \in \mathbb{Z}, \quad k, l \leq N,$$

is "not too large" compared with N ([2]; cf. also [1], [3]). It is reasonable to assume that under a similar number-theoretic condition it is possible to prove an exact law of the iterated logarithm also for systems of the form $(c_k f(n_k x))_{k\geq 1}$.

2 Preliminaries

Without loss of generality we assume that f is an even function, i.e. the Fourier series of f can be written in the form

$$f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j x$$

(the proof in the general case is exactly the same). Since by assumption $\mathrm{Var}_{[0,1]}\ f \leq 2$ the Fourier coefficients of f satisfy

$$|a_j| \leq j^{-1}, \qquad j \geq 1 \tag{12}$$

(cf. Zygmund [14, p. 48]). We write $p(x)$ for the J-th partial sum of the Fourier series of f, and $r(x)$ for the remainder term, i.e.

$$p(x) = \sum_{j=1}^{J} a_j \cos 2\pi jx, \qquad r(x) = \sum_{j=J+1}^{\infty} a_j \cos 2\pi jx.$$

The value of J will be determined later. Throughout this section we will assume that N is fixed. For the function p we have

$$\|p\|_\infty \leq \|f\|_\infty + \mathrm{Var}_{[0,1]}\ f \leq 3, \tag{13}$$

by (4.12) of Chapter II and (1.25) and (3.5) of Chapter III of Zygmund [14], independent of J.

Lemma 1

$$\left\| \max_{1\leq M\leq N} \left| \sum_{k=1}^{M} c_k r(n_k x) \right| \right\| \ll B_N J^{-1/2}.$$

(here and in the sequel, the constant implied by the symbol "$\ll$" must not depend on N, J, but may depend on q, f).

Proof: By the orthogonality of the trigonometric system, (12), Minkowski's inequality and the Carleson-Hunt inequality (see e.g. Mozzochi [9] or Arias de Reyna [5]) we have

$$\begin{aligned}
& \left\| \max_{1\leq M\leq N} \left| \sum_{k=1}^{M} c_k r(n_k x) \right| \right\| \\
\leq & \sum_{i=0}^{\infty} \left\| \max_{1\leq M\leq N} \left| \sum_{k=1}^{M} c_k \sum_{j\in[Jq^i, Jq^{i+1})} a_j \cos(2\pi j n_k x) \right| \right\| \qquad (14) \\
\ll & \sum_{i=0}^{\infty} \left\| \sum_{k=1}^{M} c_k \sum_{j\in[Jq^i, Jq^{i+1})} a_j \cos(2\pi j n_k x) \right\| \\
\ll & \sum_{i=0}^{\infty} \left\| \sum_{k=1}^{M} c_k \sum_{j\in[Jq^i, Jq^{i+1})} j^{-1} \cos(2\pi j n_k x) \right\| \\
\ll & \frac{B_N}{\sqrt{J}} \sum_{i=0}^{\infty} q^{-i/2} \\
\ll & B_N J^{-1/2}.
\end{aligned}$$

Observe that the Carleson-Hunt inequality allows us to eliminate the "max" in (14). This is possible because splitting the Fourier series of $r(x)$ into parts containing only frequencies in one interval of the form $[Jq^i, Jq^{i+1})$ for some $i \geq 0$ guarantees that for $k_1 > k_2$ always

$j_1 n_{k_1} \geq J q^i n_{k_1} > J q^i q n_{k_2} \geq J q^{i+1} n_{k_2}$, provided $j_1, j_2 \in [Jq^i, Jq^{i+1})$ for some $i \geq 0$. $\square$

Now we choose

$$J = J(N) = \lceil \log B_N \rceil^6. \tag{15}$$

As a consequence of Lemma 1 we easily get

Lemma 2

$$\mathbb{P}\left\{ x \in (0,1) : \max_{1 \leq M \leq N} \left| \sum_{k=1}^{M} r(n_k x) \right| > B_N \right\} \ll J^{-1} \ll (\log B_N)^{-6}.$$

Here and in the sequel, $\mathbb{P}$ denotes the Lebesgue-measure on $(0,1)$.

3 Exponential inequality

We still assume that N is fixed. By (11) there exists a constant C_1 such that

$$c_N \leq \frac{C_1 B_N}{(\log\log B_N)^{3/2}}, \qquad N \geq 1. \tag{16}$$

By the choice of J in (15) it is possible to find a "small" number C_2, which must not depend on N, J, such that

$$\frac{C_2 \sqrt{\log\log B_N}}{B_N} \leq \frac{(\log\log B_N)^{3/2}}{6 C_1 B_N} \frac{1}{\lceil \log_q (4J) \rceil}. \tag{17}$$

We write $e(x) = e^x$.

Lemma 3

$$\int_0^1 e\left(\lambda \sum_{k=1}^{N} c_k p(n_k x) \right) \, dx \leq e\left(\lambda^2 C B_N^2 \right),$$

for some number C (independent of J, N), provided

$$0 \leq \lambda \leq \frac{C_2 \sqrt{\log\log B_N}}{B_N}. \tag{18}$$

Proof: Divide the integers $1, 2, \ldots, N$ into blocks $\Delta_1, \Delta_2, \ldots, \Delta_w$ (for some appropriate w), such that every block contains $\lceil \log_q(4J) \rceil$ numbers (the last block may contain less), i.e. $\Delta_1 = \{1, \ldots, \lceil \log_q(4J) \rceil\}$, $\Delta_2 = \{\lceil \log_q(4J) \rceil + 1, \ldots, 2\lceil \log_q(4J) \rceil\}$, ... Write

$$\begin{aligned} I_1 &= \int_0^1 e\left(2\lambda \sum_{\substack{1 \leq i \leq w \\ i \text{ even}}} \sum_{k \in \Delta_i} c_k p(n_k x) \right) dx \\ I_2 &= \int_0^1 e\left(2\lambda \sum_{\substack{1 \leq i \leq w \\ i \text{ odd}}} \sum_{k \in \Delta_i} c_k p(n_k x) \right) dx. \end{aligned}$$

Then by the Cauchy-Schwarz-inequality

$$\int_0^1 e\left(\lambda \sum_{k=1}^{N} c_k p(n_k x)\right) \, dx \le (I_1 I_2)^{1/2}. \tag{19}$$

Writing

$$U_i = \sum_{k \in \Delta_i} c_k p(n_k x), \qquad i \ge 1,$$

we have

$$\begin{aligned} I_1 &= \int_0^1 \prod_{\substack{1 \le i \le w \\ i \text{ even}}} e\left(2\lambda \sum_{k \in \Delta_i} c_k p(n_k x)\right) \, dx \\ &\le \int_0^1 \prod_{\substack{1 \le i \le w \\ i \text{ even}}} \left(1 + 2\lambda \sum_{k \in \Delta_i} c_k p(n_k x) + 4\lambda^2 \left(\sum_{k \in \Delta_i} c_k p(n_k x)\right)^2\right) \, dx \\ &= \int_0^1 \prod_{\substack{1 \le i \le w \\ i \text{ even}}} \left(1 + 2\lambda U_i + 4\lambda^2 U_i^2\right) \, dx, \end{aligned} \tag{20}$$

where we used the inequality

$$e^x \le 1 + x + x^2, \qquad \text{valid for } |x| \le 1,$$

and the fact that (11), (13), (17) and (18) imply

$$\begin{aligned} |2\lambda U_i| &\le 2\lambda \left| \sum_{k \in \Delta_i} c_k \|p\|_\infty \right| \\ &\le 6\lambda |\Delta_i| \left(\max_{k \in \Delta_i} |c_k| \right) \\ &\le 6\lambda \lceil \log_q (4J) \rceil \frac{C_1 B_N}{(\log \log B_N)^{3/2}} \\ &\le 1 \end{aligned}$$

($|\Delta_i|$ denotes the number of elements in Δ_i). Now

$$\begin{aligned} U_i^2 &= \left(\sum_{k \in \Delta_i} c_k \sum_{j=1}^{J} a_j \cos 2\pi j n_k x \right)^2 \\ &= \sum_{k_1, k_2 \in \Delta_i} \sum_{1 \le j_1, j_2 \le J} c_{k_1} c_{k_2} a_{j_1} a_{j_2} \\ &\qquad \left(\frac{\cos 2\pi (j_1 n_{k_1} + j_2 n_{k_2}) x + \cos 2\pi (j_1 n_{k_1} - j_2 n_{k_2}) x}{2} \right) \qquad (21) \\ &= V_i + W_i, \qquad (22) \end{aligned}$$

say, where V_i is a sum of trigonometric functions having frequencies in $[n_i^-, 2Jn_i^+]$, and W_i is a sum of trigonometric functions having frequencies in $[0, n_i^-)$ (here n_i^- denotes the smallest and n_i^+ the largest number in Δ_i). No other frequencies can occur, since the largest possible frequency in (21) is

$$Jn_i^+ + Jn_i^+ = 2Jn_i^+.$$

We note that the frequencies of the trigonometric functions in U_i are also in the interval $[n_i^-, 2Jn_i^+]$, and write

$$X_i = 2\lambda U_i + V_i \tag{23}$$

Using Minkowski's inequality we have

$$\begin{aligned}
W_i &\leq \frac{1}{2} \underbrace{\sum_{k_1,k_2 \in \Delta_i} \sum_{1 \leq j_1, j_2 \leq J}}_{|j_1 n_{k_1} - j_2 n_{k_2}| < n_i^-} c_{k_1} c_{k_2} a_{j_1} a_{j_2} \\
&\leq \underbrace{\sum_{k_1,k_2 \in \Delta_i, k_1 \leq k_2} \sum_{1 \leq j_1, j_2 \leq J}}_{j_1 > j_2 n_{k_2}/n_{k_1} - 1} c_{k_1} c_{k_2} \frac{1}{j_1 j_2} \\
&\leq \sum_{k_1,k_2 \in \Delta_i, k_1 \leq k_2} \sum_{j=1}^{J} c_{k_1} c_{k_2} \frac{2 n_{k_1}}{j^2 n_{k_2}} \\
&\leq \frac{2\pi^2}{6} \sum_{k \in \Delta_i} c_k^2 \sum_{v=0}^{\infty} \frac{1}{q^v} \\
&\leq \frac{4q}{q-1} \sum_{k \in \Delta_i} c_k^2.
\end{aligned} \tag{24}$$

Let $i_1 < i_2$ be two distinct even numbers. Then the frequency of any trigonometric function in X_{i_2} is at least twice as large as the frequency of any trigonometric function in X_{i_1}. In fact, the largest trigonometric function in W_{i_1} is at most $2Jn_{i_1}^+$, and the smallest trigonometric function in X_{i_2} at least $n_{i_2}^-$, and since

$$\min\{k \in \Delta_{i_2}\} - \max\{k \in \Delta_{i_1}\} \geq \lceil \log_q(4J) \rceil$$

we have

$$n_{i_2}^- > q^{\lceil \log_q(4J) \rceil} n_{i_1}^+ \geq 4Jn_{i_1}^+.$$

This implies that for any distinct $i_1, \ldots, \ldots, i_v$ (v is arbitrary), all even, the functions $X_{i_1}, \ldots, X_{i_v}$ are multiplicatively orthogonal, i.e.

$$\int_0^1 X_{i_1} \cdot \dots \cdot X_{i_v} \, dx = 0. \tag{25}$$

From (20), (22), (23), (24) and (25) we conclude

$$\begin{aligned}
I_1 &\leq \int_0^1 \prod_{\substack{1 \leq i \leq w \\ i \text{ even}}} \left(1 + X_i + 4\lambda^2 W_i\right) dx \\
&\leq \int_0^1 \prod_{\substack{1 \leq i \leq w \\ i \text{ even}}} \left(1 + X_i + \frac{16\lambda^2 q}{q-1} \sum_{k \in \Delta_i} c_k^2 \right) dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \prod_{\substack{1\leq i\leq w\\ i \text{ even}}} \left(1+\frac{16\lambda^2 q}{q-1}\sum_{k\in\Delta_i} c_k^2\right) dx\\
&\leq \int_0^1 \prod_{\substack{1\leq i\leq w\\ i \text{ even}}} e\left(\frac{16\lambda^2 q}{q-1}\sum_{k\in\Delta_i} c_k^2\right) dx\\
&= e\left(\sum_{\substack{1\leq i\leq w\\ i \text{ even}}} \frac{16\lambda^2 q}{q-1}\sum_{k\in\Delta_i} c_k^2\right)
\end{aligned}
$$

A similar estimate for I_2 can be obtained in the same way, and finally (19) yields

$$
\begin{aligned}
&\int_0^1 e\left(\lambda\sum_{k=1}^N c_k p(n_k x)\right) dx\\
&\leq \left(e\left(\sum_{\substack{1\leq i\leq w\\ i \text{ even}}} \frac{16\lambda^2 q}{q-1}\sum_{k\in\Delta_i} c_k^2\right) e\left(\sum_{\substack{1\leq i\leq w\\ i \text{ odd}}} \frac{16\lambda^2 q}{q-1}\sum_{k\in\Delta_i} c_k^2\right)\right)^{1/2}\\
&= e\left(\sum_{1\leq i\leq w} \frac{8\lambda^2 q}{q-1}\sum_{k\in\Delta_i} c_k^2\right)\\
&= e\left(\frac{8\lambda^2 q}{q-1} B_N^2\right),
\end{aligned}
$$

which proves the lemma. $\square$

Lemma 4 *There exists a "large" number C_3, independent of J, N, such that*

$$
\mathbb{P}\left\{\left|\sum_{k=1}^N p(n_k x)\right| > C_3\sqrt{B_N^2 \log\log B_N}\right\} \leq 2(\log B_N)^{-6}
$$

Proof: In Lemma 3 we choose

$$
\lambda = \frac{C_2\sqrt{\log\log B_N}}{B_N},
$$

which is consistent with (18), and get

$$
\begin{aligned}
&\mathbb{P}\left\{\sum_{k=1}^N p(n_k x) > C_3\sqrt{B_N^2\log\log B_N^2}\right\}\\
&= \mathbb{P}\left\{e\left(\lambda\sum_{k=1}^N p(n_k x)\right) > e\left(\lambda C_3\sqrt{B_N^2\log\log B_N^2}\right)\right\}\\
&\leq e\left(\frac{8\lambda^2 q}{q-1}B_N^2 - \lambda C_3\sqrt{B_N^2\log\log B_N^2}\right)\\
&= e\left(\frac{8qC_2^2\log\log B_N}{q-1} - C_2C_3\log\log B_N\right)
\end{aligned}
$$

$$\begin{aligned} &\le e\left(-6\log\log B_N\right) \\ &= (\log B_N)^{-6} \end{aligned} \tag{26}$$

for sufficiently large C_3. A results similar to Lemma 3 is possible for $-\sum_{k=1}^N p(n_k x)$ instead of $\sum_{k=1}^N p(n_k x)$, which yields

$$\mathbb{P}\left\{-\sum_{k=1}^N p(n_k x) > C_3\sqrt{B_N^2 \log\log B_N^2}\right\} \le (\log B_N)^{-6}. \tag{27}$$

Combining (26) and (27) we get the lemma. $\square$

4 Proof of Theorem 1

We define a sequence $N_1, N_2, \dots$ recursively in the following way:
Let

$$N_1 = 1,$$

and for $m \ge 1$ let

$$N_{m+1} = \begin{cases} N_m + 1 & \text{if } c_{N_m+1}^2 \ge 2^{(m^{1/3})} \\ \max\left\{M > N_m : \sum_{k=N_m+1}^M c_k^2 < 2^{(m^{1/3})}\right\} & \text{otherwise} \end{cases}$$

This means that always

$$\sum_{k=N_m+1}^{N_{m+1}-1} c_k^2 < 2^{(m^{1/3})} \qquad \text{and} \qquad \sum_{k=N_m+1}^{N_{m+1}} c_k^2 \ge 2^{(m^{1/3})} \tag{28}$$

(the sum on the left side may be over an empty index set), and in particular

$$B_{N_m}^2 \ge \sum_{v=1}^m 2^{(v^{1/3})} \gg 2^{(m^{1/3})} m^{2/3}. \tag{29}$$

Also, (16) guarantees, together with (28), that

$$B_{N_{m+1}}^2 \le B_{N_m}^2 + 2^{(m^{1/3})} + \frac{C_1 B_{N_{m+1}}}{(\log\log B_{N_{m+1}})^{3/2}}$$

which in particular implies

$$B_{N_m}^2 \ll 2^{(m^{1/3})} m^{2/3}$$

and

$$\frac{B_{N_{m+1}}}{B_{N_m}} \to 1 \quad \text{as} \quad m \to \infty. \tag{30}$$

For $C_4 > C_3 - 3$ we apply the results from the previous two sections (for $N = N_{m+1}$) and get

$$\mathbb{P}\left\{\max_{N_m+1 \le M \le N_{m+1}} \left|\sum_{k=1}^M c_k f(n_k x)\right| > C_4\sqrt{B_{N_m}^2 \log\log B_{N_m}^2}\right\} \tag{31}$$

$$
\begin{aligned}
&\leq \mathbb{P}\left\{\max_{N_m+1\leq M\leq N_{m+1}}\left|\sum_{k=1}^{M} c_k p(n_k x)\right| > (C_4-1)\sqrt{B_{N_m}^2 \log\log B_{N_m}^2}\right\} \\
&\quad +\mathbb{P}\left\{\max_{N_m+1\leq M\leq N_{m+1}}\left|\sum_{k=1}^{M} c_k r(n_k x)\right| > B_{N_m}\right\} \qquad (32)\\
&\leq \mathbb{P}\left\{\left|\sum_{k=1}^{N_m} c_k p(n_k x)\right| > (C_4-3)\sqrt{B_{N_m}^2 \log\log B_{N_m}^2}\right\} \qquad (33)\\
&\quad +\mathbb{P}\left\{\max_{N_m+1\leq M\leq N_{m+1}-1}\left|\sum_{k=N_m+1}^{M} c_k p(n_k x)\right| > \sqrt{B_{N_m}^2 \log\log B_{N_m}^2}\right\} \\
&\quad +\mathbb{P}\left\{\left|c_{N_{m+1}} p(n_k x)\right| > \sqrt{B_{N_m}^2 \log\log B_{N_m}^2}\right\} \qquad (34)\\
&\quad +(\log B_{N_m})^{-6} \\
&\ll \mathbb{P}\left\{\max_{N_m+1\leq M\leq N_{m+1}-1}\left|\sum_{k=N_m+1}^{M} c_k p(n_k x)\right| > \sqrt{B_{N_m}^2 \log\log B_{N_m}^2}\right\} \qquad (35)\\
&\quad +(\log B_{N_m})^{-6}.
\end{aligned}
$$

Here we used Lemma 2 to estimate (32), Lemma 4 to estimate (33), and (34) vanishes for sufficiently large m because of (11) and (30). It remains to find an appropriate estimate for (35). We have

$$
\begin{aligned}
&\left\|\max_{N_m+1\leq M\leq N_{m+1}-1}\left|\sum_{k=N_m+1}^{M} c_k p(n_k x)\right|\right\|^4 \\
= &\left\|\max_{N_m+1\leq M\leq N_{m+1}-1}\left|\sum_{k=N_m+1}^{M} c_k \sum_{j=1}^{\lceil \log B_{N_{m+1}}\rceil^6} a_j \cos(2\pi j n_k x)\right|\right\|^4 \\
\leq &\sum_{j=1}^{\lceil \log B_{N_{m+1}}\rceil^6} |a_j| \left\|\max_{N_m+1\leq M\leq N_{m+1}-1}\left|\sum_{k=N_m+1}^{M} c_k \cos(2\pi j n_k x)\right|\right\|^4 \\
\leq &\sum_{j=1}^{\lceil \log B_{N_{m+1}}\rceil^6} \frac{1}{j} \left\|\max_{N_m+1\leq M\leq N_{m+1}-1}\left|\sum_{k=N_m+1}^{M} c_k \cos(2\pi n_k x)\right|\right\|^4 \\
\ll &\log\log B_{N_{m+1}} \left\|\max_{N_m+1\leq M\leq N_{m+1}-1}\left|\sum_{k=N_m+1}^{M} c_k \cos(2\pi n_k x)\right|\right\|^4 \\
\ll &\log\log B_{N_{m+1}} \left\|\sum_{s=0}^{\lceil \log_q 2\rceil-1} \max_{N_m+1\leq M\leq N_{m+1}-1}\left|\sum_{\substack{N_m+1\leq k\leq M\\ k\equiv s \mod \lceil \log_q 2\rceil}} c_k \cos(2\pi n_k x)\right|\right\|^4
\end{aligned}
$$

$$\ll \log\log B_{N_{m+1}} \sum_{s=0}^{\lceil \log_q 2\rceil -1} \left\| \max_{N_m+1 \le M \le N_{m+1}-1} \left| \sum_{\substack{N_m+1 \le k \le M \\ k \equiv s \mod \lceil \log_q 2\rceil}} c_k \cos(2\pi n_k x) \right| \right\|^4$$

$$\ll \log\log B_{N_{m+1}} \sum_{s=0}^{\lceil \log_q 2\rceil -1} \left\| \sum_{\substack{N_m+1 \le k \le N_{m+1}-1 \\ k \equiv s \mod \lceil \log_q 2\rceil}} c_k \cos(2\pi n_k x) \right\|^4, \tag{36}$$

where the last estimate follows from the Carleson-Hunt inequality. If two distinct integers k_1, k_2 are in the same residue class (mod $\lceil \log_q 2\rceil$) then necessarily $n_{k_1}/n_{k_2} \notin [1/2, 2]$. Thus

$$\begin{aligned}
&\int_0^1 \left(\sum_{\substack{N_m+1 \le k \le N_{m+1}-1 \\ k \equiv s \mod \lceil \log_q 2\rceil}} c_k \cos(2\pi n_k x) \right)^4 dx \\
\ll\ & \sum_{\substack{N_m+1 \le k_1,k_2,k_3,k_4 \le N_{m+1}-1 \\ k_1,k_2,k_3,k_4 \equiv s \mod \lceil \log_q 2\rceil}} c_{k_1} c_{k_2} c_{k_3} c_{k_4} \cdot \mathbb{1}(n_{k_1} \pm n_{k_2} \pm n_{k_3} \pm n_{k_4} = 0) \\
\ll\ & \sum_{\substack{N_m+1 \le k_1,k_2,k_3,k_4 \le N_{m+1}-1 \\ k_1,k_2,k_3,k_4 \equiv s \mod \lceil \log_q 2\rceil}} c_{k_1} c_{k_2} c_{k_3} c_{k_4} \cdot \mathbb{1}(n_{k_1} - n_{k_2} = 0) \cdot \mathbb{1}(n_{k_3} - n_{k_4} = 0) \\
\ll\ & \sum_{\substack{N_m+1 \le k_1,k_2 \le N_{m+1}-1 \\ k_1,k_2 \equiv s \mod \lceil \log_q 2\rceil}} c_{k_1}^2 c_{k_2}^2 \\
\ll\ & \left(\sum_{\substack{N_m+1 \le k \le N_{m+1}-1 \\ k \equiv s \mod \lceil \log_q 2\rceil}} c_k^2 \right)^2 \\
\ll\ & \left(\sum_{N_m+1 \le k \le N_{m+1}-1} c_k^2 \right)^2 \\
\ll\ & \left(2^{(m^{1/3})} \right)^2 \\
\ll\ & \left(\frac{B_{N_m}^2}{m^{2/3}} \right)^2
\end{aligned}$$

by the definition of N_m, N_{m+1}, which implies, in view of (30) and (36),

$$\left\| \max_{N_m+1 \le M \le N_{m+1}-1} \left| \sum_{k=N_m+1}^{M} c_k p(n_k x) \right| \right\|^4 \ll \frac{B_{N_m} \log\log B_{N_m}}{m^{1/3}}.$$

Thus

$$\mathbb{P}\left\{\max_{N_m+1\le M\le N_{m+1}-1}\left|\sum_{k=N_m+1}^{M} c_k p(n_k x)\right| > \sqrt{B_{N_m}^2 \log\log B_{N_m}^2}\right\}$$

$$\ll \frac{(\log\log B_{N_m})^2}{\left(m^{1/3}\right)^4} \ll (\log m)^2 m^{-4/3}. \tag{37}$$

Writing A_m for the set in (31), $m \ge 1$, i.e.

$$A_m = \left\{x \in (0,1) : \max_{N_m+1\le M\le N_{m+1}}\left|\sum_{k=1}^{M} c_k f(n_k x)\right| > C_4\sqrt{B_{N_m}^2 \log\log B_{N_m}^2}\right\},$$

by (35) and (37) we have

$$\mathbb{P}(A_m) \ll (\log B_{N_m})^{-6} + (\log m)^2 m^{-4/3} \ll \left(m^{1/3}\right)^{-6} + (\log m)^2 m^{-4/3}$$

and thus

$$\sum_{m=1}^{\infty} \mathbb{P}(A_m) < \infty.$$

Therefore the Borel-Cantelli-lemma implies that

$$\max_{N_m+1\le M\le N_{m+1}-1}\left|\sum_{k=1}^{M} c_k f(n_k x)\right| > C_4\sqrt{B_{N_m}^2 \log\log B_{N_m}^2}$$

for at most finitely many m for all $x \in (0,1)$, except a set of measure zero. Thus, in view of (30), if $C_5 > C_4$ we also have

$$\left|\sum_{k=1}^{N} c_k f(n_k x)\right| > C_5\sqrt{B_N^2 \log\log B_N^2}$$

for only finitely many values of N, again for all $x \in (0,1)$ except a set of measure zero. Therefore we have

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} c_k f(n_k x)\right|}{C_5\sqrt{B_N^2 \log\log B_N^2}} \le 1 \qquad \text{a.e.},$$

which proves Theorem 1.

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