SOME REMARKS ON STRONGLY COMPACT SPACES AND SEMI COMPACT SPACES

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Abstract

We consider two rather strong conditions on topological spaces and use them to characterize strongly compact spaces and semi compact spaces. As a consequence we obtain that there exist no infinite spaces which are both strongly compact and semi compact.

1 Introduction

Let (X, τ) be a topological space and let A be a subset of X . We denote the closure of A (resp. the interior of A) by clA (resp. $intA$) . A subset S of (X, τ) is called *semi–open* (resp. *preopen, somewhat preopen)* if $S \subseteq cl(intS)$ (resp. $S \subseteq int(clS)$, $int(clS) \neq \emptyset$) . These notions were introduced by Levine[9], Mashhour et al. [10] and Piotrowski [12], respectively. Piotrowski used the term "somewhat nearly open" instead of "somewhat preopen". A space (X, τ) is said to be *semi compact* (resp. *strongly compact*) if every cover of X by semi–open (resp. preopen) sets has a finite subcover. Semi compactness was studied by Dorsett [5] , [6] and [7] , while the concept of strong compactness is due to Mashhour et al. [11] .

2 Strong Compactness

Definition 1 A space (X, τ) is said to satisfy condition $(C1)$ if every infinite subset of X has nonempty interior.

The following result is easily established. Its proof is hence omitted.

Proposition 2.1 For a space (X, τ) the following are equivalent:

(1) (X, τ) satisfies $(C1)$.

(2) For any $A \subseteq X$, if $intA = \emptyset$ then A is finite.

(3) For any $A \subseteq X$, $A \setminus intA$ is finite.

(4) For any $A \subseteq X$, $clA \setminus A$ is finite.

It is clear that every finite space and every discrete space satisfies $(C1)$. Our next result shows that spaces satisfying $(C1)$ are not far away from being discrete.

Theorem 2.2 For a space (X, τ) let I_X be the set of isolated points of (X, τ) . Then (X, τ) satisfies $(C1)$ if and only if $X \setminus I_X$ is finite.

Proof. It is obvious that, if $X \setminus I_X$ is finite then (X, τ) satisfies $(C1)$. To prove the converse, let $A = X \setminus I_X$ and suppose that A is infinite. Then A can be represented as a disjoint union $A = \bigcup\{A_n \ : \ n \in \mathbb{N}\}$ where each A_n is infinite. Since (X, τ) satisfies $(C1)$ there is a point $a_n \in intA_n$ for each $n \in \mathbb{N}$. If $B = \{a_n \ : \ n \in \mathbb{N}\}$, then B is infinite and hence there exists an $m \in \mathbb{N}$ such that $a_m \in intB$. Clearly $intB \cap intA_m = \{a_m\}$ so that $a_m \in I_X$, contradicting the fact that $a_m \in A$. $\square$

Remark 2.3 The previous result has been obtained independently also by Jankovic, Reilly and Vamanamurthy [8] who used a completely different approach.

Recall that a space (X, τ) is said to be *quasi H–closed* if every open cover of X has a finite subfamily the closures of whose members cover X .

Theorem 2.4 For a space (X, τ) the following are equivalent:

(1) (X, τ) is strongly compact.

(2) (X, τ) is compact and satisfies $(C1)$.

(3) (X, τ) is quasi H–closed and satisfies $(C1)$.

Proof. $(1) \Rightarrow (2)$: It is obvious that every strongly compact space is compact. Let $A \subseteq X$ such that $intA = \emptyset$, i.e. $X \setminus A$ is dense. For each $x \in A$, if $S_x = (X \setminus A) \cup \{x\}$ then S_x is preopen. By assumption, the preopen cover $\{S_x \ : \ x \in A$ has a finite subcover. This shows that A is finite and (X, τ) satisfies $(C1)$ by Proposition 2.1 .

$(2) \Rightarrow (3)$ is obvious.

$(3) \Rightarrow (1)$: Let $\{S_\alpha \ : \ \alpha \in I\}$ be a preopen cover of (X, τ) . Then $\{int(clS_\alpha) \ : \ \alpha \in I\}$ is an open cover of (X, τ) . Since (X, τ) is quasi H–closed there is a finite subset $I' \subseteq I$ such that $X = \bigcup\{clS_\alpha \ : \ \alpha \in I'\}$. By $(C1)$, $clS_\alpha \setminus S_\alpha$ is finite for each $\alpha \in I'$. Hence there is a finite subset F of X such that $X = \bigcup\{S_\alpha \ : \ \alpha \in I'\} \cup F$. This shows that (X, τ) is strongly compact. $\square$

Corollary 2.5 The 1–point–compactification of any discrete space is strongly compact.

3 Semi Compactness

We now proceed by considering a less restrictive condition on topological spaces.

Definition 2 A space (X, τ) is said to satisfy condition $(C2)$ if every infinite subset is somewhat preopen.

It is clear that condition $(C1)$ is stronger than condition $(C2)$. The cofinite topology τ on an infinite set S provides an example of a space (X, τ) which satisfies $(C2)$ but not $(C1)$.

The proof of the following result is straightforward and hence omitted.

Proposition 3.1 For a space (X, τ) the following are equivalent:

(1) (X, τ) satisfies $(C2)$.

(2) For any $A \subseteq X$, if $int(clA) = \emptyset$ then A is finite.

(3) For any open set $U \subseteq X$, $clU \setminus U$ is finite.

(4) For any $A \subseteq X$, $clA \setminus int(clA)$ is finite.

In analogy to the well known "countable chain condition" in General Topology we say that space (X, τ) satisfies the "*finite chain condition*", abbreviated FCC, if every disjoint family of nonempty open sets is finite. Dorsett's [5] characterization of semi compactness may then be stated in the following form.

Theorem 3.2 [5] A space (X, τ) is semi compact if and only if it satisfies both $(C2)$ and FCC.

We are now going to improve Theorem 3.2 . Recall that a space (X, τ) is *S–closed* [13] if every semi–open cover of (X, τ) has a finite subfamily the closures of whose members cover X . Cameron [4] has shown (X, τ) is S–closed if and only if every cover of X by regular closed subsets has a finite subcover, where a subset S of X is called *regular closed* if $S = cl(intS)$.

Proposition 3.3 Every space (X, τ) which satisfies FCC is S–closed.

Proof. Suppose that (X, τ) satisfies FCC and there is a regular closed cover $\{F_\alpha : \alpha \in I\}$ of (X, τ) having no finite subcover. By induction we shall construct a sequence $(\alpha_n) \subseteq I$ and a disjoint family $\{U_n : n \in \mathbb{N}\}$ of nonempty open sets such that $U_n \subseteq F_{\alpha_n}$ and $U_n \cap (F_{\alpha_1} \cup ... F_{\alpha_{n-1}}) = \emptyset$ for each $n \in \mathbb{N}$. Pick $\alpha_1 \in I$ such that F_{α_1} is nonempty and let $U_1 = intF_{\alpha_1}$. Given $\{\alpha_i : 1 \leq i \leq n\}$ and nonempty disjoint open sets $\{U_i : 1 \leq i \leq n\}$ such that $U_i \subseteq F_{\alpha_i}$ and $U_i \cap (F_{\alpha_1} \cup ... F_{\alpha_{i-1}}) = \emptyset$ for each $1 < i \leq n$, we observe that there is an $\alpha_{n+1} \in I$ such that $(X \setminus (F_{\alpha_1} \cup ... \cup F_{\alpha_n})) \cap intF_{\alpha_{n+1}}$ is nonempty. Let $U_{n+1} = (X \setminus (F_{\alpha_1} \cup ... \cup F_{\alpha_n})) \cap intF_{\alpha_{n+1}}$. This produces an infinite family of nonempty disjoint open sets contradicting the fact that (X, τ) satisfies FCC. $\Box$

Remark 3.4 The converse of Proposition 3.3 is false. The Stone Cech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ is S–closed [13] and $\{\ \{n\} : n \in \mathbb{N}\}$ is an infinite family of nonempty disjoint open sets. Thus $\beta\mathbb{N}$ does not satisfy FCC .

Question. Under what circumstances does an S–closed space satisfy FCC ?

In view of Proposition 3.3 the following result is an improvement of Theorem 3.2 .

Theorem 3.5 A topological space (X, τ) is semi compact if and only if it is S–closed and satisfies $(C2)$.

Proof. The 'only if' part follows from Theorem 3.2 and Proposition 3.3 . Assume that (X, τ) is S–closed and satisfies $(C2)$, and let $\{S_\alpha\ :\ \alpha \in I\}$ be a semi–open cover of (X, τ) . By the S–closedness there is a finite subset $I' \subseteq I$ such that $X = \bigcup\{clS_\alpha\ :\ \alpha \in I'\}$. Since (X, τ) satisfies $(C2)$ and $clS_\alpha \setminus S_\alpha = cl(intS_\alpha) \setminus S_\alpha \subseteq cl(intS_\alpha) \setminus intS_\alpha$, we have that $clS_\alpha \setminus S_\alpha$ is finite for each $\alpha \in I'$. Hence there is a finite subset $F \subseteq X$ such that $X = \bigcup\{S_\alpha\ :\ \alpha \in I'\} \cup F$. This shows that (X, τ) is semi compact. $\square$

4 Concluding Remark

Following Abd El-Monsef et al. [1] , a subset S of a space (X, τ) is called *β–open* if $S \subseteq cl(int(clS))$. β–open sets have been called *semi–preopen* by Andrijevic [3] . A space (X, τ) is said to be β–compact [2] if every cover of (X, τ) by β–open sets has a finite subcover. Since preopen sets and semi–open sets are clearly β–open, every β–compact space has to be strongly compact and semi–compact. Now, if (X, τ) is an infinite strongly compact space, the set I_X of isolated points of (X, τ) is finite by Theorem 2.2. It follows that (X, τ) does not satisfy FCC and so is not semi compact by Theorem 3.2 . Consequently, infinite β–compact spaces do not exist.

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