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ASYMPTOTIC BEHAVIOUR OF THE POLES OF A SPECIAL GENERATING FUNCTION FOR ACYCLIC DIGRAPHS

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ABSTRACT. Let z_k be the k-th zero of $\phi(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!2^{\binom{n}{2}}}$ sorted increasingly by modulus from the origin, for $k \geq 0$. $\frac{1}{\phi}$ appears as special generating function for acyclic digraphs and ϕ satisfies the functional differential equation $\phi'(z) = -\phi\left(\frac{z}{2}\right)$. It is a conjecture of R.W. Robinson that $z_k = (k+1)2^k + o(2^k)$. We show that there is a K such that $z_{K+k} = (k+1)2^k + o\left(\frac{2^k}{(k+1)^{1-\epsilon}}\right)$ for all $\epsilon > 0$.

1. INTRODUCTION

Acyclic digraphs have been studied by several authors, e.g., Robinson [3],[4] and Stanley [5]. Let

$$S(z) = \sum_{n=0}^{\infty} \frac{s_n z^n}{n!}.$$

Robinson defined Δ in [3] to be the linear operation on exponential generating functions which divides x^n by $2^{\binom{n}{2}}$, i.e.,

$$\Delta S(z) = \sum_{n=0}^{\infty} \frac{s_n z^n}{n!2^{\binom{n}{2}}}.$$

He called $\Delta S(z)$ the *special generating function* for S, which he used, for example, to count labelled directed acyclic graphs in [3]. Let a_n be the number of labelled directed acyclic graphs with n vertices and

$$A(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}.$$

In this paper we will analyse the zeros of the function

$$\Delta e^{-z} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!2^{\binom{n}{2}}}$$

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which we denote by $\phi(z)$. Robinson observed that $\phi(z)$ satisfies the functional differential equation

$$\phi'(z) = -\phi\left(\frac{z}{2}\right)$$

and that

$$(\Delta e^{-z})(\Delta A(z)) = 1.$$

Figure 1, which we produced with `Mathematica`, shows a plot of this function. Let z_k be

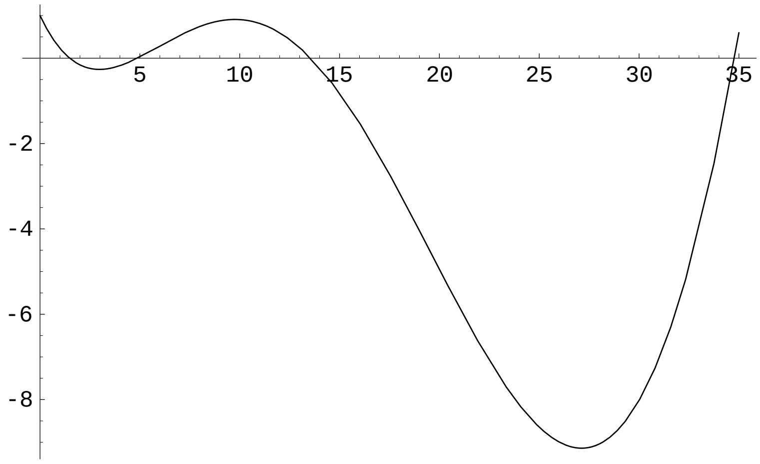

FIGURE 1. $\phi(z)$

the k-th zero of $\phi(z)$ sorted increasingly by modulus from the origin for $k \geq 0$. Robinson proved that all zeros of $\phi(z)$ are simple, positive real and satisfy $z_{k+1} > 2z_k$ for $k \geq 0$. Furthermore he conjectured that $z_k = (k+1)2^k + o(2^k)$ in [3]. We prove a slightly different result, namely Theorem 2 and present an asymptotic representation of the function ϕ in Theorem 1. Table 1 gives a comparison between the k-th zero z_k and $(k+1)2^k$.

2. The Zeros of $\phi(z)$

First we state

Lemma 1. *The function $\phi(z)$ can be written as*

$$\phi(z) = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{\pi}{\sin(\pi s)} \frac{1}{\Gamma(1-s) 2^{\frac{s(s+1)}{2}} z^s} ds,$$

where the Hankel-contour $\mathcal{H}$ encircles the negative integers and 0.

Proof. We observe

$$\phi(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! 2^{\binom{n}{2}}} = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{\Gamma(s)}{2^{\frac{s(s+1)}{2}} z^s} ds, \tag{1}$$

k	z_k	$2^{-k}z_k - (k+1)$
0	1.48807854559971029465...	0.488079
1	4.88114089489665468076...	0.44057
2	13.560408527963422630046...	0.390102
3	34.775316247750907131517...	0.346915
4	84.977290107207686550531...	0.311081
5	201.002876160804469905966...	0.28134
6	464.412828169898831201845...	0.25645
7	1054.131896685496993769517...	0.235405
8	2359.661041195481673546868...	0.217426
9	5223.380439502564290102436...	0.201915
10	11456.93506311994880902310...	0.188413
11	24937.60380074991930495522...	0.176564
12	53928.30177867352491973944...	0.166089
13	115972.233276771948755379770...	0.156767
14	248191.706915839202599726065...	0.14842
15	528905.163307529387654815292...	0.140905
16	1122900.700985837445143423214...	0.134105
17	2376063.281403420485104790828...	0.127924
18	5012791.625019875403953253170...	0.122283
19	10547160.909492859829473254757...	0.117113
20	22137913.099346041141374364701...	0.112359
21	46363780.401215272096183164933...	0.107973
22	96904841.306209099621461578720...	0.103915
23	202166693.910570234291360571678...	0.100148
24	421051803.671740123329889100282...	0.0966432
25	875548345.417406748367314625259...	0.0933741

TABLE 1. The zeros z_k compared to $(k+1)2^k$.

since

$$\operatorname{Res}\left(\Gamma(s)2^{-\frac{s(s+1)}{2}}z^{-s}, -n\right) = \frac{(-1)^n}{n!}2^{-\frac{n(n-1)}{2}}z^n, \text{ for } n \geq 0,$$

and Stirling's formula guarantees that the integral converges. The result follows with the functional equation $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$. □

We notice that the saddle point method does not work if we use (1), since there are oscillating terms, as we will see later.

We rewrite $\phi(z)$ with

$$w(s,t) = -\log\Gamma(1-s) - \frac{s(s+1)}{2}\log 2 - st$$

in the form

$$\phi(z) = \frac{1}{2\pi i}\int_{\mathcal{H}} \frac{\pi}{\sin(\pi s)} e^{w(s,\log z)}ds,$$

where log, in the whole paper, means the principal value of the complex logarithm. To simplify the following notation we write $x = \log z$ and

$$\tilde{\phi}(x) = \frac{1}{2\pi i}\int_{\mathcal{H}} \frac{\pi}{\sin(\pi s)} e^{w(s,x)}ds.$$

Since we will use the saddle point method, which is discussed for example in [1] or [2], in a slightly modified way, we are now interested in zeros of $\partial w(s,x)/\partial s$. First, we rewrite

$$\begin{aligned} w(s,x) &= -\log\Gamma(1-s) - \frac{s(s+1)}{2}\log 2 - sx \\ &= -\log((-s)\Gamma(-s)) - \frac{s(s+1)}{2}\log 2 - sx \\ &= -\log(-s) - \log\Gamma(-s) - \frac{s(s+1)}{2}\log 2 - sx \end{aligned}$$

at least if $\operatorname{Re} s < 0$, and it will turn out that this is all we need. Therefore, we search for solutions of

$$-\frac{1}{s} + \psi(-s) - \left(s + \frac{1}{2}\right)\log 2 - x = 0, \tag{2}$$

where $\psi(s)$ is the first derivative of $\log\Gamma(s)$. Let $\sigma(x)$ denote the solution of (2). Then the following asymptotic expansion holds.

We remark that throughout the whole article all error terms are considered as x goes to infinity.

Lemma 2. *The solution $\sigma(x)$ of the equation (2) satisfies*

$$\sigma(x) = \frac{1}{\log 2}\left(-x + \log x + C_1 + \frac{\log x}{x} + \frac{C_2}{x}\right) + O\left(\frac{\log^2 x}{x^2}\right), \text{ where}$$

$$C_1 = -\frac{1}{2} - \frac{\log\log 2}{2} \quad \text{and } C_2 = \frac{3}{2}\log 2 + \log\log 2.$$

Proof. We find the constants in

$$\sigma(x) = Ax + B\log x + C + \frac{D\log x}{x} + \frac{E}{x}$$

by inserting $\sigma(x)$ into (2). First, we use the asymptotic expansion

$$\psi(s) = \log s - \frac{1}{2s} + O\left(\frac{1}{s^2}\right)$$

to obtain

$$\log(-\sigma(x)) - \frac{1}{2\sigma(x)} + O\left(\frac{1}{\sigma(x)^2}\right) - \left(\sigma(x) + \frac{1}{2}\right)\log 2 - x = 0$$

from (2). Now, we have on the one hand

$$\begin{aligned}
&\log\left(Ax + B\log x + C + \frac{D\log x}{x} + \frac{E}{x}\right)\\
&= \log(Ax) + \frac{B\log x + C + \frac{D\log x}{x} + \frac{E}{x}}{Ax}\\
&+ O\left(\left(\frac{B\log x + C + \frac{D\log x}{x} + \frac{E}{x}}{Ax}\right)^2\right)\\
&= \log(Ax) + \frac{B\log x + C + \frac{D\log x}{x} + \frac{E}{x}}{Ax} + O\left(\left(\frac{\log x}{x}\right)^2\right),
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\frac{1}{2\left(Ax + B\log x + C + \frac{D\log x}{x} + \frac{E}{x}\right)}\\
&= \frac{1}{2Ax} + O\left(\frac{B\log x + C + \frac{D\log x}{x} + \frac{E}{x}}{(Ax)^2}\right) = \frac{1}{2Ax} + O\left(\frac{\log x}{x^2}\right).
\end{aligned}$$

The calculation of the constants is now straight forward. □

Lemma 3. *Let* $0 < \alpha < \frac{1}{3}$, $\beta = 2 - 3\alpha$, $\sigma = \sigma(x)$, $|t| < x^{\alpha}$, *and* $h(x) = \frac{\log 2}{2} - \frac{1}{2(Ax+B\log x)}$. *Then*

$$w(\sigma + t, x) = w(\sigma, x) - h(x)t^2 + O\left(\frac{1}{x^{\beta}}\right).$$

Proof. It is more convenient to write

$$w(s, x) = g(s) - xs \text{ where } g(s) = -\log(-s) + \log\Gamma(-s) - \frac{s(s+1)}{2}\log 2.$$

By Taylor expansion around σ we have

$$\begin{aligned}
w(\sigma + t, x) &= g(\sigma + t) - x(\sigma + t)\\
&= g(\sigma) - x\sigma + \frac{1}{2}g''(\sigma)t^2\\
&+ t^3\left(\frac{1}{6}g'''(\sigma) + \frac{1}{24}g''''(\sigma)t + \dots\right)\\
&= w(\sigma, x) + \left(-\frac{1}{\sigma} + \psi(-\sigma) - \left(\sigma + \frac{1}{2}\right)\log 2 - x\right)t\\
&+ \frac{1}{2}\left(\frac{1}{\sigma^2} - \psi'(-\sigma) - \log 2\right)t^2\\
&+ t^3\left(\frac{1}{6}g'''(\sigma) + \frac{1}{24}g''''(\sigma)t + \dots\right).
\end{aligned}$$

Since $\psi^{(n)}(\sigma) = O(1/\sigma^n)$, for $n \geq 1$, we have $g^{(n+1)}(\sigma) = O(1/\sigma^n)$ and therefore

$$t^3\left(\frac{1}{6}g'''(\sigma) + \frac{1}{24}g''''(\sigma)t + \dots\right) = O\left(\frac{t^3}{\sigma^2}\right).$$

Now, we use Lemma 2, $\psi'(\sigma) = 1/\sigma + O(1/\sigma^2)$, and $\psi''(\sigma) = -1/\sigma^2 + O(1/\sigma^3)$ to obtain

$$w(\sigma + t, x) = w(\sigma, x) + \left(-\frac{\log 2}{2} + \frac{1}{2\sigma} + O\left(\frac{1}{\sigma^2}\right)\right)t^2 + O\left(\frac{t^3}{\sigma^2}\right).$$

We use $|t| \leq x^\alpha$ to estimate the error term

$$O\left(\frac{t^3}{\sigma^2}\right) = O\left(\frac{1}{x^{2-3\alpha}}\right).$$

If we expand $\sigma = Ax + B\log x + O(1)$ we have

$$\left(\frac{1}{2\sigma} + O\left(\frac{1}{\sigma^2}\right)\right)t^2 = \frac{t^2}{2(Ax + B\log x)} + O\left(\frac{1}{x^{2-2\alpha}}\right).$$

Therefore, $1 < \beta = 2 - 3\alpha < 2 - 2\alpha < 2 - \alpha < 2$. □

Lemma 4. *The real part of* $w(\sigma(x) \pm i + t, x)$ *is*

$$\frac{x^2}{2\log 2} - \frac{x\log x}{\log 2} + x\left(\frac{1}{2} + \frac{1}{\log 2} - \frac{\log\log 2}{\log 2}\right) - t^2\left(\frac{\log 2}{2} + \frac{\log 2}{x}\right) + tO\left(\frac{\log x}{x}\right) + O\left((\log x)^2\right).$$

Proof. We use Stirling's formula to gain the asymptotic expansion

$$w(s, x) = \left(s - \frac{1}{2}\right)\log s - s + \frac{\log(2\pi)}{2} + \frac{1}{12s} - \frac{s(s+1)}{2}\log 2 - xs + O\left(\frac{1}{s^3}\right). \tag{3}$$

Using Lemma 2 and (3), we obtain $w(\sigma + t, x)$, a rather lengthy expression, we omit here. Similar as in the proof of Lemma 2, we substitute in this expression

$$\log(Ax) + \frac{B\log x + C + \frac{D\log x}{x} + \frac{E}{x} + t}{Ax} - \frac{1}{2}\left(\frac{B\log x + C + \frac{D\log x}{x} + \frac{E}{x} + t}{Ax}\right)^2 + O\left(\left(\frac{\log x}{x}\right)^3\right)$$

for $\log(\sigma + t)$ and

$$\frac{1}{Ax} + O\left(\frac{\log x}{x^2}\right)$$

for $1/(\sigma + t)$. We use `Mathematica` for the algebraic simplifications. □

Now, we let the curves $\mathcal{C}_1(t) = \sigma(x) + t$ and $\mathcal{C}_2(t) = \sigma(x) - t$, for $t \in [-x^\alpha, x^\alpha]$, where $0 < \alpha < \frac{1}{3}$ and

$$\tilde{\phi}_{\mathcal{C}_{1/2}}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_{1/2}} \frac{\pi}{\sin(\pi s)} e^{w(s,x)} ds.$$

Lemma 5.

$$\tilde{\phi}(x) = \tilde{\phi}_{\mathcal{C}_1}(x) + \tilde{\phi}_{\mathcal{C}_2}(x) + e^{w(\sigma(x),x)} O\left(\frac{1}{x^\beta}\right).$$

Proof. We define the Hankel contour $\mathcal{H}$ in Lemma 1 to consist of three curves H_1, H_2, and H_3, where $H_1(t) = t + i$ for $t \in (-\infty, 1]$, $H_2(t) = 1 - it$ for $t \in (-1, 1)$ and $H_3(t) = -t - i$ for $t \in (-1, \infty]$. Therefore, $\mathcal{H}$ circulates the negative integers and 0. We abbreviate

$$s(t) = \frac{\pi}{\sin(\pi(\sigma + t + i))},$$

and estimate

$$\begin{aligned}
&\frac{1}{2\pi} \left| \int_{-\infty}^{-x^\alpha} \frac{\pi}{\sin(\pi(\sigma + t + i))} e^{w(\sigma+t+i,x)} dt - \int_{-\infty}^{-x^\alpha} \frac{\pi}{\sin(\pi(\sigma + t - i))} e^{w(\sigma+t-i,x)} dt \right| \\
&= \frac{1}{2\pi} \left| \int_{-\infty}^{-x^\alpha} e^{\operatorname{Re} w(\sigma+t+i,x)} \left(s(t) e^{i \operatorname{Im} w(\sigma+t+i,x)} - \overline{s(t)} e^{-i \operatorname{Im} w(\sigma+t+i,x)} \right) dt \right| \\
&= \frac{1}{2\pi} \left| \int_{-\infty}^{-x^\alpha} e^{\operatorname{Re} w(\sigma+t+i,x)} \operatorname{Re}(s(t)) 2i \sin(\operatorname{Im} w(\sigma + t + i, x)) dt \right| \\
&+ \frac{1}{2\pi} \left| \int_{-\infty}^{-x^\alpha} e^{\operatorname{Re} w(\sigma+t+i,x)} \operatorname{Im}(s(t)) 2 \cos(\operatorname{Im} w(\sigma + t + i, x)) dt \right| \\
&\leq c_1 \int_{-\infty}^{-x^\alpha} e^{\operatorname{Re} w(\sigma+t+i,x)} dt
\end{aligned}$$

for a positive constant c_1. Now, we use Lemma 4 to continue with

$$\begin{aligned}
&\int_{-\infty}^{-x^\alpha} e^{\operatorname{Re} w(\sigma+t+i,x)} dt \\
&= e^{\operatorname{Re} w(\sigma+i,x) + O((\log x)^2)} \int_{-\infty}^{-x^\alpha} e^{-t^2\left(\frac{\log 2}{2} + \frac{\log 2}{x}\right) + tO\left(\frac{\log x}{x}\right)} dt \\
&\leq e^{\operatorname{Re} w(\sigma+i,x) + O((\log x)^2)} \int_{-\infty}^{-x^\alpha} e^{-c_2 t x^\alpha} \leq \frac{1}{c_2 x^\alpha} e^{\operatorname{Re} w(\sigma+i,x) + O((\log x)^2)} e^{-c_2 x^{2\alpha}}
\end{aligned}$$

for a positive constant c_2. Lemma 3 yields $w(\sigma + i, x) = w(\sigma, x) + h(x) + O\left(\frac{1}{x^\beta}\right) = w(\sigma, x) + O\left(\frac{1}{x}\right)$. Therefore,

$$e^{\operatorname{Re} w(\sigma+i,x) + O((\log x)^2)} e^{-c_2 x^\alpha} = e^{w(\sigma,x)} O\left(\frac{1}{x^\beta}\right),$$

which is all we need. The proof works analogously for the integrals on the right side of the interval $[-x^\alpha, x^\alpha]$. □

Theorem 1.

$$\tilde{\phi}(x) = e^{w(\sigma,x)}\left(2\sqrt{\frac{\pi}{h(x)}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2(2n+1)^2}{4h(x)}}\cos((2n+1)\pi\sigma) + O\left(\frac{1}{x^\beta}\right)\right).$$

Proof. We use Lemma 3 to obtain

$$\begin{aligned}
&\tilde{\phi}_{\mathcal{C}_1}(x) + \tilde{\phi}_{\mathcal{C}_2}(x) = \\
&= \frac{1}{2\pi i}\int_{-x^\alpha}^{x^\alpha} \frac{\pi}{\sin(\pi(\sigma+t+i))} e^{w(\sigma+t+i,x)}dt \\
&- \frac{1}{2\pi i}\int_{-x^\alpha}^{x^\alpha} \frac{\pi}{\sin(\pi(\sigma+t-i))} e^{w(\sigma+t-i,x)}dt \\
&= e^{w(\sigma,x)+O\left(\frac{1}{x^\beta}\right)}\frac{1}{2\pi i}\int_{-x^\alpha}^{x^\alpha} \frac{\pi}{\sin(\pi(\sigma+t+i))} e^{-h(x)t^2}dt \\
&- e^{w(\sigma,x)+O\left(\frac{1}{x^\beta}\right)}\frac{1}{2\pi i}\int_{-x^\alpha}^{x^\alpha} \frac{\pi}{\sin(\pi(\sigma+t-i))} e^{-h(x)t^2}dt \\
&= e^{w(\sigma,x)} \sum_{n\in[\sigma-x^\alpha,\sigma+x^\alpha]} (-1)^n e^{-h(x)(n-\sigma)^2} + e^{w(\sigma,x)} O\left(\frac{1}{x^\beta}\right),
\end{aligned}$$

using the residue theorem. Furthermore, we have

$$\begin{aligned}
&\sum_{\substack{n\in(\sigma-x^\alpha,\sigma+x^\alpha)\\ n\in\mathbb{Z}}} (-1)^n e^{-h(x)(n-\sigma)^2} = \sum_{n\in\mathbb{Z}} (-1)^n e^{-h(x)(n-\sigma)^2} + O\left(\frac{1}{x^\beta}\right) \\
&= 2\sqrt{\frac{\pi}{h(x)}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2(2n+1)^2}{4h(x)}}\cos((2n+1)\pi\sigma) + O\left(\frac{1}{x^\beta}\right),
\end{aligned}$$

by Fourier expansion. We finish the proof with Lemma 5. □

Let $1 < \delta < \beta$ and

$$j(k) = k\log 2 + \log k - \log 2 + \frac{d}{k^\delta} \text{ for an arbitrary real } d.$$

Lemma 6.

$$\pi\sigma(j(k)) = k\pi + \frac{\pi}{2} - \frac{d\pi}{\log 2}\frac{1}{k^\delta} + O\left(\frac{1}{k^\beta}\right).$$

Proof. We substitute

$$(\log 2)k + \frac{\log k - \log 2 + \frac{d}{k^\delta}}{(\log 2)k} + O\left(\left(\frac{\log k}{k}\right)^2\right)$$

for $\log(j(k))$ and

$$\frac{1}{(\log 2)k} + O\left(\frac{\log k}{k^2}\right)$$

for $1/j(k)$. The proof ends after some simplifications. ☐

Theorem 2. *There is an integer K such that for all $\epsilon > 0$*

$$z_{K+k} = (k+1)2^k + o\left(\frac{2^k}{(k+1)^{1-\epsilon}}\right).$$

Remark. From the theorem it follows immediately that $z_k = k2^k + O(2^k)$. Numerical computations (see Table 1) support the conjecture $K = 0$, which is indeed Robinson's conjecture.

Proof. First we choose $0 < \alpha < \min\left\{\frac{\epsilon}{3}, \frac{1}{3}\right\}$ and $\beta = 2 - 3\alpha$, i.e., the constants of Lemma 3, and $2 - \epsilon < \delta < \beta$. By Theorem 1

$$\tilde{\phi}(x) = e^{w(\sigma,x)}\left(2\sqrt{\frac{\pi}{h(x)}}f(\sigma) + O\left(\frac{1}{x^\beta}\right)\right),$$

where

$$f(\sigma) = \sum_{n=0}^{\infty} e^{-\frac{\pi^2(2n+1)^2}{4h(x)}}\cos((2n+1)\pi\sigma).$$

Let $R_k = \left(k\log 2 + \log k - \log 2 - \frac{1}{k^\delta}, k\log 2 + \log k - \log 2 + \frac{1}{k^\delta}\right)$. Lemma 6 guarantees that there exists an N_1 such that a zero of $f(\sigma(x))$ lies in R_k, for $k \geq N_1$, since a zero of $\cos(\pi\sigma(x))$ is also a zero of $\cos((2n+1)\pi\sigma(x))$ for all $n \geq 0$. We have

$$|f'(\sigma)| \geq e^{-\frac{\pi^2}{4h(x)}}|\sin(\pi\sigma)| - \pi\left|\sum_{n=1}^{\infty}(2n+1)e^{-\frac{\pi^2(2n+1)^2}{4h(x)}}\sin((2n+1)\pi\sigma)\right|.$$

We notice that

$$c_2 = 0.34 \leq h(x) \leq 0.42 = c_1$$

for $x \geq 10$. Now, we estimate

$$\left|\sum_{n=1}^{\infty}(2n+1)e^{-\frac{\pi^2(2n+1)^2}{4h(x)}}\sin((2n+1)\pi\sigma)\right|$$
$$\leq \sum_{n=1}^{\infty}(2n+1)e^{-\frac{\pi^2(2n+1)^2}{4c_1}} \leq e^{-6c_3}\sum_{n=1}^{\infty}(2n+1)e^{-c_3(2n+1)} \leq e^{-6c_3}\frac{e^{-c_3}}{(1-e^{-c_3})^2},$$

where $c_3 = \frac{\pi^2}{4c_1}$. Therefore,

$$|f'(\sigma)| \geq e^{-\frac{\pi^2}{4c_2}}|\sin(\pi\sigma)| - e^{-6c_3}\frac{e^{-c_3}}{(1-e^{-c_3})^2} > 0.0007,$$

if $\pi\sigma \in [k\pi + \frac{\pi}{2} - 0.1, k\pi + \frac{\pi}{2} + 0.1]$. With the mean value theorem and Lemma 6 we obtain that

$$|f(k\log 2 + \log k - \log 2 \pm \frac{1}{k^\delta})| \geq \frac{c}{k^\delta} + O\left(\frac{1}{k^\beta}\right),$$

where $c > 0$. Thus, we observe that there exists an N such that a zero of $f(x) + O\left(\frac{1}{x^\beta}\right)$, which is also a zero of $\tilde{\phi}(x)$, is contained in R_k, for all $k \geq N$. Let $k \geq N$ and x_k be the zero of $\tilde{\phi}$ which lies in in R_k. As Robinson stated in [3, p. 258] we also have $z_i > 2z_{i-1}$, for all $i \geq 1$. This means that there can be no zero of $\tilde{\phi}$ between x_k and x_{k+1}. Since otherwise $x_{k+1} - x_k > 2\log 2$, which is impossible, because we showed that $x_{k+1} - x_k \leq (k+1)\log 2 + \log(k+1) - \log 2 + \frac{1}{(k+1)^\delta} - \left(k\log 2 + \log k - \log 2 - \frac{1}{k^\delta}\right) = \log 2 + \log\left(\frac{k+1}{k}\right) + \frac{1}{(k+1)^\delta} + \frac{1}{k^\delta}$. Let M be the index of the zero which corresponds to the zero of $\tilde{\phi}(x)$ which lies in R_N, i.e., $\log z_M \in R_N$. Then $\log z_{M+i} \in R_{N+i}$, for all $i \geq 1$, and therefore $\log z_{K+k} \in R_k$, for $k \geq N$, where $K = M - N$. This means that

$$\log z_{K+k} = (k+1)\log 2 + \log(k+1) - \log 2 + O\left(\frac{1}{(k+1)^\delta}\right),$$

or in other words

$$\begin{aligned} z_{K+k} &= (k+1)2^k e^{O\left(\frac{1}{(k+1)^\delta}\right)} = (k+1)2^k + O\left(\frac{2^k}{(k+1)^{\delta-1}}\right) \\ &= (k+1)2^k + o\left(\frac{2^k}{(k+1)^{1-\epsilon}}\right), \end{aligned}$$

since $1 - \epsilon < \delta - 1$, finishing the proof. □

References

[1] N. G. De Bruijn, Asymptotic methods in analysis, (North-Holland Publishing Co.-Amsterdam, 1958)
[2] P. Henrici, Applied and Computational Complex Analysis, (Pure and Applied Mathematics, A Wiley Interscience Series of Texts, Monographs & Tracts, John Wiley & Sons, 1977)
[3] R.W. Robinson, Counting labeled acyclic digraphs, in: F. Harary, ed., *New Directions in the Theory of Graphs* (Academic Press, New York, 1973) 239–279.
[4] R.W. Robinson, Counting unlabeled acyclic digraphs, *Lect. Notes Math.* **622** (1977) 28–43.
[5] R. P. Stanley, Acyclic orientations of graphs, *Discrete Mathematics*, **5**, No. 2 (1973), 171-178.

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