Average Case Analysis of Numerical Integration

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Abstract

We extend the average case analysis of numerical integration in the sense of Traub and Woźniakowski to homogeneous spaces. Various examples are described and the connections to minimal energy point sets are outlined.

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1 Introduction

Persi Diaconis [5] considered numerical integration of real valued functions on the unit interval $[0,1]$ from a Bayesian point of view. He surveyed on methods how to get information on the integral, provided that the prior distribution of the functions is known (for instance Brownian motion). He also mentions several applications such as random surfaces (as used in tomography) and statistical geodesy. In particular, Lauritzen [20, 21, 22] used such methods and ideas to estimate the gravity potential of the earth from the knowledge of a certain number of measurements.

From a different point of view numerical integration of functions on the s-dimensional unit cube $[0,1]^s$ was analyzed by Traub, Wasilkowski, and Woźniakowski (cf. [33, 32, 34]). These authors approximated the integral by the arithmetic mean of N function values. They equipped the space of continuous functions on $[0,1]^s$ with Wiener measure and investigated the average case error in the sense of quadratic mean. Furthermore, they proved that the average case complexity is optimal, provided that the integration points have L^2-discrepancy of optimal order of magnitude.

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We extend these ideas to functions defined on homogeneous spaces X. In section 2 we construct a Gaussian stochastic process on X by prescribing the covariance function. Of course, a very interesting special case is the sphere, and in this case such processes have been investigated in [2, 3]. For similar processes in the more general setting of hypergroups we refer to [1].

In section 3 we outline several examples and related concepts of discrepancy. We start with a discontinuous process defined via the Walsh orthogonal system. Then we introduce the diaphony, the L^2-discrepancy, and various kinds of geodesic discrepancies and relate them to suitably defined Gaussian processes. For a more detailed study of different notions of discrepancies on spheres we refer to our earlier paper [11].

In the final section 4 we outline an extension of the Traub Woźniakowski approach to numerical integration of functions on homogeneous spaces and its connection to minimal energy point sets.

2 Construction of the Process

Let G be a compact group acting transitively and continuously on a metric space X and μ_0 the normalized Haar measure on G. Then X can be topologically identified with the quotient G/K of G modulo the stabilizer $K \subseteq G$ of a point. A natural probability measure on $X \cong G/K$ is given by $\mu(M) = \mu_0(\{g \in G \mid gK \in M\})$ (cf. [15, 6, 27]).

For any $\rho \in G^*$ (the dual of G modulo equivalence of representations) let

$$m_{ij}^{(\rho)}(g) = (U^{(\rho)}(g)a_j, a_i), \quad i, j = 1, \ldots, n_\rho$$

be the coordinate functions of the irreducible unitary representation $U^{(\rho)}$ in the finite dimensional Hilbert space generated by an orthonormal system $a_1, \ldots, a_{n_\rho}$. We will denote the trivial representation by ρ_0. This system can be chosen such that for a certain $d_\rho \leq n_\rho$

$$\mathrm{Sp}(\{m_{ij}^{(\rho)} \mid i, j = 1, \ldots, n_\rho\}) \cap L^2(G/K) = \mathrm{Sp}(\{m_{ij}^{(\rho)} \mid 1 \leq i \leq n_\rho, 1 \leq j \leq d_\rho\}),$$

where $\mathrm{Sp}(S)$ denotes the linear space generated by S. By the theorem of Peter-Weyl and well-known algebraic tools it can be deduced that for $J_\rho = \mathrm{Sp}(\{m_{ij}^{(\rho)} \mid i = 1, \ldots, n_\rho, j = 1, \ldots, m_\rho\})$

$$L^2(G/K) = \widehat{\bigoplus_{\rho \in G^*}} J_\rho \quad \text{(Hilbert orthogonal sum)}.$$

By reordering orthonormal bases of the J_ρ's we obtain a complete orthonormal system $\psi_{n,\rho}(x)$ ($\rho \in G^*$ $n = 1, \ldots, d_\rho = \dim(J_\rho)$) of continuous real functions.

Define now a symmetric kernel function by

$$K(x, y) = \sum_{\rho \in G^*} a_\rho \sum_{n=1}^{m_\rho} \psi_{n,\rho}(x)\psi_{n,\rho}(y) \tag{2.1}$$

with strictly positive coefficients a_ρ and

$$\sum_{\rho \in G^*} a_\rho m_\rho = 1.$$

Furthermore, $K(x, y) = 1$, if and only if $x = y$, the series (2.1) is uniformly convergent and the kernel function is positive definite. The term "positive definite" refers to the fact that

$$\sum_{i,j=1}^{M} K(x_i, x_j)u_i u_j \geq 0$$

for all choices of $x_i \in X$ and $u_i \in \mathbb{R}$, $i = 1, \ldots, M$. This is the generalization of the usual concept of positive definite functions to harmonic spaces (cf. [15], [30]). If K acts transitively on every geodesic sphere around the point stable under K, then the kernel can be written as function of the distance $d(x, y)$, which would give the notion of positive definiteness studied in [29] and [30]. This will always be the case in our examples.

Remark 1 *The function $d(x, y) = \sqrt{1 - K(x, y)}$ defines a G-invariant metric on X. Conversely, for an arbitrarily given metric d the function $1 - d(x, y)^2$ can be considered, and it is a natural question, whether it is positively definite. The identity*

$$\begin{vmatrix} 1 & 1 - d(x,y)^2 & 1 - d(x,z)^2 \\ 1 - d(x,y)^2 & 1 & 1 - d(y,z)^2 \\ 1 - d(x,z)^2 & 1 - d(y,z)^2 & 1 \end{vmatrix} =$$
$$(d(x,y) + d(y,z) - d(x,z))(d(x,z) + d(z,y) - d(x,y)) \times$$
$$(d(x,z) + d(x,y) - d(y,z))(d(x,y) + d(x,z) + d(y,z))$$
$$-2d(x,y)^2 d(x,z)^2 d(y,z)^2$$

shows that in the case of a geodesic metric (take $d(x, z) = d(x, y) + d(y, z)$) there is no positive definiteness of the function $1 - d(x, y)^2$.

Definition 1 *Let $A_{n,\rho}$, $\rho \in G^*$, $n = 1, \ldots, m_\rho$ be independent normal random variables with mean 0 and variance a_ρ. Define the process*

$$Y(x) = \sum_{\rho\in G^*} \sum_{n=1}^{m_\rho} A_{\rho,n}\psi_{n,\rho}(x). \tag{2.2}$$

Equip now the space of continuous functions $C(X)$ with the Lévy measure λ defined by this process.

Remark 2 *The series (2.2) is a.s. uniformly convergent by Kolmogorov's three series theorem.*

Remark 3 *The process $Y(x)$ could be defined alternatively as a Gaussian process with mean $\mathbb{E}Y(x) = 0$ and covariance function $\mathbb{E}Y(x)Y(y) = K(x,y)$. This equivalent definition would avoid Fourier series.*

We are now ready to compute the integral of the square of the integration error with respect to λ.

Proposition 1

$$\int_{C(X)} \left(\frac{1}{N}\sum_{n=1}^{N} y(x_n) - \int_X y(x)\,d\mu(x)\right)^2 d\lambda(y) = \tag{2.3}$$

$$\sum_{\rho\in G^*\setminus\{\rho_0\}} a_\rho \sum_{m=1}^{m_\rho} \left(\frac{1}{N}\sum_{n=1}^{N} \psi_{m,\rho}(x_n)\right)^2. \tag{2.4}$$

Proof: Observe first that $\int_X Y(x)\,d\mu(x) = A_{\rho_0,1}$. Thus we can rewrite the first integral as

$$\int_{C(X)} \left(\frac{1}{N}\sum_{n=1}^{N}\sum_{\rho\neq\rho_0}\sum_{m=1}^{m_\rho} A_{m,\rho}\psi_{m,\rho}(x_n)\right)^2 \lambda(dY).$$

Interchanging the orders of summation and using the independence of the $A_{m,\rho}$'s gives the result.

3 Examples and Geodesic L^2-Discrepancy

We will now present a list of examples which will show that several concepts of diaphony and L^2-discrepancy fit into the general approach decribed above.

3.1 Example 1, Dyadic Diaphony

In [14] a concept of diaphony based on the orthonormal system of Walsh-functions on the unit cube $[0,1)^s$ is introduced. As general references for Walsh functions we refer to [10] and [28].

For $x = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{2^{k+1}}$ given by its binary digital expansion (in case of ambiguity we take the finite version of the digital expansion) and $m = \sum_{k=0}^{K} \delta_k 2^k$ we define the m-th Walsh function by

$$w_m(x) = (-1)^{\sum_{k=0}^{K} \varepsilon_k \delta_k}. \tag{3.1}$$

For $\mathbf{x} \in [0,1)^s$ and $\mathbf{m} \in \mathbb{N}^s$ we define

$$w_{\mathbf{m}}(\mathbf{x}) = \prod_{j=1}^{s} w_{m_j}(x_j).$$

The dyadic diaphony of a sequence $\omega = (\mathbf{x}_1, \mathbf{x}_2, \ldots)$ is then given by

$$F_N(\omega) = \left(\frac{1}{3^s - 1} \sum_{\mathbf{m} \neq 0} \rho(\mathbf{m}) |S_N(w_{\mathbf{m}}, \omega)|^2 \right)^{\frac{1}{2}}, \tag{3.2}$$

where $\rho(\mathbf{m}) = \prod_{j=1}^{s} \rho(m_j)$ with

$$\rho(m) = \begin{cases} 2^{-2g} & \text{for } 2^g \leq m < 2^{g+1}, \quad g \in \mathbb{N} \\ 1 & \text{for } m = 0 \end{cases} \tag{3.3}$$

and

$$S_N(w_{\mathbf{m}}, \omega) = \frac{1}{N} \sum_{n=1}^{N} w_{\mathbf{m}}(\mathbf{x}_n)$$

denotes the Weyl sum with respect to the Walsh functions.

At first sight this concept does not fit into the scheme introduced above, since the Walsh functions are not the group characters on $[0,1)^s$. But there is the bijection

$$\xi : \sum_{k=0}^{\infty} \frac{\varepsilon_k}{2^{k+1}} \in [0,1) \mapsto (\varepsilon_0, \varepsilon_1, \ldots) \in \mathbb{F}_2^{\mathbb{N}}$$

between $[0,1)$ and a subset of measure 1 of the group $\mathbb{F}_2^{\mathbb{N}}$ (the group of infinite sequences of $0,1$ equipped with the product topology and the Haar measure).

This map is also continuous except for the dyadic rational points. Clearly the component-wise map between s-tuples enjoys the same properties. $[0,1)$ also inherits an addition law from $\mathbb{F}_2^{\mathbb{N}}$, which is just $x\dot{+}y=\xi^{-1}(\xi(x)+\xi(y))$. Again, this addition will be performed component-wise on the s-tuples.

Since the function

$$\varphi(x)=\begin{cases}3-3\cdot 2^{1+\lfloor\log_2 x\rfloor} & \text{for } x\in(0,1)\\ 3 & \text{for } x=0\end{cases}\tag{3.4}$$

has the Walsh expansion

$$\varphi(x)=\sum_{k=0}^{\infty}\rho(k)w_k(x)$$

we can write (under an additional condition on the values of the $\mathbf{x}_n$)

$$F_N^2(\omega)=\frac{1}{3^s-1}\frac{1}{N^2}\sum_{n,m=1}^{N}\phi(\mathbf{x}_n\dot{+}\mathbf{x}_m)\tag{3.5}$$

with

$$\phi(\mathbf{x})=\prod_{j=1}^{s}\varphi(x_j)-1.$$

Notice, that $w_k(x\dot{+}y)=w_k(x)w_k(y)$, if either one of x and y is a dyadic rational or $x\dot{+}y$ is not a dyadic rational (this describes the condition indicated above).

With the restriction on the choice of the values $\mathbf{x}_n$ formula (3.5) fits into the general theory developed above. We will now derive some properties of the stochastic process behind this type of diaphony. The process is given as the Gaussian process with covariance function

$$\mathbb{E}X(\mathbf{x})X(\mathbf{y})=\phi(\mathbf{x}\dot{+}\mathbf{y}),\tag{3.6}$$

which is equivalent to

$$\mathbb{E}(X(\mathbf{x})-X(\mathbf{y}))^2=2(3^s-1)-2\phi(\mathbf{x}\dot{+}\mathbf{y})\le 2\cdot 3^s\sum_{j=1}^{s}\varphi(x_j\dot{+}y_j).\tag{3.7}$$

Equation (3.7) implies that the process is continuous whenever $\mathbf{x}\to\mathbf{y}$ implies $\mathbf{x}\dot{+}\mathbf{y}\to\mathbf{0}$. This is the case exactly when $\mathbf{y}$ has no dyadic rational entry. It remains to consider the case, when one of the components of the vector $\mathbf{y}$ is a dyadic rational. Then it follows from the right continuity of the function ϕ that $\mathbf{x}\dot{+}\mathbf{y} \to \mathbf{0}$, if the components of $\mathbf{x}$ with the same indices as the dyadic rational components of $\mathbf{y}$ converge monotonically decreasing to this value. It follows from the fact that the Walsh functions only have jump discontinuities, that the limits in the other $2^s - 1$ orthants exist. For these limits to be the same, the Fourier coefficients would have to satisfy linear relations, which are certainly only satisfied with probability 0. Thus we have proved

Proposition 2 *The Gaussian process $X(\mathbf{x})$, $\mathbf{x} \in [0,1)^s$ given by $\mathbb{E}X(\mathbf{x}) = 0$ and (3.6) gives the dyadic diaphony as defined in [14] and has trajectories which are continuous in every point $\mathbf{x}$ with no dyadic rational component. In all points with at least one dyadic rational component the limits in the different orthants exist but are different with probability* 1*, if a rational component is approached from two different directions.*

Remark 4 *Similar results could be derived for arbitrary Cantor-type digital expansions and the characters of* $\mathbf{A}$*-adic numbers as defined in [15]. Especially, there is a generalization of the above results to arbitrary q-adic digital expansions.*

3.2 Example 2, Euclidian L^2-Discrepancy

It was first observed by O. Strauch [31] that the usual L^2-discrepancy on the interval can be written as a Wiener integral

$$\int_0^1 \left(\frac{1}{N} \sum_{n=1}^{N} \chi_{[0,x)}(x_n) - x \right)^2 dx = 2 \int \left(\frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\, dx \right)^2 df,$$

where the second integral is extended over all continuous functions on $[0,1]$ with $f(0) = 0$.

We give an alternative interpretation of L^2-discrepancy which is based on a process with periodic trajectories. The corresponding diaphony can be written as

$$\frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2 = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\left| \frac{1}{N} \sum_{n=1}^{N} \sin(2\pi k x_n) \right|^2 + \left| \frac{1}{N} \sum_{n=1}^{N} \cos(2\pi k x_n) \right|^2 \right).$$

This yields the corresponding kernel function

$$\begin{aligned} K(x,y) &= \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos(2\pi k(x-y)) \\ &= 6\left(\|x-y\|-\frac{1}{2}\right)^2-\frac{1}{2}, \end{aligned}$$

where $\|x-y\|$ denotes the chordal distance on the circle $\mathbb{R}/\mathbb{Z}$: $\|x-y\| = \min_{n\in\mathbb{Z}}|x-y-n|$. This definition of the process is equivalent to

$$\mathbb{E}\left(X(x)-X(y)\right)^2 = 12\|x-y\|\left(1-\|x-y\|\right),$$

which implies that the trajectories are continuous functions.

Remark 5 *Note that the simpler kernel function $K(x,y) = 1-\alpha\|x-y\|$ (with $0<\alpha<4$ for positive definiteness) would give trajectories $X(x)$ with the unwanted property that $X(x)+X(x+\frac{1}{2}) = const.$ This is a consequence of*

$$\mathbb{E}\left(X(x)-X(x+\frac{1}{2})-X(y)+X(y+\frac{1}{2})\right)^2 = 2\alpha\left(\|x-y\|+\|x-y+\frac{1}{2}\|-1\right),$$

which vanishes for geometric reasons.

The computations above immediately generalize to the s-dimensional torus. Here the kernel function corresponding to the usual L^2-discrepancy is given by

$$\begin{aligned} K_s(\mathbf{x},\mathbf{y}) &= \frac{1}{2^s-1}\left(\prod_{j=1}^{s}\left(6\left(\|x_j-y_j\|-\frac{1}{2}\right)^2+\frac{1}{2}\right)-1\right) \\ &= \frac{1}{2^s-1}\left(\prod_{j=1}^{s}(K(x_j,y_j)+1)-1\right). \end{aligned}$$

We once again have continuous periodic trajectories.

Remark 6 *It is also known that the discrepancy function*

$$D_N(x,x_n) = \frac{1}{\sqrt{N}}\left(\sum_{n=1}^{N}\chi_{[0,x)}(x_n)-x\right)$$

behaves like a trajectory of the Brownian bridge as $N\to\infty$. This yields to a new proof of Kolmogorov's theorem (cf. [16]) on the distribution of the discrepancy (cf. [7, 8]). It is possible to use this fact to study the distribution of L^2-discrepancy of sequences (cf. [12]).

3.3 Example 3, Geodesic L^2-Discrepancy

Since X is supposed to be compact and metrizable with G-invariant metric $d : X \times X \to [0, R]$ it is natural to define a discrepancy by

$$D_N^G(x_n) = \int\limits_X \int\limits_0^R \left(\frac{1}{N} \sum_{n=1}^N \chi_{B(x,r)}(x_n) - \mu(B(x,r)) \right)^2 dr\, d\mu(x),$$

where R denotes the diameter of X. Expanding and integrating term by term yields

$$\begin{aligned} D_N^G(x_n) &= \frac{1}{N^2} \sum_{m,n=1} \int\limits_X \int\limits_0^R \chi_{B(x_m,r)}(x)\chi_{B(x_n,r)}(x)\, dr\, d\mu(x) \\ -& \frac{2}{N} \sum_{n=1}^N \int\limits_X \int\limits_0^R \chi_{B(x_n,r)}(x)\mu(B(x,r))\, dr\, d\mu(x) + \int\limits_X \int\limits_0^R \mu(B(x,r))^2\, dr\, d\mu(x) \\ =& \frac{1}{N^2} \sum_{m,n=1}^N \int\limits_{\frac{1}{2}d(x_m,x_n)}^R \mu(B(x_m,r) \cap B(x_n,r))\, dr - \int\limits_0^R \mu(B(\cdot,r))^2\, dr. \end{aligned} \tag{3.8}$$

We take the function

$$K(x,y) = f(d(x,y)) = \int\limits_{\frac{1}{2}d(x,y)}^R \mu(B(x,r) \cap B(y,r))\, dr - \int_0^R \mu(B(\cdot,r))^2\, dr \tag{3.9}$$

as the kernel function.

As a first special case let us consider the circle $\mathbb{R}/\mathbb{Z}$. Then we have by (3.8)

$$\begin{aligned} D_N^G(x_n) &= \int_0^1 \int_0^{\frac{1}{2}} \left(\frac{1}{N} \sum_{n=1}^N \chi_{(x-r,x+r)}(x_n) - 2r \right)^2 dr\, dx \\ &= \frac{1}{N^2} \sum_{m,n=1}^N \left(\int_{\frac{1}{2}\|x_m-x_n\|}^{\frac{1}{2}(1-\|x_m-x_n\|)} (2r - \|x_m - x_n\|)\, dr + \right. \\ &\quad \left. \int_{\frac{1}{2}(1-\|x_m-x_n\|)}^{\frac{1}{2}} (4r-1)\, dr \right) - \frac{1}{6}, \end{aligned}$$

since

$$\begin{aligned}
&\tfrac{1}{2\pi}\int_0^{2\pi}\chi_{(x_m-r,x_m+r)}(x)\chi_{(x_n-r,x_n+r)}(x)\,dx\\
&=\begin{cases} 0 & \text{if } r\le \frac12\|x_m-x_n\|\\[2ex] 2r-\|x_m-x_n\| & \text{if } \frac12\|x_m-x_n\|<r\\ & \quad\le \frac12-\frac12\|x_m-x_n\|\\ 4r-1 & \text{if } r>\frac12-\frac12\|x_m-x_n\|.\end{cases}
\end{aligned}$$

Computing the remaining integrals and collecting terms yields

$$D_N^G(x_n)=\frac{1}{N^2}\sum_{m,n=1}^{N}\left(\frac16-\frac12\|x_m-x_n\|+\frac12\|x_m-x_n\|^2\right). \tag{3.10}$$

The next special case which will be considered is the sphere S^2. Here we have to compute the kernel function

$$K(\mathbf{x},\mathbf{y})=\int\limits_{\frac12\arccos\langle\mathbf{x},\mathbf{y}\rangle}^{\pi}\mu\left(B(\mathbf{x},\varphi)\cap B(\mathbf{y},\varphi)\right)\,d\varphi-\frac14\int_0^{\pi}(1-\cos\varphi)^2\,d\varphi. \tag{3.11}$$

In order to compute the normalized surface measure of $B(\mathbf{x},\varphi)\cap B(\mathbf{y},\varphi)$, we express the characteristic function of the cap $B(\mathbf{u},\varphi)$ by its Fourier series (cf. [23, 24])

$$\chi_{B(\mathbf{u},\varphi)}(\mathbf{x})=\frac{1-\cos\varphi}{2}+\sum_{n=1}^{\infty}\left(P_{n+1}(\cos\varphi)-P_{n-1}(\cos\varphi)\right)\sum_{k=-n}^{n}K_{nk}(\mathbf{u})K_{nk}(\mathbf{x}).$$

P_n denotes the n-th Legendre polynomial. Integrating $\chi_{B(\mathbf{x},\varphi)}(\mathbf{u})\chi_{B(\mathbf{y},\varphi)}(\mathbf{u})$ we obtain

$$\mu\left(B(\mathbf{x},\varphi)\cap B(\mathbf{y},\varphi)\right)=\frac{(1-\cos\varphi)^2}{4}+\sum_{n=1}^{\infty}\frac{\left(P_{n+1}(\cos\varphi)-P_{n-1}(\cos\varphi)\right)^2}{2n+1}P_n(\langle\mathbf{x},\mathbf{y}\rangle).$$

Inserting this into (3.11) and integrating with respect to φ yields

$$K(\mathbf{x},\mathbf{y})=\sum_{n=1}^{\infty}a_nP_n\left(\langle\mathbf{x},\mathbf{y}\rangle\right) \tag{3.12}$$

with

$$\begin{aligned} a_n &= 16^{-n}\left(\frac{1}{16}\sum_{m=0}^{n+1}\binom{2m}{m}^2\binom{2n+2-2m}{n+1-m}^2 + 16\sum_{m=0}^{n-1}\binom{2m}{m}^2\binom{2n-2-2m}{n-1-m}^2\right. \\ &- \left. 2\sum_{m=1}^{n}\binom{2m}{m}\binom{2m-2}{m-1}\binom{2n+2-2m}{n+1-m}\binom{2n-2m}{n-m}\right), \end{aligned}$$

where we have used the Fourier expansion of Legendre polynomials (cf. [23], p. 229). Thus we have for the geodesic discrepancy on S^2

$$D_N^G(\mathbf{x}_n) = \frac{1}{N^2}\sum_{i,j=1}^{N} K(\mathbf{x}_i, \mathbf{x}_j),$$

where K is given by the Fourier expansion (3.12).

In the case of the sphere one could obtain a more explicit formula for the L^2-discrepancy by replacing the measure dr by $\sin r\, dr$, which amounts to a change of the metric; this new metric is no more geodesic. This yields as above the Fourier coefficient of the kernel

$$a_n = \frac{2}{(2n+3)(2n-1)},$$

which leads to the explicit expression for the kernel

$$K(\mathbf{x}, \mathbf{y}) = \frac{2}{3} - \sqrt{\frac{1-\langle \mathbf{x}, \mathbf{y}\rangle}{2}},$$

where we have used [23] p. 238 for $\lambda = \frac{1}{2}$. The expression $\sqrt{\frac{1-\langle \mathbf{x}, \mathbf{y}\rangle}{2}}$ is clearly half the Euclidean distance between the points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$. We obtain for the discrepancy with respect to this new metric

$$\tilde{D}_N^G(\mathbf{x}_n) = \frac{1}{N^2}\sum_{i,j=1}^{N}\left(\frac{2}{3} - \sqrt{\frac{1-\langle \mathbf{x}_i, \mathbf{x}_j\rangle}{2}}\right).$$

4 Average Case Analysis

In this final section we outline Woźniakowski's approach of the average case analysis of numerical integration. In [34] the author considered real valued functions on the s-dimensional unit cube and the classical Wiener measure; for further surveys on this theory we refer to [33, 32, 9].

Here we will extend the basic ideas of this approach to numerical integration of real valued functions on homogeneous spaces. Let f be a continuous real-valued function on the homogeneous space $X \cong G/K$ and set

$$I(f) = \int_X f(x)\,d\mu(x), \tag{4.1}$$

with the natural probability measure μ on X associated to the Haar measure on G. We approximate $I(f)$ by the arithmetic mean

$$I_N(f) = \frac{1}{N}\sum_{n=1}^{N} f(x_n) \tag{4.2}$$

extended over the integration points $x_n \in X$.

Usually integration algorithms with low costs are desirable. In this context the costs would depend on computation time and the approximation error. In the case of the approximation of $I(f)$ by $I_N(f)$ the cost $\mathrm{cost}(I_N)$ is proportional to N, the number of function evaluations. The average error of numerical integration with respect to the Lévy measure λ as defined in section 2 is given by

$$E^{\mathrm{avg}}(I_N) = \left(\int_{C(X)} \left(I(f) - I_N(f)\right)^2\,\lambda(df)\right)^{\frac{1}{2}}. \tag{4.3}$$

The average case ε-complexity is given as the minimal cost of all algorithms with average error $\leq \varepsilon$:

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon, \mathrm{C(X)}) = \inf\{\mathrm{cost(I_N)} \mid \mathrm{E^{avg}(I_N)} \leq \varepsilon\}. \tag{4.4}$$

By Proposition 1 we have

$$E^{\mathrm{avg}}(I_N) = F_N^A(x_n), \tag{4.5}$$

where

$$F_N^A(x_n) = \left(\sum_{\rho\in G^*\setminus\{\rho_0\}} a_\rho \sum_{m=1}^{m_\rho}\left(\frac{1}{N}\sum_{n=1}^{N}\psi_{m,\rho}(x_n)\right)^2\right)^{\frac{1}{2}}$$

is the diaphony of the point set x_n with respect to the coefficients $A = (a_\rho)$. The classical notion of diaphony was introduced by Zinterhof [35] in the case of the s-dimensional unit cube (see also [9]). In this case the coefficients are given by

$$a(\mathbf{h}) = \frac{1}{\prod_{k=1}^{s} \max(1, |h_k|)}.$$

As it is shown in section 3 in many important cases the diaphony can be expressed in terms of suitably chosen concepts of L^2-discrepancies, such as the classical discrepancy on $[0,1)^s$ (cf. [19, 9]) or the geodesic discrepancy on the sphere.

In order to establish a bound for the average case complexity of numerical integration we need suitable point sets of small diaphony. Note that the diaphony can be interpreted as an energy functional.

Definition 2 *A set $z_n \in X$ is called a minimal energy set on X, if the function $F_N^A(x_n)$ attains its minimal value in the point $(x_1, \ldots, x_N) = (z_1, \ldots, z_N)$. The minimal value is denoted by $\Phi(N)$.*

In the classical case of the s-dimensional unit cube $\Phi(N) = \mathcal{O}(\frac{\log^{\frac{s-1}{2}} N}{N})$; this value is attained for instance for the Hammersley point set. In the case of the sphere various authors such as Kuijlaars, Rakhmanov, Saff, Zhou and others considered minimal energy problems for different kinds of functionals (cf. [17, 18, 25, 26]). In the spherical case explicit constructions for the point sets are presently not known. In [13] spherical designs were shown to be minimal energy point sets for certain functionals. (Just recall that a spherical t-design is a set of integration points giving exact integration for polynomials up to order t; for more details, see [4]).

Thus the problem of estimating the average case complexity of numerical integration is reduced to finding the minimal energy point configuration for the corresponding diaphony. In particular, the average case ε-complexity is given by the smallest N such that $\Phi(N) \le \varepsilon$, provided that Φ is decreasing.

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