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BLOCK ADDITIVE FUNCTIONS ON THE GAUSSIAN INTEGERS

MICHAEL DRMOTA*, PETER J. GRABNER†, AND PIERRE LIARDET‡

Abstract. We present three conceptually different methods to prove distribution results for block additive functions with respect to radix expansions of the Gaussian integers. Based on generating function approaches we obtain a central limit theorem and asymptotic expansions for the moments. Furthermore, these generating functions as well as ergodic skew products are used to prove uniform distribution in residue classes and modulo 1.

1. Introduction

Let $q = -a + i \in \mathbb{Z}[i]$ for a positive integer a and $\mathcal{N} = \{0, 1, \ldots, a^2\}$. Then every Gaussian integer $z \in \mathbb{Z}[i]$ can be uniquely represented as

$$z = \sum_{j \geq 0} \varepsilon_j(z)\, q^j$$

with $\varepsilon_j(z) \in \mathcal{N}$. Formally we set $\varepsilon_j(z) = 0$ for all negative integers $j < 0$. It will be convenient sometimes to use infinite or even doubly infinite sequences (filled with zeros) for the representation of Gaussian integers. We denote the length of the expansion by

$$\mathrm{length}_q(z) = \max\{j \in \mathbb{N}_0 \mid \varepsilon_j(z) \neq 0\} + 1$$

and $\mathrm{length}_q(0) = 0$. (We denote the positive integers by $\mathbb{N}$ and use $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for the non-negative integers.) Throughout the paper we will use the notation $\log_b$ for the logarithm to base b. The following lemma was proved in [15].

Lemma 1. *There exists a positive constant c such that for all $z \in \mathbb{Z}[i]$*

$$\left|\mathrm{length}_q(z) - \log_{|q|}|z|\right| \leq c.$$

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The fundamental domain of the base q representation on $\mathbb{Z}[i]$ is defined by

$$\mathcal{F}_q = \left\{ \sum_{\ell=1}^{\infty} \frac{\varepsilon_\ell}{q^\ell} \mid \forall \ell : \varepsilon_\ell \in \mathcal{N} \right\}.$$

This subset of $\mathbb{C}$ plays the same rôle for q-adic numeration as the unit interval does for classical number systems on the integers (cf. [10, 11, 15]). More generally, radix representations of elements of the ring of integers $\mathbb{Z}_K$ of a number field K can be considered. A base $\alpha \in \mathbb{Z}_K$ together with the digit set $D = \{0, 1, \ldots, |\mathrm{N}_{K|\mathbb{Q}}(\alpha)| - 1\}$ is called a canonical number system (cf. [19, 20]), if every $\zeta \in \mathbb{Z}_K$ has a representation of the form

$$\zeta = \sum_{\ell=0}^{n} \varepsilon_\ell \alpha^\ell$$

with $\varepsilon_\ell \in D$ for $0 \le \ell \le n$. The point 0 is an inner point of $\mathcal{F}_q$. This follows from the general fact that (α, D) is a canonical number system, if and only if the corresponding fundamental domain contains 0 as an inner point (cf. [1]).

Let $F : \mathcal{N}^{L+1} \to \mathbb{R}$ be any given function (for some $L \ge 0$) with $F(0, 0, \ldots, 0) = 0$. Furthermore, set

$$s_F(z) = \sum_{j=-L}^{\infty} F\left(\varepsilon_j(z), \varepsilon_{j+1}(z), \ldots, \varepsilon_{j+L}(z)\right).$$

This means that we consider a weighted sum over all subsequent digital patterns of length $L+1$ of the digital expansion of z. The function s_F is called block additive function of rank $L+1$. This generalises the block-additive digital functions studied in [6] for digital expansions on the rational integers. This definition readily extends to functions taking values in an arbitrary abelian group A. We will use this in the general considerations in Section 5.

For example, for $L = 0$ we obtain completely additive functions such as those studied in [18, Section 5], for instance for $F(\varepsilon) = \varepsilon$ we just have the sum-of-digits function studied in [12, 15, 16], or if $L = 1$ and $F(\varepsilon, \eta) = 1 - \delta_{\varepsilon,\eta}$ ($\delta_{x,y}$ denoting the Kronecker symbol) then $s_F(n)$ is just counting the number of times that a digit is different from the preceding one etc.

2. Overview of the Results

Our main objective is to get information on sums

$$S_N(x) = \sum_{|z|^2 < N} x^{s_F(z)}, \tag{2.1}$$

where x is a complex variable. It is clear that $S_N(x)$ encodes the distribution of $s_F(z)$. For example, if we assume that $s_F(z)$ has only non-negative integer values then we have

$$S_N(x) = \sum_{k\ge 0} \#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\}\, x^k.$$

More generally, let Y_N denote the random variable that is induced by the distribution of $s_F(z)$ for $|z|^2 < N$, that is, the distribution function of Y_N is given by

$$\mathbb{P}\{Y_N \le y\} = \frac{1}{S_N(1)} \#\{|z|^2 < N : s_F(z) \le y\}. \tag{2.2}$$

Then we have

$$\mathbb{E}\, x^{Y_N} = \frac{1}{S_N(1)} \sum_{|z|^2<N} x^{s_F(z)} = \frac{1}{S_N(1)} S_N(x). \tag{2.3}$$

In particular the moment generating function $\mathbb{E}\, e^{\lambda Y_N}$ and the characteristic function $\mathbb{E}\, e^{itY_N}$ of Y_N can be expressed with help of $S_N(x)$. (Note that $S_N(1) = \pi N + \mathcal{O}\left(N^{\frac{1}{3}}\right)$.)

In what follows we will present three different methods to obtain asymptotic information for $S_N(x)$. In Section 3 we use a measure theoretic approach showing that for real numbers x sufficiently close to 1 we have

$$S_N(x) = \Phi(x, \log_{|q|^2} N)\, N^{\log_{|q|^2} \lambda(x)} \cdot \left(1 + \mathcal{O}\left(N^{-\kappa}\right)\right), \tag{2.4}$$

where $\Phi(x,t)$ is a function that is analytic in x and periodic (with period 1) and Hölder continuous in t, and $\lambda(x)$ is the dominant eigenvalue of a certain matrix $\mathbf{A}(x)$ defined in (3.1). This representation directly implies that the random variable

$$X_N = \frac{Y_N - \mu \log_{|q|^2} N}{\sqrt{\sigma^2 \log_{|q|^2} N}},$$

with

$$\mu = \frac{\lambda'(1)}{\lambda(1)} \quad \text{and} \quad \sigma^2 = \frac{\lambda''(1)}{\lambda(1)} + \frac{\lambda'(1)}{\lambda(1)} - \frac{\lambda'(1)^2}{\lambda(1)^2}$$

satisfies a central limit theorem and we have convergence of all moments. More precisely we get (uniformly in y)

$$\frac{1}{\pi N} \# \left\{ |z|^2 < N : s_F(z) \le \mu \log_{|q|^2} N + y\sqrt{\sigma^2 \log_{|q|^2} N} \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}u^2}\, du + o(1)$$

and (for every $L \ge 0$)

$$\frac{1}{\pi N} \sum_{|z|^2<N} \left(s_F(z) - \mu \log_{|q|^2} N\right)^L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^L e^{-\frac{1}{2}u^2}\, du + o(1).$$

The drawback of the method given in Section 3 is that it only works for real numbers x. In Section 4 we present a method that is based on Dirichlet series that extends (2.4) to a complex neighbourhood of $x = 1$. Furthermore, we provide upper bounds for $S_N(x)$ for complex x with modulus $|x|$ close to 1. With the help of this extension we are able to provide more precise distributional results. Besides the central limit theorem we also get a local limit theorem, that is, asymptotic expansions for the numbers

$$\#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\}$$

if k is close to the mean $\mu \log_{|q|^2} N$ and if $s_F(z)$ is integer valued. Furthermore, we obtain very precise asymptotic expansions of the moments.

Next we consider the sequence $s_F(z)$ taking values in a compact abelian group A. Then the closure of the set $\{s_F(z) : z \in \mathbb{Z}[i]\}$ is a subgroup of A denoted by $A(F)$. The results on exponential sums obtained in Section 4 are used to prove uniform distribution of $(s_F(z))_{z\in\mathbb{Z}[i]}$ in the groups $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}/M\mathbb{Z}$ with respect to the Haar measure λ_A under according natural conditions. The method gives results for uniform distribution of the values of s_F in large circles, i. e.

$$\lim_{N\to\infty} \frac{1}{\pi N} \# \left\{z \in \mathbb{Z}[i]; |z|^2 < N, s_F(z) \in B\right\} = \lambda_A(B)$$

for all $B \subseteq A$ with $\lambda_A(\partial B) = 0$.

In Section 5 we use an approach based on ergodic $\mathbb{Z}[i]$-actions and skew products to extend the distribution results for group valued s_F to well uniform distribution with respect to Følner sequences $(Q_n)_{n\in\mathbb{N}}$, i. e.

$$\lim_{n\to\infty} \frac{1}{\#Q_n} \# \{z \in Q_n; s_F(z+y) \in B\} = \lambda_A(B)$$

uniformly in $y \in \mathbb{Z}[i]$. This generalises the results on uniform distribution of $(s_F(z))_{z\in\mathbb{Z}[i]}$ obtained in Section 4. On the other hand methods from ergodic theory do not allow to obtain error terms, which come as a natural by-product of the method used in Section 4.

3. A measure theoretic method

In the following we use ideas developed in [13, 14]. The measure theoretic approach to asymptotic questions about digital functions gives a smooth proof for a real version of the asymptotic representation (2.4) for $S_N(x)$.

In order to formulate our results we have to introduce some notations.

For every block $B = (\eta_0, \eta_1, \ldots, \eta_L) \in \mathcal{N}^{L+1}$ we set $B' = (\eta_1, \ldots, \eta_L) \in \mathcal{N}^L$, that is, the block without the first digit, and $\eta(B) = \eta_0$, the first digit of the block B. Furthermore, set

$$g_F(B) = \sum_{i=0}^{L} \left(F(0,\ldots,0,\eta_0,\eta_1,\ldots,\eta_i) - F(0,\ldots,0,0,\eta_1,\ldots,\eta_i)\right).$$

Note that $g_F(B) = 0$ if $\eta_0 = 0$.

By the definition of block additive function we directly get the following property.

Lemma 2. *For $z \in \mathbb{Z}[i]$ let $B = B(z) = (\varepsilon_0(z), \ldots, \varepsilon_L(z))$ the block of the first $L+1$ digits of the q-ary digital expansion of z. Then*

$$s_F(z) = g_F(B) + s_F(v),$$

where $z = \varepsilon_0(z) + qv$.

Now define a matrix $\mathbf{A}(x) = (A_{B,C}(x))_{B,C \in \mathcal{N}^{L+1}}$ by

$$A_{B,C}(x) = \begin{cases} x^{g_F(B)} & \text{if } C = (B', \ell) \text{ for some } \ell \in \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Finally, let $\lambda(x)$ be the dominant eigenvalue of the matrix $\mathbf{A}(x)$ that surely exists if x is close to the positive real axis, in particular, if x is close to 1, compare with Lemma 4. Note that $\lambda(1) = |q|^2$.

Theorem 1. *The following asymptotic relation holds uniformly for x in some interval I containing* 1

$$\sum_{|z|^2 < N} x^{s_F(z)} = \Phi(x, \log_{|q|^2} N)\, N^{\log_{|q|^2} \lambda(x)} \cdot \left(1 + \mathcal{O}\left(N^{-\kappa}\right)\right) \tag{3.2}$$

with some $\kappa > 0$, where $\Phi(x,t)$ is a periodic (with period 1) and Hölder continuous function in t and continuous in x.

Proof. Before we present the proof of Theorem 1 we derive some direct corollaries.

Corollary 1. *Set*

$$\mu = \frac{\lambda'(1)}{\lambda(1)} \quad \text{and} \quad \sigma^2 = \frac{\lambda''(1)}{\lambda(1)} + \mu - \mu^2.$$

If $\sigma^2 > 0$ then we have uniformly for real y

$$\begin{aligned} &\frac{1}{\pi N} \# \left\{ |z|^2 < N : s_F(z) \le \mu \log_{|q|^2} N + y\sqrt{\sigma^2 \log_{|q|^2} N} \right\} \\ &\qquad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}u^2}\, du + o(1). \end{aligned} \tag{3.3}$$

and for every $L \ge 0$

$$\frac{1}{\pi N} \sum_{|z|^2 < N} \left(s_F(z) - \mu \log_{|q|^2} N \right)^L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^L e^{-\frac{1}{2}u^2}\, du + o(1). \tag{3.4}$$

Furthermore, we have exponential tail estimates of the form

$$\begin{aligned} &\frac{1}{\pi N} \# \left\{ |z|^2 < N : \left| s_F(z) - \mu \log_{|q|^2} N \right| \ge \eta \sqrt{\log_{|q|^2} N} \right\} \\ &\qquad \ll \min\left(e^{-c\eta}, e^{-c\eta^2 + \mathcal{O}\left(\eta^3/\sqrt{\log N}\right)} \right) \end{aligned} \tag{3.5}$$

for some constant $c > 0$.

Remark 1. The above result suggests that the distribution of $s_F(z)$, $|z|^2 < N$, can be approximated by a sum of (weakly dependent) random variables. This is in fact a possible approach to this problem. Observe that the constant μ can be explicitly calculated by

$$\mu = \frac{\lambda'(1)}{\lambda(1)} = \frac{1}{|q|^{L+1}} \sum_{B\in\mathcal{N}^{L+1}} s_F(B).$$

Of course, this *mean value* corresponds to the contribution of one block of length $L+1$ in the digital expansion of z that has approximately $\log_{|q|^2} N$ digits. In a similar way it is also possible to represent σ^2 but this is much more involved.

Proof of Corollary 1. Let Y_N denote the random variable that is induced by the distribution of $s_F(z)$ for $|z|^2 < N$ given by (2.2). Then the moment generating function of Y_N is given by (using (2.3))

$$\begin{aligned}\mathbb{E}\, e^{tY_N} &= \frac{1}{S_N(1)} S_N(e^t)\\ &= \frac{1}{\pi}\Phi(e^t, \log_{|q|^2} N) N^{\log_{|q|^2}\lambda(e^t)-1}\cdot\left(1+\mathcal{O}\left(N^{-\eta}\right)\right).\end{aligned}$$

Hence, by using the local expansion (recall that $\lambda(1) = |q|^2$)

$$\log\lambda(e^t) = \log|q|^2 + \mu t + \frac{\sigma^2}{2}t^2 + \mathcal{O}\left(t^3\right)$$

we directly obtain that the moment generating function of the normalised random variable

$$Z_N = \frac{Y_N - \mu\log_{|q|^2} N}{\sqrt{\sigma^2\log_{|q|^2} N}}$$

is given by

$$\begin{aligned}\mathbb{E}\, e^{tZ_N} &= e^{-t(\mu/\sigma)\sqrt{\log_{|q|^2} N}}\,\mathbb{E}\, e^{(t/\sqrt{\sigma^2\log_{|q|^2} N})Y_N}\\ &= e^{\frac12 t^2 + \mathcal{O}\left(t^3/\sqrt{\log N}\right)}.\end{aligned}$$

Of course, this directly translates to (3.3).

Furthermore, convergence of the moment generating function also implies convergence of all moments, that is, we get (3.4). Finally, the tail estimates (3.5) are a direct consequence of Chernov type inequalities. □

The proof of Theorem 1 runs along the lines of [14, Sections 4 and 5] and is organised in four steps.

Step 1 defines a sequence of discrete measures, which are obtained by suitably rescaling point masses $x^{s_F(z)}$. Let δ_z denote the Dirac measure supported at $\{z\}$. Then we define a family of measures (depending on n

and x) by setting

$$\mu_{n,x} = \frac{\sum_{z\in\mathcal{B}_n} x^{s_F(z)}\delta_{\frac{z}{q^n}}}{\sum_{z\in\mathcal{B}_n} x^{s_F(z)}}, \tag{3.6}$$

where

$$\mathcal{B}_n = \{z \in \mathbb{Z}[i] \mid \mathrm{length}(z) \le n\}.$$

Using the matrix $\mathbf{A}(x)$ introduced in (3.1), we can write the denominator in (3.6) as

$$(x^{g_F(B)})_B \cdot \mathbf{A}(x)^n \cdot (\delta_{\mathbf{0},C})_C^T,$$

$\delta_{\mathbf{0},C}$ denoting the Kronecker symbol, and ^{T} the transposition.

Step 2 uses characteristic functions to show that the sequence $\mu_{n,x}$ has a weak limit. The fact that the values $x^{s_F(z)}$ are formed from the digital expansion of z can be used to express the Fourier transforms $\hat{\mu}_{n,x}$ of the measures $\mu_{n,x}$

$$\hat{\mu}_{n,x}(t) = \frac{\sum_{z\in\mathcal{B}_n} x^{s_F(z)} e\left(\Re\left(\frac{tz}{q^n}\right)\right)}{\sum_{z\in\mathcal{B}_n} x^{s_F(z)}}. \tag{3.7}$$

in terms of matrix products. Here $t \in \mathbb{C}$ and as usual $e(\cdot) = e^{2\pi i(\cdot)}$. We define the matrix $\mathbf{H}(x,t)$ by setting

$$H_{B,C}(x,t) = A_{B,C}(x) e\left(\Re\left(tB_0\right)\right).$$

This allows us to write

$$\hat{\mu}_{n,x}(t) = \frac{\mathbf{v}_1(x, tq^{-n}) \cdot \mathbf{H}(x, \frac{t}{q^{n-1}}) \cdots \mathbf{H}(x, \frac{t}{q}) \cdot \mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^n \cdot \mathbf{v}_2} \tag{3.8}$$

with

$$\mathbf{v}_1(x,t) = \left(x^{s_F(B)} e\left(\Re\left(tB_0\right)\right)\right)_B \quad \text{and } \mathbf{v_2} = (\delta_{\mathbf{0},C})_C^T.$$

The matrices $\frac{1}{\lambda(x)}\mathbf{H}(x,t)$ satisfy the conditions of [14, Lemma 5] (*mutatis mutandis*) and therefore, the sequence of matrices

$$\mathbf{P}_n(x,t) = \lambda(x)^{-n}\mathbf{H}(x, \frac{t}{q^{n-1}}) \cdots \mathbf{H}(x, \frac{t}{q})$$

converges to a limit $\mathbf{P}(x,t)$ and

$$\begin{aligned} \|\mathbf{P}_n(x,t) - \mathbf{P}_n(x,0)\| &\ll |t| \text{ for } |t| \le 1 \\ \|\mathbf{P}_n(x,t) - \mathbf{P}(x,t)\| &\ll (1+|t|)^{\eta(x)}|q|^{-\eta(x)n} \text{ for all } t, \end{aligned} \tag{3.9}$$

where

$$\eta(x) = \frac{\log\lambda(x) - \log|\lambda_1(x)|}{\log|q| + \log\lambda(x) - \log|\lambda_1(x)|},$$

where $\lambda_1(x)$ denotes the second largest eigenvalue of $\mathbf{A}(x)$. These relations hold uniformly for x in compact subsets of $(0,\infty)$.

For $|t| \ge 1$ (3.9) together with (3.8) implies

$$|\hat{\mu}_{n,x}(t) - \hat{\mu}_x(t)| \ll |t|^{\eta(x)} q^{-n\eta(x)}. \tag{3.10}$$

For $|t| \le 1$ and $L > K > \ell$ we estimate using (3.8)

$$
\begin{aligned}
&|\hat{\mu}_{K,x}(t) - \hat{\mu}_{L,x}(t)| \\
&= \Bigg|\lambda(x)^K \frac{\mathbf{v}_1(x, tq^{-K}) \cdot \mathbf{P}_{K-\ell}(q^{-\ell}t)\mathbf{P}_\ell(t)\mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^K \cdot \mathbf{v}_2} \\
&\qquad\qquad - \lambda(x)^L \frac{\mathbf{v}_1(x, tq^{-L}) \cdot \mathbf{P}_{L-\ell}(q^{-\ell}t)\mathbf{P}_\ell(t)\mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^L \cdot \mathbf{v}_2}\Bigg| \\
&\ll \Bigg|\lambda(x)^K \frac{\mathbf{v}_1(x, tq^{-K}) \cdot \mathbf{P}_{K-\ell}(0)\mathbf{P}_\ell(t)\mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^K \cdot \mathbf{v}_2} \\
&\qquad\qquad - \lambda(x)^L \frac{\mathbf{v}_1(x, tq^{-L}) \cdot \mathbf{P}_{L-\ell}(0)\mathbf{P}_\ell(t)\mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^L \cdot \mathbf{v}_2}\Bigg| + |q|^{-\ell}|t| \\
&= \Bigg|\lambda(x)^K \frac{\mathbf{v}_1(x, tq^{-K}) \cdot \mathbf{P}_{K-\ell}(0)(\mathbf{P}_\ell(t) - \mathbf{P}_\ell(0))\mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^K \cdot \mathbf{v}_2} \\
&\qquad\qquad - \lambda(x)^L \frac{\mathbf{v}_1(x, tq^{-L}) \cdot \mathbf{P}_{L-\ell}(0)(\mathbf{P}_\ell(t) - \mathbf{P}_\ell(0))\mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^L \cdot \mathbf{v}_2}\Bigg| + |q|^{-\ell}|t| \\
&\ll |t| \left(\left(\frac{\lambda_1(x)}{\lambda(x)} \right)^{K-\ell} + |q|^{-\ell} \right) \ll |t||q|^{-\eta(x)K},
\end{aligned}
$$

where we have chosen $\ell = \lceil \eta K \rceil$. Passing L to infinity yields

$$
|\hat{\mu}_{n,x}(t) - \hat{\mu}_x(t)| \ll |t| q^{-n\eta(x)} \tag{3.11}
$$

for $|t| \le 1$. Especially, (3.10) and (3.11) establish the existence of a (weak) limiting measure μ_x.

Remark 2. What we have proved up to now is enough to have the asymptotic relation (3.2) without error term for all $x > 0$.

Step 3 establishes estimates for the measure dimension of μ_x, which will be needed in Step 4. We define the matrices $\mathbf{I}_\varepsilon$ by setting

$$
(\mathbf{I}_\varepsilon)_{B,C} = \begin{cases} \delta_{B,C} & \text{if the block } B \text{ starts with the digit } \varepsilon \\ 0 & \text{otherwise.} \end{cases}
$$

Clearly $\mathbf{I}_0 + \mathbf{I}_1 + \cdots + \mathbf{I}_{a^2}$ is the identity matrix. Furthermore, we have

$$
\begin{aligned}
&\mu_x \left(\frac{\varepsilon_1}{q} + \frac{\varepsilon_2}{q^2} + \cdots + \frac{\varepsilon_k}{q^k} + q^{-k}\mathcal{F} \right) \\
&\qquad = \lim_{n\to\infty} \frac{(\mathbf{v}_1(x,0) \cdot \mathbf{I}_{\varepsilon_1}\mathbf{A}(x)\mathbf{I}_{\varepsilon_2}\mathbf{A}(x) \cdots \mathbf{I}_{\varepsilon_k}\mathbf{A}(x)\mathbf{A}(x)^{n-k} \cdot \mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^n \cdot \mathbf{v}_2}.
\end{aligned}
$$

The limit can be computed by the Perron-Frobenius theorem and equals

$$
\lambda(x)^{-k}\mathbf{v}_1(x,0) \cdot \mathbf{I}_{\varepsilon_1}\mathbf{A}(x)\mathbf{I}_{\varepsilon_2}\mathbf{A}(x) \cdots \mathbf{I}_{\varepsilon_k}\mathbf{v}(x),
$$

where $\mathbf{v}(x)$ denotes the (Perron-Frobenius) eigenvector of $\mathbf{A}(x)$ associated to the eigenvalue $\lambda(x)$ normalised so that

$$\mathbf{v}_1(x,0)\cdot\mathbf{v}(x)=1.$$

Now we define

$$\xi(x)=\max_{\varepsilon}\max_{B}\frac{(\mathbf{A}(x)\mathbf{I}_\varepsilon\mathbf{v})_B}{(\mathbf{v}(x))_B}$$

(this is always finite, since all coordinates of $\mathbf{v}(x)$ are strictly positive). Clearly, $\xi(x)<\lambda(x)$ and $\xi(1)=1$. By definition of $\xi(x)$ we have the component-wise inequality

$$\mathbf{A}(x)\mathbf{I}_\varepsilon\mathbf{v}(x)\le\xi(x)\mathbf{v}(x),$$

from which we conclude

$$\mathbf{v}_1(x,0)\cdot\mathbf{I}_{\varepsilon_1}\mathbf{A}(x)\mathbf{I}_{\varepsilon_2}\mathbf{A}(x)\cdots\mathbf{I}_{\varepsilon_k}\mathbf{v}(x)\le\xi(x)^k\mathbf{v}_1(x,0)\cdot\mathbf{v}(x)=\xi(x)^k$$

and

$$\mu_x\left(\frac{\varepsilon_1}{q}+\frac{\varepsilon_2}{q^2}+\cdots+\frac{\varepsilon_k}{q^k}+q^{-k}\mathcal{F}\right)\ll\left(\frac{\xi(x)}{\lambda(x)}\right)^k. \tag{3.12}$$

Since $(q,\{0,\dots,a^2\})$ is a canonical number system, every ball $B(z,r)$ can be covered by an absolutely bounded number of sets of the form

$$\frac{\varepsilon_1}{q}+\frac{\varepsilon_2}{q^2}+\cdots+\frac{\varepsilon_k}{q^k}+q^{-k}\mathcal{F}$$

for $k=\lfloor-\log_{|q|}r\rfloor$ and $r<1$. This together with (3.12) implies

$$\mu_x(B(z,r))\ll r^{\beta(x)} \tag{3.13}$$

with

$$\beta(x)=\frac{\log\lambda(x)-\log\xi(x)}{\log|q|}.$$

Notice that $\beta(1)=2$, which is no surprise, since μ_1 is Lebesgue measure restricted to $\mathcal{F}$.

Furthermore, we need at most $\mathcal{O}(|q|^{2n})$ times the area of the annulus $B(0,r+\varepsilon+|q|^{-n})\setminus B(0,r-|q|^{-n})$ copies of $q^{-n}\mathcal{F}$ to cover the annulus $B(0,r+\varepsilon)\setminus B(0,r)$. This together with (3.12) implies

$$\mu_x\left(B(0,r+\varepsilon)\setminus B(0,r)\right)\ll|q|^{-n\beta(x)}|q|^{2n}(2r+\varepsilon)(\varepsilon+|q|^{-n})$$

for all n. Setting $n=-\lceil\log_{|q|}\varepsilon\rceil$ gives

$$\mu_x\left(B(0,r+\varepsilon)\setminus B(0,r)\right)\ll(r+\varepsilon)\varepsilon^{\beta(x)-1}. \tag{3.14}$$

This gives a sensible estimate, if $\beta(x)>1$ or equivalently $\log\xi(x)<\log\lambda(x)-\log|q|$. Since this inequality is satisfied for $x=1$ and $\beta(x)$ depends continuously on x, there exists an interval I around $x=1$, such that $\beta(x)\ge\beta_0>1$ for some $\beta_0<2$.

Step 4 uses the estimates for the measure dimension of μ_x and a suitable version of the Berry-Esseen inequality to provide bounds for $|\mu_{n,x}(B(0,r))-$

$\mu_x(B(0,r))|$. Since $\mu_{n,x}(B(0,r))$ can be easily related to the sum occurring in (3.2), this gives the error term in (3.2).

We recall the following result obtained in [14]. The statement uses the notation $\mathbf{c}(\phi) = (\cos\phi, \sin\phi)^T$.

Proposition 1 ([14, Proposition 1])**.** *Let ν_1 and ν_2 be two probability measures in $\mathbb{R}^2$ with their Fourier transforms defined by*

$$\hat{\nu}_k(\mathbf{t}) = \int\limits_{\mathbb{R}^2} e\left(\langle \mathbf{x}, \mathbf{t}\rangle\right)\, d\nu_k(\mathbf{x}).$$

Suppose that ν_2 satisfies

$$\nu_2\left(B(\mathbf{0}, r+\varepsilon) \setminus B(\mathbf{0}, r)\right) \ll \varepsilon^\theta \tag{3.15}$$

for some $0<\theta<1$ and all $r \geq 0$. Then the following inequality holds for all $r\geq 0$ and $T>0$

$$\begin{aligned} &|\nu_1(B(\mathbf{0},r)) - \nu_2(B(\mathbf{0},r))| \\ &\qquad \ll \int_0^T \int_0^{2\pi} K_r(t,T)\,|\hat{\nu}_1(t\mathbf{c}(\phi)) - \hat{\nu}_2(t\mathbf{c}(\phi))|\, t\, d\phi\, dt + T^{-\frac{2\theta}{\theta+2}}, \end{aligned} \tag{3.16}$$

where the kernel function $K_r(t,T)$ satisfies

$$K_r(t,T) \ll \frac{1}{T^2} + \min\left(r^2, \frac{r^{\frac12}}{t^{\frac32}}\right).$$

The implied constant in (3.16) *depends only on the implied constant in* (3.15)*.*

Inserting (3.10) and (3.11) into (3.16) with $\theta = \beta(x) - 1$ yields

$$\begin{aligned} &|\mu_{n,x}(B(0,r)) - \mu_x(B(0,r))| \\ &\ll \int_0^1 K_r(t,T) t |q|^{-\eta(x)n} t\, dt + \int_1^T K_r(t,T) t^{\eta(x)} |q|^{-\eta(x)n} t\, dt + T^{-2\frac{\beta(x)-1}{\beta(x)+1}}. \end{aligned} \tag{3.17}$$

Using the bounds for $K_r(t,T)$ and setting

$$\log T = \frac{\eta(x)}{\eta(x) + \frac12 + 2\frac{\beta(x)-1}{\beta(x)+1}} n \log|q|$$

yields

$$|\mu_{n,x}(B(0,r)) - \mu_x(B(0,r))| \ll |q|^{-2\kappa(x)n}$$

uniformly in r with

$$\kappa(x) = \frac{\eta(x)(\beta(x)-1)}{(\eta(x)+\frac12)(\beta(x)+1) + 2\beta(x) - 2}.$$

Choosing κ to be the minimum attained by $\kappa(x)$ on a compact interval I, where $\beta(x) \geq \beta_0 > 1$ for some $\beta_0 < 2$, gives

$$|\mu_{n,x}(B(0,r)) - \mu_x(B(0,r))| \ll |q|^{-2\kappa n} \tag{3.18}$$

for all $x \in I$.

Now, by definition of $\mu_{k,x}$, we have

$$\sum_{|z|^2<N} x^{s_F(z)} = \mathbf{v}_1(x,0)\cdot \mathbf{A}(x)^k \cdot \mathbf{v}_2 \cdot \mu_{k,x}(B(0,|q|^{-k}\sqrt{N}))$$

for $k = \lfloor \log_{|q|^2} N\rfloor + M$ and some integer constant $M > 0$, which is chosen so that $B(0,|q|^{1-M}) \subset \mathcal{F}$. Inserting (3.18) and $\mathbf{v}_1(x,0)\cdot \mathbf{A}(x)^k \cdot \mathbf{v}_2 = C(x)\lambda(x)^k + \mathcal{O}(\lambda_1(x)^k)$ yields

$$\begin{aligned}\sum_{|z|^2<N} x^{s_F(z)} &= C(x)\lambda(x)^k \mu_x(B(0,|q|^{-k}\sqrt{N})) + \mathcal{O}(\lambda_1(x)^k) + \mathcal{O}(\lambda(x)^k|q|^{-2\kappa k})\\ &= N^{\log_{|q|^2}\lambda(x)} C(x)\lambda(x)^{\{\log_{|q|^2} N\}+M} \mu_x(B(0,q^{\{\log_{|q|^2} N\}-M}))(1+\mathcal{O}(N^{-\kappa})).\end{aligned}$$

We observe that the measure μ_x satisfies the self-similarity relation

$$\mu_x(B(0,|q|r)) = \lambda(x)\mu_x(B(0,r))$$

for r sufficiently small. Setting

$$\Phi(x,t) = C(x)\lambda(x)^{t+M}\mu_x(B(0,q^{t-M})) \text{for } t<1$$

and noting that (3.13) implies the Hölder continuity of Φ as a function of t completes the proof. □

Remark 3. For complex values of x this method breaks down, because the weak limits μ_x have infinite total variation and are therefore not complex measures.

4. A Dirichlet series method

The goal of this section is to generalise Theorem 1 to complex x. The proof relies on Dirichlet series and Mellin-Perron techniques.

Theorem 2. *There exists a complex neighbourhood of $x = 1$ (that is, $|x-1| \le \delta$ for some $\delta > 0$) such that uniformly*

$$\sum_{|z|^2<N} x^{s_F(z)} = \Phi(x, \log_{|q|^2} N)\, N^{\log_{|q|^2}\lambda(x)} \cdot \left(1 + \mathcal{O}\left(N^{-\kappa}\right)\right) \tag{4.1}$$

with some $\kappa > 0$, where $\Phi(x,t)$ is a function that is analytic in x and periodic (with period 1) and Hölder continuous in t.

Furthermore, if F is integer valued with the property that

$$d = \gcd\{g_F(B) : B \in \mathcal{N}^{L+1}\} = 1. \tag{4.2}$$

Then we have uniformly for $|x-1| \ge \delta$ and $|\Re(x) - 1| \le \delta_2$

$$\sum_{|z|^2<N} x^{s_F(z)} \ll N^{\log_{|q|^2}\lambda(|x|)-\kappa} \tag{4.3}$$

with some $\kappa > 0$ and some δ_2 with $0 < \delta_2 < \delta$.

Remark 4. In what follows we will also show that $\Phi(x,t)$ has an explicit representation (see (4.21)). For example, for the sum-of-digits function $s_q(z)$ we have

$$\Phi(x,t) = \frac{X^{-t}}{1-X^{-1}} \sum_{\ell=1}^{a^2} x^\ell X^{\left\lfloor t - \log_{|q|^2} \ell^2 \right\rfloor} + \frac{X^{-t}}{1-X^{-1}} \sum_{\ell=1}^{a^2} x^\ell \sum_{z\neq 0} x^{s_q(z)} \left(X^{\left\lfloor t-\log_{|q|^2} |qz+\ell|^2 \right\rfloor} - X^{\left\lfloor t-\log_{|q|^2} |qz|^2\right\rfloor} \right),$$

where X abbreviates

$$X = \frac{x^{|q|^2}-1}{x-1}.$$

The asymptotic representations (4.1) and (4.3) can be used in various ways (compare also with [7] and [8]). We directly derive asymptotic expansions for moments (Corollary 2) and a refinement of the central limit theorem stated in Corollary 1, further a local limit theorem (Corollary 3), uniform distribution in residue classes (Corollary 4) and uniform distribution modulo 1 (Corollary 5).

Corollary 2. *For every integer $r \geq 1$ we have*

(4.4)
$$\frac{1}{\pi N} \sum_{|z|^2<N} s_F(z)^r = \mu^r (\log_{|q|^2} N)^r + \sum_{\ell=0}^{r-1} G_{r,\ell}(\log_{|q|^2} N) \cdot (\log_{|q|^2} N)^\ell + \mathcal{O}\left(N^{-\kappa}\right),$$

where the functions $G_{r,\ell}(t)$ $(0 \leq \ell < r)$ are continuous and periodic (with period 1).

Proof. Since (4.1) is uniform in a neighbourhood of 1 and $\Phi(x,t)$ is analytic in x one can take derivatives at $x=1$ at arbitrary order by using the formula

$$G^{(r)}(1) = \frac{r!}{2\pi i} \int_{|x-1|=\delta/2} \frac{G(x)}{(x-1)^{r+1}} dx.$$

Furthermore, note that $\Phi(1,t) = \pi$. Hence, the asymptotic leading term is given by $\left(\lambda'(1)/\lambda(1)\right)^r (\log_{|q|^2} N)^r$ and has no periodic fluctuations. □

Note that if we combine Corollaries 1 and 2 then we also get error terms for the central moments of the form

$$\frac{1}{\pi N} \sum_{|z|^2<N} \left(s_F(z) - \mu \log_{|q|^2} N\right)^L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^L e^{-\frac{1}{2}u^2}\, du + \mathcal{O}\left(N^{-\kappa}\right),$$

for every integer $L \geq 0$. Furthermore, if we use the characteristic function $\mathbb{E}\, ^{itY_N} = S_N(e^{it})/S_N(1)$ instead of the moment generating function $\mathbb{E}\, e^{tY_N}$, that is, we set $x = e^{it}$ in Theorem 2, combined with Berry-Esseen techniques we get a central limit theorem with error terms, too:

$$\frac{1}{\pi N}\#\left\{|z|^2 < N : s_F(z) \le \mu \log_{|q|^2} N + y\sqrt{\sigma^2 \log_{|q|^2} N}\right\}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}u^2}\,du + \mathcal{O}\left((\log N)^{-\frac{1}{2}}\right).$$

Corollary 3. *Suppose that F is integer valued and that (4.2) holds. Set*

$$\mu(x) = \frac{x\lambda'(x)}{\lambda(x)} \quad \text{and} \quad \sigma^2(x) = \frac{x^2\lambda''(x)}{\lambda(x)} + \mu(x) - \mu(x)^2.$$

Furthermore, for $k \in K(N) = \mathbb{Z} \cap \left[\mu(1-\delta_2)\cdot \log_{|q|^2} N, \mu(1+\delta_2)\cdot \log_{|q|^2} N\right]$ we define $x_{k,N}$ by $\mu(x_{k,N}) = k/\log_{|q|^2} N$, where δ and δ_2 are from Theorem 2. Then we have uniformly for $k \in K(N)$

$$\#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\}$$
$$= \frac{\Phi(x_{k,N}, \log_{|q|^2} N)}{\sqrt{2\pi\sigma^2(x_{k,N})\log_{|q|^2} N}}\, N^{\log_{|q|^2}\lambda(x_{k,N})}\, x_{k,N}^{-k}\left(1 + \mathcal{O}\left(\frac{1}{\log N}\right)\right) \tag{4.5}$$

Furthermore, if $|k - \mu \log_{|q|^2} N| \le C\sqrt{\log_{|q|^2} N}$ (for some $C > 0$) we also have

$$\#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\}$$
$$= \frac{\pi N}{\sqrt{2\pi\sigma^2 \log_{|q|^2} N}} \exp\left(-\frac{(k - \mu\log_{|q|^2} N)^2}{2\sigma^2 \log_{|q|^2} N}\right)\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log N}}\right)\right). \tag{4.6}$$

Note that $\mu = \mu(1)$ and $\sigma^2 = \sigma^2(1)$.

Proof. We apply (4.1) and (4.3) and use Cauchy's formula:

$$\#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\} = \frac{1}{2\pi i}\int_{|x| = x_{k,N}} \left(\sum_{|z|^2<N} x^{s_F(z)}\right) x^{-k-1}\,dx,$$

where $x_{k,N}$ is the saddle point of the asymptotic leading term of the integrand:

$$N^{\log_{|q|^2}\lambda(x)} x^{-k} = e^{\log\lambda(x)\cdot\log_{|q|^2} N - k\log x}.$$

We do not work out the details of standard saddle point techniques. We just refer to [8], where problems of almost the same kind have been discussed. □

Corollary 4. *Suppose that F is integer valued and that (4.2) holds. Then for every integer $M \ge 1$ and all $m \in \{0, 1, \ldots, M-1\}$ we have*

$$\frac{1}{\pi N}\#\left\{|z|^2 < N : s_F(z) \equiv m \bmod M\right\} = \frac{1}{M} + \mathcal{O}\left(N^{-\eta}\right)$$

for some $\eta > 0$.

Remark 5. Alternatively to condition (4.2) we can assume that s_F attains a value that is relatively prime to M. Then the same assertion holds, compare with Corollary 8.

Proof. We use (4.3) for all M-th roots of unity $x = e^{2\pi im/M}$ and apply simple discrete Fourier techniques. □

Corollary 5. *Let s_F be a block-additive function which attains one irrational value. Then the sequence $(s_F(z))_{z\in\mathbb{Z}[i]}$ is uniformly distributed modulo 1.*

Remark 6. Note that Corollary 5 in particularly applies to sequences of the kind $(\alpha s_F(z))_{z\in\mathbb{Z}[i]}$ if s_F is integer valued and if α is irrational.

Proof. We only have to prove that there exists a block B of length $L+1$ such that $g_F(B)$ is irrational. For this purpose we find a $z_0 \in \mathbb{Z}[i]$ with $s_F(z_0)$ irrational and with base q representation of minimal length. Then by Lemma 2 we write $z_0 = \varepsilon_0 + qv$ and $g_F(B) = s_F(z_0) - s_F(v)$. Since the base q representation of v has one digit less than the representation of z_0, $s_F(v)$ is rational, and therefore $g_F(B)$ is irrational.

Choosing $x^{g_F(B)} = e(hg_F(B))$ for $h \in \mathbb{Z}\setminus\{0\}$ gives a matrix $\mathbf{A}(x)$ with eigenvalues strictly less than $|q|^2$. By Weyl's criterion this implies the assertion. □

We now turn to the proof of Theorem 2. For this purpose we will consider the Dirichlet series

$$G_B(x,s) = \sum_{z\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(z),\ldots,\varepsilon_L(z))=B} \frac{x^{s_F(z)}}{|z|^{2s}}$$

for $B \in \mathcal{N}^{L+1}$. It is easy to see that these series are well defined in a certain range. Set $A_1 = \max_{B\in\mathcal{N}^{L+1}} F(B)$ and $A_2 = \min_{B\in\mathcal{N}^{L+1}} F(B)$. Then we have $A_2 \log_{|q|^2}|z| - \mathcal{O}(1) \le s_F(z) \le A_2 \log_{|q|^2}|z| + \mathcal{O}(1)$. Hence, if $|x| \ge 1$ then $G_B(x,s)$ is surely absolutely convergent for $\Re(s) > 1 + \frac{1}{2}A_1 \log_{|q|^2}|x|$. Similarly, if $|x| \le 1$ then $G_B(x,s)$ is absolutely convergent for $\Re(s) > 1 - \frac{1}{2}A_2 \log_{|q|^2}(1/|x|)$.

Next we provide a representation for $G_B(x,s)$ that can be used for analytic continuation.

Lemma 3. *Define the vectors* $\mathbf{G}(x,s) = (G_B(x,s))_{B\in\mathcal{N}^{L+1}}$ *and* $\mathbf{H}(x,s) = (H_B(x,s))_{B\in\mathcal{N}^{L+1}}$*, where*

$$H_B(x,s) = \begin{cases} 0, & \text{if } \eta(B)=0,\\ \dfrac{x^{s_F(\eta_0)}}{|\eta_0|^{2s}} + \dfrac{x^{g_F(B)}}{|q|^{2s}} \displaystyle\sum_{\substack{v\in\mathbb{Z}[i]\setminus\{0\}\\ (\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=(0,\dots,0)}} x^{s_F(v)}\left(\frac{1}{|v+\eta_0/q|^{2s}} - \frac{1}{|v|^{2s}}\right), & \\ & \text{if } \eta_0=\eta(B)\neq 0 \text{ and } B'=(0,\dots,0),\\ \dfrac{x^{g_F(B)}}{|q|^{2s}} \displaystyle\sum_{\substack{v\in\mathbb{Z}[i]\setminus\{0\}\\ (\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=B'}} x^{s_F(v)}\left(\frac{1}{|v+\eta_0/q|^{2s}} - \frac{1}{|v|^{2s}}\right), & \\ & \text{if } \eta_0=\eta(B)\neq 0 \text{ and } B'\neq(0,\dots,0). \end{cases}$$

Then $H_B(x,s)$ *is absolutely convergent for* $\Re(s) > \frac{1}{2} + \frac{1}{2}A_1\log_{|q|^2}|x|$ *(if* $|x|\ge 1$*) and for* $\Re(s) > \frac{1}{2} - \frac{1}{2}A_2\log_{|q|^2}(1/|x|)$ *(if* $|x|\le 1$*). More precisely, we have in that range*

$$H(x,\sigma+it) \ll \begin{cases} (1+|t|)^{2(1-\sigma)+A_1\log_{|q|^2}|x|} & \text{if } |x|\ge 1,\\ (1+|t|)^{2(1-\sigma)-A_2\log_{|q|^2}(1/|x|)} & \text{if } |x|\le 1, \end{cases} \tag{4.7}$$

and a meromorphic continuation of $\mathbf{G}(x,s) = (G_B(x,s))_{B\in\mathcal{N}^{L+1}}$ *given by*

$$\mathbf{G}(x,s) = \left(\mathbf{I} - \frac{1}{|q|^{2s}}\mathbf{A}(x)\right)^{-1}\mathbf{H}(x,s), \tag{4.8}$$

where $\mathbf{A}(x)$ *is defined in (3.1).*

Proof. We use the substitution $z = \eta_0 + qv$. If $\varepsilon_0(z) = \eta_0 = 0$ we have $s_F(z) = s_F(q)$ and consequently

$$\begin{aligned} G_B(x,s) &= \frac{1}{|q|^{2s}} \sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(v),\dots,\varepsilon_L(v))=B'} \frac{x^{s_F(v)}}{|v|^{2s}}\\ &= \frac{1}{|q|^{2s}}\sum_{\ell=0}^{a^2} G_{(B',\ell)}(x,s). \end{aligned}$$

Similarly, if $\eta_0 > 0$ and $B' = (0,\dots,0)$ we get

$$\begin{aligned} G_B(x,s) &= \frac{x^{s_F(\eta_0)}}{|\eta_0|^{2s}} + \frac{x^{g_F(B)}}{|q|^{2s}} \sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=(0,\dots,0)} \frac{x^{s_F(v)}}{|v+\eta_0/q|^{2s}}\\ &= \frac{x^{g_F(B)}}{|q|^{2s}} \sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=(0,\dots,0)} \frac{x^{s_F(v)}}{|v|^{2s}} + H_B(x,s)\\ &= \frac{x^{g_F(B)}}{|q|^{2s}}\sum_{\ell=0}^{a^2} G_{(0,\dots,0,\ell)}(x,s) + H_B(x,s). \end{aligned}$$

Finally, if $\eta_0 > 0$ and $B' \ne (0,\dots,0)$ then the case $v=0$ cannot appear and we also get

$$G_B(x,s) = \frac{x^{g_F(B)}}{|q|^{2s}} \sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=B'} \frac{x^{s_F(v)}}{|v+\eta_0/q|^{2s}}$$
$$= \frac{x^{g_F(B)}}{|q|^{2s}} \sum_{\ell=0}^{a^2} G_{(B',\ell)}(x,s) + H_B(x,s).$$

Now with $\mathbf{A}(x) = (A_{B,C}(x))_{B,C\in\mathcal{N}^{L+1}}$ this directly translates to

$$\mathbf{G}(x,s) = \frac{1}{|q|^{2s}}\mathbf{A}(x)\mathbf{G}(x,s) + \mathbf{H}(x,s)$$

which implies (4.8).

Set $s = \sigma + it$. Since

$$\left||v+\ell/\eta_0|^{2s} - |v|^{2s}\right| \ll |v|^{2\sigma} \min\left(1, \frac{1+|t|}{|v|}\right)$$

it easily follows that $H_B(x,s)$ is absolutely convergent for $\Re(s) > \frac12 + \frac12 A_1 \log_{|q|^2}|x|$ (if $|x|\ge 1$) and for $\Re(s) > \frac12 - \frac12 A_2 \log_{|q|^2}(1/|x|)$ (if $|x|\le 1$) and that $H(x,s)$ is bounded by (4.7). □

If we set $a_n = \sum_{|z|^2=n} x^{s_F(z)}$ then $G(s,x) = \sum_{n\ge1} a_n n^{-s}$ and Mellin-Perron's formula gives (for non-integral N)

$$\sum_{n<N} a_n = \sum_{0\ne|z|^2<N} x^{s_q(z)} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} G(x,s)\frac{N^s}{s}\,ds \tag{4.9}$$

for any sufficiently large c such that the line $\Re(s)=c$ is contained in the half-plane of convergence of $G(x,s)$.

We will first use this representation to get upper bounds for the sum $\sum_{0\ne|z|^2<N} x^{s_q(z)}$. For this purpose we have to know something on the dominant eigenvalue $\lambda(x)$ of $\mathbf{A}(x)$.

Lemma 4. *If x is sufficiently close to the positive real axis then $\lambda(x)$ is a simple eigenvalue of $\mathbf{A}(x)$ and all other eigenvalues have smaller modulus.*

Furthermore, if F is integer valued such that (4.2) holds and if $x\ne0$ is not a positive real number then all eigenvalues β of $\mathbf{A}(x)$ are bounded by

$$|\beta| < \lambda(|x|). \tag{4.10}$$

Proof. Suppose first that x is a positive real number. Then it easily follows that $\mathbf{A}(x)$ is a primitive irreducible nonnegative matrix. We just have to observe that for every pair of blocks $B, C \in \mathcal{N}_{L+1}$ there exists a Gaussian integer z such that both, B and C, occur in the q-ary digital expansion of z. Hence, all elements of $\mathbf{A}(x)^{L+1}$ are positive and consequently by [24, Theorem 2.1, p. 49] $\mathbf{A}(x)$ is primitive and irreducible. Thus, $\lambda(x) > 0$ is simple and all other eigenvalues have smaller modulus. By continuity, this property remains true if x is sufficiently close to the positive real axis.

Next, suppose that $x = |x|e^{i\varphi}$ with $0 < \varphi < 2\pi$ is not a positive real number. Since $|x^{g_F(B)}| = |x|^{g_F(B)}$ [24, Theorem 2.1, p. 36] implies that all eigenvalues β of $\mathbf{A}(x)$ are bounded by $|\beta| \le \lambda(|x|)$. Furthermore, equality $|\beta| = \lambda(|x|)$ holds if and only if there exists a complex number μ with $|\mu| = 1$ and a diagonal matrix $D = \mathrm{diag}(\mu_B)_{B\in\mathcal{N}_{L+1}}$ with complex numbers μ_B of modulus $|\mu_B| = 1$ such that

$$\mathbf{A}(x) = \lambda\, D\, \mathbf{A}(|x|)\, D^{-1}.$$

Without loss of generality we may assume that $\mu_{00\cdots 0} = 1$.

We now show that in this case $\mu = 1$ and $\mu_B = 1$ for all $B \in \mathcal{N}_{L+1}$, resp. $\mathbf{A}(x) = \mathbf{A}(|x|)$. First observe that $A_{00\cdots 0,\, 00\cdots 0}(x) = 1$ (for all x). Thus, $\mu = 1$. Furthermore, observe that $A_{B,C}(x) = A_{B,C}(|x|) \ne 0$ implies $\mu_B = \mu_C$. Obviously, we have $A_{B,C}(x) = A_{B,C}(|x|) \ne 0$ if $C = (B', \ell)$ (for some ℓ) and $\eta_B = 0$. Thus, if $B = (\eta_1, \ldots, \eta_L)$ is any block in $\mathcal{N}_{L+1}$ then we can consider the sequence of blocks

$$B_0 = (0,0,\ldots,0),\ B_1 = (0,\ldots,0,\eta_1),\ B_2 = (0,\ldots,0,\eta_1,\eta_2),\ \ldots,\ B_L = B$$

and can conclude inductively that

$$1 = \mu_{B_0} = \mu_{B_1} = \cdots = \mu_B.$$

However, if (4.2) holds then for every φ, $0 < \varphi < 2\pi$, there exists $B \in \mathcal{N}_{L+1}$ with $e^{i\varphi g_F(B)} \ne 1$ and, thus, $x^{g_F(B)} \ne |x|^{g_F(B)}$. Consequently, all eigenvalues β of $\mathbf{A}(|x|e^{i\varphi})$ are strictly bounded by $|\beta| < \lambda(|x|)$. □

Next note that the inverse matrix $((\mathbf{I} - u\mathbf{A}(x))^{-1}$ can be written as

$$(\mathbf{I} - u\mathbf{A}(x))^{-1} = \frac{1}{\det(\mathbf{I} - u\mathbf{A}(x))}\,(P_{BC}(u,x))_{B,C\in\mathcal{N}^{L+1}}$$

with polynomials $P_{BC}(u,x)$ having degree in u smaller than $D := |\mathcal{N}^{L+1}| = |q|^{2L+2}$. As above let $\lambda(x)$ be the dominating eigenvalue of $\mathbf{A}(x)$ and $\lambda_2(x), \ldots, \lambda_D(x)$ the remaining ones (where we assume that x is sufficiently close to the real axis and that all roots are simple.) Then by the partial fraction decomposition we have

$$\frac{P_{BC}(u,x)}{\det(\mathbf{I} - u\mathbf{A}(x))} = \frac{C_{BC}(x)}{1 - u\lambda(x)} + \sum_{j=2}^{D} \frac{C_{j,BC}(x)}{1 - u\lambda_j(x)} \tag{4.11}$$

for certain (analytic) function $C_{BC}(x)$ and $C_{j,BC}(x)$. This also shows that $G(x,s)$ can be represented as

$$G(x,s) = \frac{K(x,s)}{1 - \frac{1}{|q|^{2s}}\lambda(x)} + \sum_{j=2}^{D} \frac{K_j(x,s)}{1 - \frac{1}{|q|^{2s}}\lambda_j(x)}, \tag{4.12}$$

where $K(x,s)$ resp. $K_j(x,s)$ are linear combination of the functions $H_B(x,s)$ with coefficients that are analytic in x, compare also with (4.20).

This shows that (4.8) provides an analytic continuation of $G(s,x)$ to the the range $\Re(s) > \log_{|q|^2}|\lambda(x)|$ if x is sufficiently close to 1, say $|x-1| \le \delta$. Furthermore, if $|x-1| \ge \delta$ and $|\Re(x)-1| \le \delta_2$ then Lemma 4 shows that all eigenvalues β of $\mathbf{A}(x)$ satisfy $|\beta| \le \lambda(|x|) - \eta'$ for some η'. Consequently, for all x in that range the function $G(x,s)$ is analytic in the half plane $\Re(s) > \log_{|q|^2}(\lambda(|x|) - \eta')$.

With help of this knowledge we are now ready to prove the second part of Theorem 2. The proof method is close to that of [16].

Lemma 5. *Suppose that F is integer valued and that (4.2) holds. Then there exists $\delta > 0$ and $\kappa > 0$ such that*

$$\sum_{|z|^2<N} x^{s_F(z)} \ll N^{\log_{|q|^2}\lambda(|x|)-\kappa} \tag{4.13}$$

uniformly for $|x-1| \ge \delta$ and $|\Re(x)-1| \le \delta_2$

Proof. Our starting point is formula (4.9). Observe that the integral is not absolutely convergent. However, a slight variation of the Mellin-Perron formula gives

(4.14)
$$S_N^{(2)}(x) = \sum_{0\ne|z|^2<N} x^{s_q(z)}\left(1-\frac{|z|^2}{N}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(x,s)\frac{N^s}{s(s+1)}\,ds$$

with an integral that will be absolutely convergent in the range of interest.

Suppose now that $|x-1| \ge \delta$ and $|\Re(x)-1| \le \delta_2$. Then we already know that $G(x,s)$ is analytic for $\Re(s) > \log_{|q|^2}(\lambda(|x|)-\eta')$ and that it can be estimated by

$$|G(x,s)| \ll (1+|t|)^{2(1-\sigma)+\eta''}$$

if $\sigma = \Re(s) \ge \log_{|q|^2}(\lambda(|x|)-\eta'/2) > \log_{|q|^2}\lambda(|x|)-\eta'''$. Thus, it follows that

$$S_N^{(2)}(x) \ll N^{\log_{|q|^2}\lambda(|x|)-\eta'''}.$$

It is now easy to derive proper upper bounds for

$$S_N(x) = \sum_{0\ne|z|^2<N} x^{s_q(z)}$$

Observe that for every factor $\rho > 1$ we have

$$S_N(x) = \frac{\rho S_{\rho N}^{(2)}(x) - S_N^{(2)}(x)}{\rho-1} + \frac{1}{\rho-1}\sum_{N\le|z|^2<\rho N} x^{s_F(z)}\left(1-\frac{|z|^2}{N}\right).$$

Set $c = \log_{|q|^2}\lambda(|x|) - \eta'''$. By adjusting δ_2 we can assume that $c < 1$. Finally with

$$\rho = 1 + N^{-(1-c)/2}$$

it follows that

$$S_N(x) \ll N^{(1+c)/2} N^{\max\left(A_1 \log_{|q|^2}(1+\delta_2), A_2 \log_{|q|^2}(1-\delta_2)\right)}.$$

Since δ_2 can be chosen arbitrarily small it finally follows that

$$S_N(x) \ll N^{\log_{|q|^2} \lambda(|x|) - \eta},$$

where $\eta > 0$. □

In order to prove the asymptotic expansion (4.1) for complex x (close to 1) we will use the following properties (see also [3, p. 243]).

Lemma 6. *Suppose that a and c are positive real numbers. Then*

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} - 1 \right| \le \frac{a^c}{\pi T \log a} \qquad (a > 1), \tag{4.15}$$

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} \right| \le \frac{a^c}{\pi T \log(1/a)} \qquad (0 < a < 1), \tag{4.16}$$

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} - \frac{1}{2} \right| \le \frac{C}{T} \qquad (a = 1). \tag{4.17}$$

Proof. Suppose first that $a > 1$. By considering the contour integral of the function $F(s) = a^s/s$ around the rectangle with vertices $-A-iT, c-iT, c+iT, -A+iT$ and passing A to infinity one directly gets the representation

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} &= \mathrm{Res}(a^s/s; s=0) \\ &\quad + \frac{1}{2\pi i} \int_{-\infty}^{c} \frac{a^{x+iT}}{x+iT}\, dx \\ &\quad + \frac{1}{2\pi i} \int_{-\infty}^{c} \frac{a^{x-iT}}{x-iT}\, dx. \end{aligned}$$

Since

$$\left| \frac{1}{2\pi i} \int_{-\infty}^{c} \frac{a^{x\pm iT}}{x \pm iT}\, dx \right| \le \frac{a^c}{\pi T \log a}$$

we directly obtain the bound in the case $a > 1$.

The case $0 < a < 1$ can be handled in the same way. And finally, in the case $a = 1$ the integral can be explicitly calculated (and estimated). □

For the formulation of the next lemma we use Iverson's notation $[\![p]\!]$ which is 1 if p is a true proposition and 0 else.

Lemma 7. *Suppose that ℓ is a positive real number, λ a non-zero complex number c a real number with $c > \log_{|q|^2} |\lambda|$. Then we have for all real* *numbers* $N > \ell^2$

(4.18)
$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{\frac{1}{\ell^{2s}}}{1-\frac{1}{|q|^{2s}}\lambda}\frac{N^s}{s}\,ds = \frac{\lambda^{\lfloor \log_{|q|^2}(N/\ell^2)\rfloor+1}-1}{\lambda-1} - \frac{1}{2}\lambda^{\lfloor \log_{|q|^2}(N/\ell^2)\rfloor}[\![\log_{|q|^2}(N/\ell^2)\in\mathbb{Z}]\!]$$

Furthermore, if $c > \max\{1, \log_{|q|^2}|\lambda|\}$ *and* x *is sufficiently close to* 1 *then we have for every set of* S *of Gaussian integers with* $0 \notin S$ *and all irrational numbers* $N > 1$

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{\sum_{z\in S}x^{s_F(z)}\left(\frac{1}{|qz+\ell|^{2s}}-\frac{1}{|qz|^{2s}}\right)}{1-\frac{1}{|q|^{2s}}\lambda}\frac{N^s}{s}\,ds$$
(4.19)
$$= \frac{1}{1-\lambda^{-1}}\sum_{z\in S}x^{s_F(z)}\left(\lambda^{\lfloor \log_{|q|^2}(N/|qz+\ell|^2)\rfloor}-\lambda^{\lfloor \log_{|q|^2}(N/|qz|^2)\rfloor}\right)$$
$$-\frac{1}{2}\sum_{z\in S}x^{s_F(z)}\lambda^{\lfloor \log_{|q|^2}(N/|qz+\ell|^2)\rfloor}[\![\log_{|q|^2}(N/|qz+\ell|^2)\in\mathbb{Z}]\!]$$
$$+\frac{1}{2}\sum_{z\in S}x^{s_F(z)}\lambda^{\lfloor \log_{|q|^2}(N/|qz|^2)\rfloor}[\![\log_{|q|^2}(N/|qz|^2)\in\mathbb{Z}]\!]+\mathcal{O}(1).$$

Proof. By assumption we have $|\lambda/|q|^{2s}| < 1$. Thus, by using a geometric series expansion and Lemma 6 we get for all $N > 1$ such that $\log_{|q|^2}(N/\ell^2)$ is not an integer

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\frac{1}{\ell^{2s}}}{1-\frac{1}{|q|^{2s}}\lambda}\frac{N^s}{s}\,ds = \sum_{k\ge 0}\lambda^k\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{N}{|q|^{2k}\ell^2}\right)^s\frac{ds}{s}$$
$$= \sum_{k\le \log_{|q|^2}(N/\ell^2)}\lambda^k + \mathcal{O}\left(\frac{1}{T}\sum_{k\ge 0}\frac{|\lambda|^k\left(\frac{N}{|q|^{2k}\ell^2}\right)^c}{\left|\log\left(\frac{N}{|q|^{2k}\ell^2}\right)\right|}\right)$$
$$= \frac{\lambda^{\lfloor \log_{|q|^2}(N/\ell^2)\rfloor+1}-1}{\lambda-1} + \mathcal{O}\left(\frac{1}{T}\frac{(N/\ell^2)^c}{1-\frac{1}{|q|^{2c}}|\lambda|}\right).$$

Similarly we can proceed if $\log_{|q|^2}(N/\ell^2)$ is an integer. Of course, this implies (4.18).

Next assume that neither $\log_{|q|^2}(N/|qz+\ell|^2)$ nor $\log_{|q|^2}(N/|qz|^2)$ are integers for all $z \in S$. Hence, if $N > |qz+\ell|^2$ then we have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\frac{1}{|qz+\ell|^{2s}}}{1-\frac{1}{|q|^{2s}}\lambda}\frac{N^s}{s}\,ds = \frac{\lambda^{\lfloor \log_{|q|^2}(N/|qz+\ell|^2)\rfloor+1}-1}{\lambda-1} + \mathcal{O}\left(\frac{1}{T}\sum_{k\ge 0}\frac{|\lambda|^k\left(\frac{N}{|q|^{2k}|qz+\ell|^2}\right)^c}{\left|\log\left(\frac{N}{|q|^{2k}|qz+\ell|^2}\right)\right|}\right)$$

and if $N < |qz+\ell|^2$ then we just have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\frac{1}{|qz+\ell|^{2s}}}{1-\frac{1}{|q|^{2s}}\lambda}\frac{N^s}{s}\,ds = \mathcal{O}\left(\frac{1}{T}\sum_{k\ge 0}\frac{|\lambda|^k\left(\frac{N}{|q|^{2k}|qz+\ell|^2}\right)^c}{\left|\log\left(\frac{N}{|q|^{2k}|qz+\ell|^2}\right)\right|}\right).$$

Furthermore, for given N there are only finitely many pairs (k,z) with

$$\left|\frac{N}{|q|^{2k}|qz+\ell|^2}-1\right| < \frac{1}{2}.$$

Hence, the series

$$\sum_{z\in S}\sum_{k\ge 0}\frac{|\lambda|^k\left(\frac{N}{|q|^{2k}|qz+\ell|^2}\right)^c}{\left|\log\left(\frac{N}{|q|^{2k}|qz+\ell|^2}\right)\right|}$$

is convergent if x is sufficiently close to 1. Consequently we get

$$\begin{aligned}
&\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{\sum\limits_{z\in S}x^{s_F(z)}\left(\frac{1}{|qz+\ell|^{2s}}-\frac{1}{|qz|^{2s}}\right)}{1-\frac{1}{|q|^{2s}}\lambda}\frac{N^s}{s}\,ds\\
&=\frac{1}{1-\lambda^{-1}}\sum_{z\in S,|z|^2<N}x^{s_F(z)}\left(\lambda^{\left\lfloor\log_{|q|^2}(N/|qz+\ell|^2)\right\rfloor}-\lambda^{\left\lfloor\log_{|q|^2}(N/|qz|^2)\right\rfloor}\right)+\mathcal{O}(1).
\end{aligned}$$

Finally, since $|qz+\ell|^2 = |qz|^2(1+\mathcal{O}(1/|z|))$ it follows that for x sufficiently close to 1 we have

$$\sum_{z\in S,|z|^2\ge N}x^{s_F(z)}\left(\lambda^{\left\lfloor\log_{|q|^2}(N/|qz+\ell|^2)\right\rfloor}-\lambda^{\left\lfloor\log_{|q|^2}(N/|qz|^2)\right\rfloor}\right)=\mathcal{O}(1).$$

This proves (4.19) if neither $\log_{|q|^2}(N/|qz+\ell|^2)$ nor $\log_{|q|^2}(N/|qz|^2)$ are integers. It is, however, easy to adapt the above calculation in the general case. □

We now come back to the representation (4.12) for $G(s,x)$. We already mentioned that $K(s,x)$ and $K_j(s,x)$ are linear combinations of the functions $H_B(x,y)$ with coefficients that are analytic in x. We make this explicit for $K(s,x)$ in the following form:

(4.20)

$$\begin{aligned}
K(s,x) &= \sum_{\ell=1}^{a^2}\frac{c'_\ell(x)}{\ell^{2s}}\\
&\quad+\sum_{\ell=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c''_{\ell,B'}(x)\sum_{\substack{z\in\mathbb{Z}[i]\setminus\{0\}\\(\varepsilon_0(z),\dots,\varepsilon_{L-1}(z))=B'}}x^{s_F(z)}\left(\frac{1}{|qz+\ell|^{2s}}-\frac{1}{|qz|^{2s}}\right)
\end{aligned}$$

Hence, we obtain for $N>1$

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{K(x,s)}{1-\frac{1}{|q|^{2s}}\lambda(x)}\frac{N^s}{s}\,ds=\frac{1}{1-\lambda(x)^{-1}}\sum_{\ell=1}^{a^2}c'_\ell(x)\lambda(x)^{\left\lfloor\log_{|q|^2}\frac{N}{\ell^2}\right\rfloor}$$
$$+\sum_{\ell=1}^{a^2}\sum_{B'\in\mathcal{L}^L}\frac{c''_{\ell,B'}(x)}{1-\lambda(x)^{-1}}\sum_{z\neq0}x^{s_F(z)}\left(\lambda(x)^{\left\lfloor\log_{|q|^2}\frac{N}{|qz+\ell|^2}\right\rfloor}-\lambda(x)^{\left\lfloor\log_{|q|^2}\frac{N}{|qz|^2}\right\rfloor}\right)$$
$$-\frac12\sum_{\ell=1}^{a^2}c'_\ell(x)\lambda(x)^{\left\lfloor\log_{|q|^2}\frac{N}{\ell^2}\right\rfloor}[\![\log_{|q|^2}\frac{N}{\ell^2}\in\mathbb{Z}]\!]$$
$$-\frac12\sum_{\ell=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c''_{\ell,B'}(x)\sum_{z\neq0}x^{s_F(z)}\lambda(x)^{\left\lfloor\log_{|q|^2}\frac{N}{|qz+\ell|^2}\right\rfloor}[\![\log_{|q|^2}\frac{N}{|qz+\ell|^2}\in\mathbb{Z}]\!]$$
$$+\frac12\sum_{\ell=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c''_{\ell,B'}(x)\sum_{z\neq0}x^{s_F(z)}\lambda(x)^{\left\lfloor\log_{|q|^2}\frac{N}{|qz|^2}\right\rfloor}[\![\log_{|q|^2}\frac{N}{|qz|^2}\in\mathbb{Z}]\!]+\mathcal{O}(1),$$

where the $\mathcal{O}(1)$-term is uniform for $N>1$ and for x in a complex neighbourhood of $x=1$. Note that the *correction terms* vanish if N is, for example, irrational. Actually we will prove in Lemma 8 that these correction terms can be always neglected since they sum up to zero in all cases.

Furthermore, note that the right hand side of this representation is of order $\mathcal{O}\left(N^{\log_{|q|^2}\Re(\lambda(x))}\right)$. Thus, if we do corresponding calculations for $K_j(x,s)$ and $\lambda_j(x)$ also get

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{K_j(x,s)}{1-\frac{1}{|q|^{2s}}\lambda_j(x)}\frac{N^s}{s}\,ds=\mathcal{O}\left(N^{\log_{|q|^2}\Re(\lambda_j(x))}\right).$$

Hence, setting

$$\overline{\Phi}(x,t)=\frac{\lambda(x)^{-t}}{1-\lambda(x)^{-1}}\sum_{\ell=1}^{a^2}c'_\ell(x)\lambda(x)^{\left\lfloor t-\log_{|q|^2}\ell^2\right\rfloor}\tag{4.21}$$
$$+\sum_{\ell=1}^{a^2}\sum_{B'\in\mathcal{L}^L}\frac{\lambda(x)^{-t}c''_{\ell,B'}(x)}{1-\lambda(x)^{-1}}\sum_{z\neq0}x^{s_F(z)}\left(\lambda(x)^{t-\left\lfloor\log_{|q|^2}|qz+\ell|^2\right\rfloor}-\lambda(x)^{\left\lfloor t-\log_{|q|^2}|qz|^2\right\rfloor}\right)$$

and

$$\overline{\overline{\Phi}}(x,t)=-\frac{\lambda(x)^{-t}}{2}\sum_{\ell=1}^{a^2}c'_\ell(x)\lambda(x)^{\left\lfloor t-\log_{|q|^2}\ell^2\right\rfloor}[\![t-\log_{|q|^2}\ell^2\in\mathbb{Z}]\!]\tag{4.22}$$
$$-\frac{\lambda(x)^{-t}}{2}\sum_{\ell=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c''_{\ell,B'}(x)\sum_{z\neq0}x^{s_F(z)}\lambda(x)^{\left\lfloor t-\log_{|q|^2}|qz+\ell|^2\right\rfloor}[\![t-\log_{|q|^2}|qz+\ell|^2\in\mathbb{Z}]\!]$$
$$+\frac{\lambda(x)^{-t}}{2}\sum_{\ell=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c''_{\ell,B'}(x)\sum_{z\neq0}x^{s_F(z)}\lambda(x)^{\left\lfloor t-\log_{|q|^2}|qz|^2\right\rfloor}[\![t-\log_{|q|^2}|qz|^2\in\mathbb{Z}]\!]$$

we end up with the representation

(4.23)

$$S_N(x) = \left(\overline{\Phi}(x, \log_{|q|^2} N) + \overline{\overline{\Phi}}(x, \log_{|q|^2} N)\right) N^{\log_{|q|^2} \lambda(x)} \cdot \left(1 + \mathcal{O}\left(N^{-\kappa}\right)\right),$$

where $\kappa > 0$ is just the minimal difference between $\Re(\lambda(x))$ and $\Re(\lambda_j(x))$ $(j \geq 2)$ when x varies in a sufficiently small neighbourhood of $x = 1$. By definition it is clear that $\overline{\Phi}(x,t) = \overline{\Phi}(x,t+1)$, $\overline{\overline{\Phi}}(x,t) = \overline{\overline{\Phi}}(x,t+1)$ and that $\overline{\Phi}(x,t)$ and $\overline{\overline{\Phi}}(x,t)$ represent analytic functions in x if t is fixed. However, $\overline{\overline{\Phi}}(x, \log_{|q|^2} N) = 0$ if N is irrational. Thus, it is natural to expect that $\overline{\overline{\Phi}}(x,t) = 0$ for all t which is in fact true. The next lemma provides this fact and also the continuity of $\overline{\Phi}(x,t)$. The proof of Theorem 2 is then completed.

Lemma 8. *The function $\overline{\Phi}(x,t)$ is Hölder continuous in t and analytic for x in a complex neighbourhood of $x = 1$. Furthermore, $\overline{\overline{\Phi}}(x,t) = 0$ for all t.*

Remark 7. In particular this shows that $\Phi(x,t)$ from Theorem 1 equals $\overline{\Phi}(x,t)$ for real x.

Proof. First assume that x is real. By considering $N = |q|^{2(n+t)}$ for $n = 0, 1, 2\ldots$ it follows from Theorem 1 and (4.23) that $\Phi(x,t) = \overline{\Phi}(x,t) + \overline{\overline{\Phi}}(x,t)$. Furthermore, we have $\overline{\overline{\Phi}}(x,t) = 0$ if t is not of the form $t = \log_{|q|^2} m - k$ for some positive integers m and k. (This occurs, for example, if $t = \log_{|q|^2} T$ for some irrational number T.) Since t with this property are dense in $[0,1)$ it follows that $\overline{\Phi}(x,t)$ is continuous in t if and only if $\overline{\overline{\Phi}}(x,t) = 0$ for all t. This observation can be also deduced from the subsequent inequality (4.24) which is also true for complex x. Hence, continuity of the mapping $t \mapsto \overline{\Phi}(x,t)$ follows from $\overline{\overline{\Phi}}(x,t) = 0$ even if x is a complex number.

We now suppose that $t \in [0,1)$ is of the form $t = \log_{|q|^2} m - k$ for some positive integers m and k where we assume that k is chosen to be minimal. If $s \neq t$ is also of that form, that is $s = \log_{|q|^2} n - j \in [0,1)$ for positive integers n and j, then we have (for a properly chosen constant $c > 0$)

$$|s - t| \geq c \cdot \left||q|^{2s} - |q|^{2t}\right| \geq \frac{1}{|q|^{2(k+j)}}.$$

In particular we obtain

$$|q|^{2j} \geq c\frac{1}{|q|^{2k}|s-t|}.$$

Observe that only terms of the form $\lambda(x)^{-j}$ contribute to $\overline{\overline{\Phi}}(x,s)$; here $n = |qz + \ell|^2$ resp. $n = |qz|^2$ for some $z \in \mathbb{Z}[i]$. Thus, if we fix some $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\overline{\overline{\Phi}}(x,s)\right| < \varepsilon$$

for all s with $|s-t|<\delta$.

Next observe that if $t=\log_{|q|^2} m - k$ then for $0<\theta<1$

$$\begin{aligned}\frac{\lambda(x)^{\left\lfloor (t+\theta)-\log_{|q|^2}\ell^2\right\rfloor}-\lambda(x)^{\left\lfloor (t-\theta)-\log_{|q|^2}\ell^2\right\rfloor}}{1-1/\lambda(x)} &= \frac{\lambda^{-k}-\lambda^{-k-1}}{1-1/\lambda(x)}\\ &=\lambda^{-k}=\lambda(x)^{\left\lfloor t-\log_{|q|^2}\ell^2\right\rfloor}.\end{aligned}$$

Thus, by a similar reasoning as above we also get

$$\left|\overline{\Phi}(x,t+\theta)-\overline{\Phi}(x,t-\theta)+\frac{1}{2}\overline{\overline{\Phi}}(x,t)\right|<\varepsilon \tag{4.24}$$

if $0<\theta<\delta$. Furthermore, by continuity of $\Phi(x,t)=\overline{\Phi}(x,t)+\overline{\overline{\Phi}}(x,t)$ we have

$$\begin{aligned}&|\Phi(x,t+\theta)-\Phi(x,t-\theta)|\\ &\qquad=\left|\overline{\Phi}(x,t+\theta)+\overline{\overline{\Phi}}(x,t+\theta)-\overline{\Phi}(x,t-\theta)+\overline{\overline{\Phi}}(x,t-\theta)\right|<\varepsilon\end{aligned}$$

in that range. Consequently

$$\begin{aligned}\left|\overline{\overline{\Phi}}(x,t)\right| &\le 2\left|\overline{\Phi}(x,t+\theta)-\overline{\Phi}(x,t-\theta)\right|+2\varepsilon\\ &\le 2\left|\Phi(x,t+\theta)-\Phi(x,t-\theta)\right|+6\varepsilon\\ &<7\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ can be chosen arbitrarily small it follows that $\overline{\overline{\Phi}}(x,t)=0$.

Thus, we have shown that $\overline{\overline{\Phi}}(x,t)=0$ for all t if x is a real number close to 1. Since $\overline{\overline{\Phi}}(x,t)$ is an analytic function in x we obtain $\overline{\overline{\Phi}}(x,t)=0$ for complex x close to 1, too. As mentioned above this implies that $\overline{\Phi}(x,t)$ is continuous in t even if x is a complex number close to 1.

Similarly we can argue to show that $\Phi(x,t)$ is Hölder continuous in t. Here we just have to use a quantified version of (4.24). We leave the details to the reader. □

5. A method based on ergodic $\mathbb{Z}[i]$-actions and skew products

In this section we will consider block additive functions s_F taking values in an abelian group A, hence one has $F:\mathcal{A}^{L+1}\to A$. The neutral element will be denoted by 0_A. We assume that A is compact metrisable, equipped with its Haar measure λ_A and introduce the metrisable compact space $\Omega:=A^{\mathbb{Z}[i]}$. The shift $\mathbb{Z}[i]$-action $\Sigma:\zeta\mapsto\Sigma_\zeta$ on Ω is defined by setting for all $\omega:z\mapsto\omega_z$ and all $\zeta\in\mathbb{Z}[i]$:

$$(\Sigma_\zeta(\omega))_z:=\omega_{\zeta+z}\,.$$

For any $\omega\in\Omega$, let consider its orbit closure K_ω which is the topological closure of its orbit

$$\mathcal{O}_\omega:=\{\Sigma_\zeta(\omega);\ \zeta\in\mathbb{Z}[i]\}$$

under the shift. Readily K_ω is a compact subspace of Ω and $\Sigma_\zeta(K_\omega) = K_\omega$ for all $\zeta \in \mathbb{Z}[i]$. The restriction of Σ_ζ on K_ω, still denoted by Σ_ζ, is a homeomorphism of K_ω, defining the shift $\mathbb{Z}[i]$-action $\Sigma : \zeta \mapsto \Sigma_\zeta$ on K_ω. By definition, the couple $\mathcal{K}_\omega := (\Sigma, K_\omega)$ is the flow associated to ω.

The function s_F can be viewed as an element of the compact space $\Omega := A^{\mathbb{Z}[i]}$. For short we set $K(F)$ (resp. $\mathcal{K}(F)$) in place of K_{s_F} (resp. $\mathcal{K}_{s_F}$) and we set $I(F) := \{s_F(z);\ z \in \mathbb{Z}[i]\}$.

Lemma 9. *Assume that A is a compact metrisable group, then the closure $A(F)$ of the set $I(F)$ is a subgroup of A.*

Proof. It is clear that the neutral element 0_A of A belong to $I(F)$ so that, due to compactness, it is enough to prove that for any a and a' in $A(F)$ one has $a + a'$ in $A(F)$. Let U be any neighbourhood of the neutral element 0_A of A and let V be another neighbourhood of 0_A such that $V + V \subset U$. By assumption there exists Gaussian integers z and z' such that $s_F(z) - a \in V$ and $s_F(z') - a' \in V$. Taking $z'' = z + q^{\mathrm{length}_q(z)+L+1} z'$ one gets $s_F(z'') - (a + a') = s_F(z) - a + s_F(z') - a' \in V + V$. Hence $s_F(z'') - (a + a') \in U$ proving that $a + a'$ belongs to $A[F]$. □

In the next theorem we make use of the following simple result:

Lemma 10. *For any neighbourhood V of 0_A in A there exists a finite set $B = B(V)$ of $\mathbb{Z}[i]$ such that for all $r \in \mathbb{Z}[i]$ there exists $b \in B$ such that $s_F(r + b) \in V$.*

Proof. We may assume that $V = -V$ otherwise replace V by $V \cap (-V)$. Since $I(F)$ is dense in $A(F)$ and $A(F)$ is compact there exists an integer $N = N(V)$ such that

$$A(F) \subseteq \bigcup_{z, \mathrm{length}_q(z) \le N} s_F(z) + V.$$

Giving any Gaussian integer r we use the q-adic expansion of r to write the decomposition $r = r' + q^{N+L+1} t$ with $\mathrm{length}_q(r') \le N + L + 1$ and choose r'' with $\mathrm{length}_q(r'') \le N$ such that $-s_F(t) \in V + s_F(r'')$. With $b = -r' + r''$ we get

$$s_F(r + b) = s_F(r'' + q^{N+L+1} t) = s_F(r'') + s_F(t) \in V.$$

In addition, from Lemma 1,

$$\begin{aligned} \mathrm{length}(b) &\le c + \frac{\log(|r'| + |r''|}{\log |q|} \\ &\le c + \frac{\log 2 + \log |q|(c + N + L + 1)}{\log |q|} \\ &\le c' + N + L + 1\,. \end{aligned}$$

The proof ends by taking $B := \{z \in \mathbb{Z}[i]\,;\ \mathrm{length}_q(z) \le c' + N + L + 1\}$. □

We are ready to prove the main result on the topological structure of $\mathcal{K}(F)$.

Theorem 3. *The flow $\mathcal{K}(F)$ is minimal, that is to say if M is any nonempty compact subspace of $K(F)$ such that $\Sigma_\zeta(M) \subset M$ for all $\zeta \in \mathbb{Z}[i]$ then $M = K(F)$.*

Proof. Since $K(F)$ is the orbit closure of s_F, it is enough to prove that s_F is uniformly recurrent (see [9, Section 4]). To this aim we have to show that for any neighbourhood W of 0_Ω, the neutral element of Ω, the set $S(W) := \{u \in \mathbb{Z}[i];\ \Sigma_u(s_F) - s_F \in W\}$ is syndetic, that is to say there is a finite set E such that $\mathbb{Z}[i] = S(W) + E$. We may restrict to fundamental neighbourhoods that are of the form

$$W(M,U) = \bigcap_{\mathrm{length}_q(z)\le M} \{\omega \in \Omega\,;\ \omega_z \in U\}$$

where U is any neighbourhood of 0_A. For the sequel, we introduce a neighbourhood V of 0_A such that $V + V \subset U$, a finite set $B = B(V)$ of $\mathbb{Z}[i]$ given by Lemma 10 and let $h = \max\{\mathrm{length}_q(b)\,;\ b \in B\}$. Let us start with any Gaussian integer z and write the decomposition $z = z' + q^{M+L+1}r$ with $\mathrm{length}_q(z') \le M{+}L{+}1$. By Lemma 1, $\mathrm{length}(-z') \le 2c{+}M{+}L{+}1$ and there exists $r' \in B$ such that $s_F(r + r') \in V$. Now chose $\zeta = -z' + q^{M+L+1}r'$. By construction $z+\zeta = q^{M+L+1}(r+r')$ which implies $s_F(z+\zeta+t) - s_F(t) \in V$ for all Gaussian integers t of length at most M. This means that $z + \zeta$ belongs to $S(W)$ with

$$\mathrm{length}_q(\zeta) \le c + \frac{\log(|z'| + |r'||q|^{M+L+1})}{\log|q|} \le c'' + M + L + 1 + h$$

were c'' is an absolute constant. Therefore ζ belongs to a finite subset of $\mathbb{Z}[i]$ and consequently $S(W)$ is syndetic. ☐

Now we introduce tools from ergodic theory to prove rather general distribution results on block-additive functions. We will use ideas discussed in more details in [16] and refer to this paper for a detailed exposition of the method.

The general idea of an approach motivated by ergodic theory is to build a dynamical system (X, T, μ) from the underlying digital expansion. The space X is then a suitably chosen compactification of $\mathbb{Z}[i]$, the action $T : \mathbb{Z}[i] \to \mathrm{Aut}(X)$ is simply addition by elements of $\mathbb{Z}[i]$. Since the compactification X carries a natural group structure in our case, μ is chosen as the Haar measure on this group. Since no non-trivial block additive function can be extended to a continuous or even measurable function on X (see Remark 9 below), we use a trick developed by T. Kamae [17], which overcomes this problem by constructing a suitable cocycle (we will introduce this notion below). The fact that the additive function has no extension to X is then reflected by the non-triviality of the cocycle.

Consider the infinite product space

$$\mathcal{K}_q = \{0, 1, \dots, a^2\}^{\mathbb{N}_0}$$

and embed $\mathbb{Z}[i]$ by q-adic digital expansion

$$\begin{aligned} \iota : \mathbb{Z}[i] &\to \mathcal{K}_q \\ z &\mapsto (\varepsilon_0(z), \varepsilon_1(z), \dots, \varepsilon_L(z), 0, 0 \dots). \end{aligned}$$

Then it was proved in [16] that addition in $\mathbb{Z}[i]$ can be extended continuously to $\mathcal{K}_q$. By this construction $\mathcal{K}_q$ inherits a group structure by

$$\mathcal{K}_q = \operatorname*{proj\,lim}_{n\to\infty} \mathbb{Z}[i]/q^n\mathbb{Z}[i].$$

The corresponding Haar measure μ is the infinite product measure of uniform distribution on the digits. The cylinder set of base $(x_0, \dots, x_n) \in \{0, \dots, a^2\}^{n+1}$ is given by

$$\begin{aligned} [x_0, \dots, x_n] &:= x_0 + x_1 q + \dots + x_n q^n + q^{n+1}\mathcal{K}_q \\ &= \{z \in \mathcal{K}_q; \varepsilon_0(z) = x_0, \dots, \varepsilon_n(z) = x_n\}. \end{aligned}$$

The Haar measure of such sets is given by $\mu([x_0, \dots, x_n]) = |q|^{-n-1}$. The Gaussian integers $\mathbb{Z}[i]$ act on $\mathcal{K}_q$ by addition

$$\begin{aligned} T : \mathbb{Z}[i] &\to \operatorname{Aut}(\mathcal{K}_q) \\ z &\mapsto (x \mapsto x + z). \end{aligned}$$

This continuous action is uniquely ergodic.

Definition 1. A sequence $(Q_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{Z}[i]$ is called a Følner sequence, if it has the following properties

(1) $\forall n : Q_n \subset Q_{n+1}$
(2) There exists a constant K such that $\forall n : \#(Q_n - Q_n) \le K\#Q_n$
(3) $\forall g \in \mathbb{Z}[i] : \lim_{n\to\infty} \frac{\#(Q_n \triangle (g + Q_n))}{\#Q_n} = 0$

($\triangle$ denotes the symmetric difference).

Classical examples of such sequences, are the sequence of balls of radius $\sqrt{n}$, $Q_n = \{z \in \mathbb{Z}[i]; |z|^2 < n\}$, or the squares $Q_n = \{z \in \mathbb{Z}[i]; |\Re(z)| < n, |\Im(z)| < n\}$. Another example more connected to digital expansions are the "discrete q-adic dragons" $Q_n = \{z \in \mathbb{Z}[i]; \text{length}_q(z) \le n\}$.

We recall that a point $x \in X$ is called (T, μ)-generic (or simply generic, if the underlying action is clear), if

$$\forall f \in C(X) : \lim_{n\to\infty} \frac{1}{\#Q_n} \sum_{z\in Q_n} f \circ T_z(x) = \int_X f\, d\mu \tag{5.1}$$

for a Følner sequence $(Q_n)_{n\in\mathbb{N}}$. By Tempel'man's ergodic theorem (cf.[21, Chapter 6, Theorem 4.4]) μ-almost all points are generic. Clearly, for a uniquely ergodic continuous action every point is generic and even more, the convergence in (5.1) is uniform in x.

For uniquely ergodic non-continuous actions we need additional conditions, which will be developed below, to have the same conclusion. To this aim we introduce the following definition.

Definition 2. Let X be a compact metrisable space and $T:\mathbb{Z}[i]\times X\to X$ a Borel-measurable $\mathbb{Z}[i]$-action. A subset $A\subset X$ is called *uniformly T-negligible*, if

$$\forall\varepsilon>0,\exists g\in C(X), g\geq \mathbb{1}_A : \limsup_{n\to\infty}\left\|\frac{1}{\#Q_n}\sum_{z\in Q_n} g\circ T_z\right\|_\infty<\varepsilon$$

for a Følner sequence $(Q_n)_{n\in\mathbb{N}}$.

Definition 3. Let X be a compact metrisable space and $T:\mathbb{Z}[i]\times X\to X$ a Borel-measurable $\mathbb{Z}[i]$-action. The action T is called *uniformly quasi-continuous*, if for every $z\in\mathbb{Z}[i]$ the set of discontinuity points of T_z is uniformly T-negligible.

Remark 8. If T is uniformly quasi-continuous and μ is a T-invariant Borel probability measure on X, then T is μ-continuous.

The following theorem is an adapted version of [23, Annexe, Théorème]. The proof is slightly simplified by the fact that the action is invertible.

Theorem 4. *Let T be a uniformly quasi-continuous $\mathbb{Z}[i]$-action on the compact metric space X and assume that T is uniquely ergodic with invariant measure λ. Then for any λ-continuous function f we have*

$$\lim_{n\to\infty}\frac{1}{\#Q_n}\sum_{z\in Q_n} f\circ T_z(x)=\int_X f\,d\lambda \tag{5.2}$$

uniformly in x.

Proof. Let $\mathcal{R}_\lambda$ denote the Banach space of real valued λ-continuous functions on X equipped with the uniform norm and let

$$E=\overline{\langle\{g-g\circ T_z; g\in\mathcal{R}_\lambda, z\in\mathbb{Z}[i]\}\rangle}.$$

Then λ defines a linear form on $\mathcal{R}_\lambda$ with $\ker(\lambda)\subseteq E$. We will show that we have equality in fact.

Let $L:\mathcal{R}_\lambda\to\mathbb{R}$ be a continuous linear form with $E\subseteq\ker(L)$ and $L(1)=1$. Define for $f\geq 0$

$$|L|(f)=\sup\left\{L(g); g\in\mathcal{R}_\lambda, |g|\leq f\right\}.$$

Then $|L|$ can be extended to a continuous positive linear form on $\mathcal{R}_\lambda$. Thus $|L|$ determines a measure ℓ on X.

We will now prove that $|L|$ and therefore ℓ is T-invariant. By definition we have for $f \geq 0$

$$\begin{aligned}|L|(f \circ T_z) &= \sup\{L(g); g \in \mathcal{R}_\lambda, |g| \leq f \circ T_z\} \\ &\geq \sup\{L(g \circ T_z); g \circ T_z \in \mathcal{R}_\lambda, |g \circ T_z| \leq f \circ T_z\} \geq |L|(f),\end{aligned}$$

where we have used $L(g) = L(g \circ T_z)$ by $E \subseteq \ker(L)$. Applying the same inequality to $f \circ T_{-z}$ shows the T invariance.

By unique ergodicity we have $\ell = \lambda$. On the other hand $|L| - L$ is also a T-invariant positive linear form. Thus we have $|L| - L = a\lambda$ with $a \geq 0$. Hence $L = (1-a)\lambda$ and by $L(1) = 1$ we get $a = 0$ and we have $E = \ker(L)$ by the Hahn-Banach theorem.

Summing up, for every $f \in \mathcal{R}_\lambda$ and every $\varepsilon > 0$ there exists a $k \in \mathbb{N}$, $g_1, \ldots, g_k \in \mathcal{R}_\lambda$, and $z_1, \ldots, z_k \in \mathbb{Z}[i]$ such that

$$\left\| f - \lambda(f) - \sum_{m=1}^{k} (g_m - g_m \circ T_{z_m}) \right\|_\infty < \varepsilon.$$

Applying the ergodic means to this inequality and using (3) in Definition 1 finishes the proof. □

We recall the definition of a cocycle:

Definition 4. Let (X, T, μ) be a $\mathbb{Z}[i]$-action on X and A an abelian group. A T-*cocycle* (or simply a cocycle, if the underlying action T is fixed) is a Borel map

$$a : \mathbb{Z}[i] \times X \to A$$

such that

$$\begin{aligned} &(i) \quad a(g+h, x) = a(g, T_h x) + a(h, x) \quad \mu - \text{a.e.}, \\ &(ii) \quad \mu\left(\bigcup_{g \in \mathbb{Z}[i]} (\{x \mid T_g x = x\} \cap \{x \mid a(g,x) \neq 0_A\})\right) = 0. \end{aligned}$$

If we assume that T is aperiodic, i.e. $\mu(\{x \mid \exists g \neq 0, T_g x = x\}) = 0$, then condition (ii) is always satisfied.

A cocycle a is called a *coboundary*, if there exists a Borel map $f : X \to A$, such that

$$\forall x \in X, g \in \mathbb{Z}[i] : a(g, x) = f(T_g x) - f(x).$$

The skew product $(X \times A, T^a, \mu \otimes \lambda_A)$ corresponding to the cocycle a is given by

$$\begin{aligned} T^a : \mathbb{Z}[i] &\to \operatorname{Aut}(X \times A) \\ z &\mapsto ((x, b) \mapsto (x + z, b + a(z, x))). \end{aligned} \tag{5.3}$$

Definition 5. An element $\alpha \in A$ is said to be an essential value of the cocycle a if for every neighbourhood $N(\alpha)$ of α in A and for every $B \in \mathfrak{B}(X)$ (Borel sets) with $\mu(B) > 0$,

$$\mu\left(\bigcup_{g\in\mathbb{Z}[i]} \left(B \cap T_g^{-1}(B) \cap \{x \mid a(g,x) \in N(\alpha)\}\right)\right) > 0. \tag{5.4}$$

Let

$$E(a) = \{\alpha \in A \mid \alpha \quad \text{is an essential value of } a\}.$$

This definition does not require ergodicity of T. We have the following proposition.

Proposition 2 (cf. [25])**.** *Let $a : \mathbb{Z}[i] \times X \to A$ be a cocycle, then the following properties hold:*

(1) *If $b : \mathbb{Z}[i] \times X \to A$ is a coboundary then $E(a + b) = E(a)$.*
(2) *$E(a)$ is a closed subgroup of A.*
(3) *a is a coboundary $\Leftrightarrow E(a) = \{0_A\}$.*

Let $\mathcal{I}$ be the set of T^a-invariant elements in $\mathfrak{B} \otimes \mathfrak{B}_A$ and put

$$I(a) = \{\beta \in A \mid \mu \otimes h_A(\tau_\beta B \bigtriangleup B) = 0 \quad \text{for every } B \in \mathcal{I}\}$$

where $\tau_\beta : X \times A \to X \times A$ is given by

$$\tau_\beta(x, \alpha) = (x, \alpha + \beta).$$

The set of essential values is directly related to the ergodicity of the skew product action T^a by the following theorem of K. Schmidt.

Theorem 5 ([25, Theorem 5.2])**.** *Let T be an ergodic action on $(X, \mathfrak{B}, \mu)$ which is assumed to be non-atomic. Then for any cocycle $a : G \times X \to A$:*

$$E(a) = I(a).$$

Corollary 6. *If T is ergodic, then*

$$T^a \quad \text{is ergodic } \Leftrightarrow E(a) = A.$$

The cocycle suitable for our purposes is defined as

$$a_F(z, x) = \begin{cases} \lim\limits_{\substack{w\to x\\ w\in\mathbb{Z}[i]}} (s_F(w + z) - s_F(w)) & \text{if the limit exists} \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

The limit exists, if the carry propagation in the addition $x + z$ terminates after finitely many steps. It was proved in [16] that for almost all $x \in \mathcal{K}_q$ the addition $x + z$ produces only finitely many carries. Thus $a_F(z, x)$ is defined for μ-almost all x. Furthermore, since $a_F(z, \cdot)$ is constant on cylinder sets defined by the different possible carries in the addition $x + z$ (cf. [16]), a_F is also μ-continuous. Furthermore, notice that the set of discontinuity points of $a_F(z,\cdot)$ is closed, hence it is also uniformly T-negligible by the unique ergodicity of the continuous action T. Thus we have proved

Lemma 11. *The skew product action* T^{a_F} *given by* (5.3) *is uniformly quasi-continuous.*

We naturally define

$$V(a_F) = \overline{\{a_F(z,x) \mid x \in \mathcal{K}_q,\, z \in \mathbb{Z}[i]\}},$$

the closed subgroup $V(a_F)$ of the values of a_F. Recalling the definition of the group $A(F) = \overline{\{s_F(z)\,;\; z \in \mathbb{Z}[i]\}}$, we have readily

Proposition 3. *The groups generated by the values of* s_F *and* a_F *are equal*

$$V(a_F) = A(F) = \overline{\{s_F(z)\,;\; z \in \mathbb{Z}[i]\}}\,.$$

Proposition 4. *Let* s_F *be a block-additive function on* $\mathbb{Z}[i]$ *and* a_F *be the corresponding cocycle on* $\mathcal{K}_q$*. Then the set of essential values of* a_F *equals the closed subgroup* $A(F)$ *of* A *generated by the values of* s_F

$$E(a_F) = A(F).$$

Proof. We need the following lemma which is the analog of [5, Lemma 12] but in the case of cocycles for $\mathbb{Z}[i]$-action.

Lemma 12. *Let* $\alpha \in A$ *and assume that for any neighbourhood* $V = V(\alpha)$ *of* α *in* A *there exists a constant* $\kappa > 0$ *such that for all non empty cylinder set* C *of* $\mathcal{K}_q$ *there exists* $\zeta \in \mathbb{Z}[i]$ *such that the inequality*

$$\mu(C \cap T_\zeta(C) \cap \{x \in \mathcal{K}_q\,;\; a_F(\zeta,x) \in V\}) \geq \kappa\mu(C).$$

holds. Then $\alpha \in E(a_F)$.

Proof of Lemma 12. Set for short $W(V,\zeta) := \{x \in \mathcal{K}_q\,;\; a_F(\zeta,x) \in V\}$. If B is a Borel subset of $\mathcal{K}_q$ due to the regularity of the Haar measure, for any $\varepsilon > 0$ (and $\varepsilon < 1$), there exists a non empty cylinder set C such that $\mu(B \cap C) \geq (1-\varepsilon)\mu(C)$, hence $\mu(C \setminus (B \cap C)) \leq \varepsilon\mu(C)$ leading to

$$\begin{aligned}\mu(B \cap T_\zeta(B) \cap W(V,\zeta)) &\geq \mu(B \cap C) \cap (T_\zeta(B \cap C)) \cap W(V,\zeta)) \\ &\geq \mu(C \cap T_\zeta(C) \cap W(V,\zeta)) - 2\varepsilon\mu(C)\,.\end{aligned}$$

Choose ζ such that $\mu(C \cap T_\zeta(C) \cap W(V,\zeta)) \geq \kappa\mu(C)$ and $\varepsilon < \kappa/2$, then we get $\mu(B \cap T_\zeta(B) \cap W(V,\zeta)) > 0$. Hence $\zeta \in E(a_F)$ as expected. □

Going back to the proof of Proposition 4, it is enough to prove that $a_F(y,z_0) \in E(a_F)$ for every $y, z_0 \in \mathbb{Z}[i]$, $y = (y_0, y_1, \ldots, y_t)_q$. Let C be any non empty cylinder set, say

$$C = [\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k].$$

Take $\zeta = q^{k+L+3} z_0$ and consider

$$C_0 = [\varepsilon_0, \varepsilon_1, \dots, \varepsilon_k, \underbrace{0, \dots, 0}_{L+2}, y_0, y_1, \dots, y_t, \underbrace{0, \dots, 0}_{M}]$$

with $M = 4 + \max(0, \mathrm{length}_q(z_0) - t)$. One has $\mu(C_0) = \kappa\mu(C)$ with $\kappa = \frac{1}{|q|^{M+t+L+2}}$ The M digits 0 in the end ensure that there is no carry propagation beyond the $k+L+t+M+4$ fixed digits. This means that for any $x \in C_0$, we have

$$a_F(\zeta, x) = a_F(z_0, y) \text{ and } C_0 \subset C \cap T_\zeta^{-1}(C)\,.$$

This implies that for any neighbourhood V of $a_F(z_0, y)$ one has

$$\mu(C \cap T_\zeta^{-1}(C) \cap W(V, \zeta)) \geq \kappa\mu(C)$$

and Lemma 12 gives $a_F(z_0, y) \in E(a_F)$. □

Remark 9. By considering both Proposition 3 and (3) in Proposition 2 one sees that if s_F can be extended to a measurable map on $\mathcal{K}_q$, then the cocycle a_F is a coboundary, hence s_F is trivial *i.e.*, $s_F(z) = 0_A$ for all $z \in \mathbb{Z}[i]$.

Putting together Proposition 4, Corollary 6, and Lemma 11 we obtain

Proposition 5. *Let s_F be a block-additive function taking its values in the compact abelian metrisable group A and let a_F be the corresponding cocycle defined by* (5.5), *and assume that $A(F) = A$. Then the skew product T^{a_F} is uniquely ergodic and more precisely, for all $\mu \otimes \lambda_A$-continuous maps $f : X \times A \to \mathbb{C}$*

$$\lim_{n\to\infty} \frac{1}{\#Q_n} \sum_{z \in Q_n} f \circ T_z^{a_F}(x, g) = \int_{X \times A} f \, d(\mu \otimes \lambda_A)$$

uniformly in $(x, g) \in \mathcal{K}_q$.

Corollary 7. *Let s_F be a real valued block additive function, which attains an irrational value. Then $(s_F(z))_{z\in\mathbb{Z}[i]}$ is well uniformly distributed modulo* 1 *with respect to any Følner sequence $(Q_n)_{n\in\mathbb{N}}$, i.e.*

$$\lim_{n\to\infty} \frac{1}{\#Q_n} \#\{z \in Q_n : \{s_F(z+y)\} \in I\} = \lambda(I)$$

for every interval $I \subset [0, 1]$ ($\{\cdot\}$ denotes the fractional part) uniformly in $y \in \mathbb{Z}[i]$.

Proof. The assumption that s_F attains an irrational value, clearly implies that $V(a_F \pmod 1) = \mathbb{R}/\mathbb{Z}$. Using Weyl's criterion (cf. [22]) the assertion is equivalent to

$$\forall k \in \mathbb{Z} \setminus \{0\} : \lim_{n\to\infty} \frac{1}{\#Q_n} \sum_{z \in Q_n} e(k s_F(z+y)) = 0$$

uniformly in $y \in \mathbb{Z}[i]$. The points $(y,0)$ are uniformly generic for T^{a_F} by Proposition 5. Now, by definition of T^{a_F} we have

$$\begin{aligned} T_z^{a_F}(y,0) &= (y+z, a_F(z,y)) \\ &= (y+z, s_F(y+z) - s_F(y) \pmod 1). \end{aligned}$$

Genericity of $(y,0)$ implies

$$\lim_{n\to\infty} \frac{1}{\# Q_n} \left| \sum_{z\in Q_n} \chi_0 \otimes e_k(T_z^{a_F}(y,0)) \right| = \lim_{n\to\infty} \frac{1}{\# Q_n} \left| \sum_{z\in Q_n} e(k s_F(y+z)) \right| = 0,$$

where χ_0 denotes the trivial character of $\mathcal{K}_q$ and $e_k(\cdot) = e(k\cdot)$. The convergence is uniform in $y \in \mathbb{Z}[i]$ □

Corollary 8. *Let s_F be an integer valued block additive function. Then for any integer $M \geq 2$ for which there exists a value $s_F(z)$ that is coprime to M the sequence $(s_F(z))_{z\in\mathbb{Z}[i]}$ is well uniformly distributed in residue classes modulo M with respect to any Følner sequence $(Q_n)_{n\in\mathbb{N}}$, i.e.*

$$\lim_{n\to\infty} \frac{1}{\# Q_n} \#\{z \in Q_n : s_F(z+y) \equiv m \bmod M\} = \frac{1}{M}$$

for $m \in \{0,1,\ldots,M-1\}$, uniformly in $y \in \mathbb{Z}[i]$.

Proof. After observing that $V(a_F \pmod M) = \mathbb{Z}/M\mathbb{Z}$, the proof runs along the same lines as the proof of Corollary 7. □

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(M. Drmota) Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8–10, A-1040 Wien, Austria
E-mail address: michael.drmota@tuwien.ac.at

(P. Grabner) Institut für Analysis und Computational Number Theory, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria
E-mail address: peter.grabner@tugraz.at

(P. Liardet) Université de Provence, CMI, UMR 6632, 39, rue Joliot-Curie, 13453 Marseille, Cedex 13, France
E-mail address: liardet@cmi.univ-mrs.fr