SOME IDENTITIES FOR CHEBYSHEV POLYNOMIALS

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Abstract. We prove a generalization of a conjectured formula of Melham and provide some background about the involved (Chebyshev) polynomials.

1. Introduction

In [3] Melham considered the two sequences

$$\begin{aligned} U_n &= pU_{n-1} - U_{n-2}, & U_0 &= 0,\ U_1 = 1, \\ V_n &= pV_{n-1} - V_{n-2}, & V_0 &= 2,\ V_1 = p, \end{aligned}$$

and conjectured the formula

$$U_n^{2k} + U_{n+1}^{2k} = \sum_{r=0}^{k} \frac{D^r V_k}{r!} U_n^{k-r} U_{n+1}^{k-r},$$

where D means differentiation with respect to p. We remark here that up to simple changes of variable these polynomials are Chebyshev polynomials. More precisely

$$\begin{aligned} U_n(p) &= \mathcal{U}_{n-1}(\tfrac{p}{2}), \\ V_n(p) &= 2\mathcal{T}_n(\tfrac{p}{2}), \end{aligned}$$

where $\mathcal{T}_n$ and $\mathcal{U}_n$ denote the classical Chebyshev polynomials of first and second kind, respectively.

The aim of this paper is to prove a general identity that contains Melham's conjecture as a special case: Set $W_n = aU_n + bV_n$ and $\Omega = a^2 + 4b^2 - b^2p^2$, then

$$W_n^{2k} + W_{n+1}^{2k} = \sum_{r=0}^{k} \Omega^{k-r} \lambda_{k,r} W_n^r W_{n+1}^r, \tag{1}$$

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with

$$\lambda_{k,r} = \sum_{0\le 2j\le r} (-1)^j \frac{k(k-1-j)!}{(k-r)!j!(r-2j)!} p^{r-2j}$$

and $\lambda_{0,0} = 2$.

From [1, 2] we know explicit expansions for Chebyshev polynomials:

$$V_k = \sum_{0\le 2j\le k} (-1)^j \binom{k-j}{j} \frac{k}{k-j}\, p^{k-2j}$$

for $k \ge 1$ and $V_0 = 2$. Then we have

$$\lambda_{k,r} = \frac{D^{k-r}V_k}{(k-r)!},$$

which links Melham's conjecture and (1).

2. Proof of the Formula

We will make use of the identity

$$\sum_{t=0}^{\infty} y^t \frac{(a+t)!}{t!}(bt+c) = a!(1-y)^{-a-2}\big(c + y(ab+b-c)\big), \tag{2}$$

which follows from

$$\sum_{t=0}^{\infty} \binom{a+t}{t} y^t = (1-y)^{-a}. \tag{3}$$

In order to prove (1) we form the generating function

$$g(z) = \sum_{k=0}^{\infty} z^k \sum_{r=0}^{k} \Omega^{k-r}\, \lambda_{k,r}\, \sigma^r$$

with $\sigma = W_n W_{n+1}$. We reorder this to obtain (setting $k - r = t$)

$$g(z) = \sum_{r\ge 0}\sum_{k\ge r} z^k \Omega^{k-r}\sigma^r \lambda_{k,r} = \sum_{r\ge 0}\sum_{t\ge 0} z^{r+t}\Omega^t\sigma^r\lambda_{r+t,r}$$

$$= \sum_{r\ge 1}\sum_{t\ge 0} z^{r+t}\Omega^t\sigma^r la_{r+t,r} + 1 + \sum_{t\ge 0} z^t\Omega^t \qquad \text{(using } \lambda_{0,0} = 2\text{)}$$

$$= \sum_{r\ge 1}\sum_{t\ge 0}\sum_{0\le 2j\le r} z^t\Omega^t(\sigma z)^r(-1)^j \frac{(r+t)(r+t-1-j)!}{t!j!(r-2j)!} p^{r-2j} + 1 + \sum_{t\ge 0} z^t\Omega^t$$

$$= \sum_{j\ge 1}\sum_{r\ge 2j}\sum_{t\ge 0} z^t\Omega^t(\sigma z)^r(-1)^j \frac{(r+t)(r+t-1-j)!}{t!j!(r-2j)!} p^{r-2j}$$

$$+ 1 + \sum_{r\ge 0}\sum_{t\ge 0} z^t\Omega^t(\sigma z)^r \frac{(r+t)!}{t!r!} p^r \qquad \text{(terms for } j = 0 \text{ plus the last sum)}$$

$$= \sum_{j\ge1}\sum_{r\ge2j}(-1)^j p^{r-2j}(\sigma z)^r(1-\Omega z)^{j-r-1}\frac{(r-j-1)!}{j!(r-2j)!}(r-j\Omega z) \quad \text{by (2)}$$

$$+\sum_{r\ge0}(p\sigma z)^r(1-\Omega z)^{-r-1}+1 \quad \text{by (3)}$$

$$= \sum_{j\ge1}\frac{(-1)^j(\sigma z)^{2j}}{(1-\Omega z)^{j+1}}\sum_{s\ge0}\left(\frac{p\sigma z}{1-\Omega z}\right)^s\frac{(j+s-1)!}{j!s!}(j(2-\Omega z)+s) \qquad (r=2j+s)$$

$$+\frac{1}{1-(\Omega+p\sigma)z}+1$$

$$= \sum_{j\ge1}\frac{(-1)^j(\sigma z)^{2j}}{(1-(\Omega+p\sigma)z)^{j+1}}\big(2-(\Omega+p\sigma)z\big) \qquad \text{by (2)}$$

$$+\frac{1}{1-(\Omega+p\sigma)z}+1$$

$$= -\frac{(\sigma z)^2\big(2-(\Omega+p\sigma)z\big)}{\big(1-(\Omega+p\sigma)z\big)\big(1-(\Omega+p\sigma)z+\sigma^2z^2\big)}+\frac{1}{1-(\Omega+p\sigma)z}+1$$

$$= \frac{2-(\Omega+p\sigma)z}{1-(\Omega+p\sigma)z+\sigma^2z^2}.$$

The generating function of the left hand side is

$$\frac{1}{1-W_n^2z}+\frac{1}{1-W_{n+1}^2z}=\frac{2-(W_n^2+W_{n+1}^2)z}{1-(W_n^2+W_{n+1}^2)z+W_n^2W_{n+1}^2z^2}$$

and the assertion follows from

$$W_n^2+W_{n+1}^2=pW_nW_{n+1}+\Omega,$$

which is easily proved e. g. by using the explicit forms

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}, \qquad V_n=\alpha^n+\beta^n,$$

with

$$\alpha=\frac{p+\sqrt{p^2-4}}{2}, \qquad \beta=\frac{p-\sqrt{p^2-4}}{2}.$$

3. Further identities

Many other similar formulæ seem to exist; we just give one other example; set

$$a_{k,r}=\sum_{0\le\lambda\le r}(-1)^\lambda p^{2k-2\lambda}\frac{k(k-\lfloor\frac{\lambda}{2}\rfloor-1)!2^{\lceil\frac{\lambda}{2}\rceil}}{(k-r)!\lambda!(r-\lambda)!}\prod_{i=0}^{\lfloor\frac{\lambda}{2}\rfloor-1}\Big(2k-2\Big\lceil\frac{\lambda}{2}\Big\rceil-1-2i\Big)$$

and $a_{0,0}=2$, then

$$W_n^{2k}+W_{n+2}^{2k}=\sum_{r=0}^{k}\Omega^{k-r}\,a_{k,r}\,W_n^rW_{n+2}^r.$$

The proof is as before.

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