LIMIT DISTRIBUTION OF Q-ADDITIVE FUNCTIONS FROM AN ERGODIC POINT OF VIEW

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Dedicated to Prof. Imre Kátai on the occasion of his 70^{th} birthday.

Abstract. We use methods from ergodic theory to prove new versions of distribution results for additive functions with respect to some numeration systems.

1. Introduction

Let $(a_n)_{n\geq 0}$ a sequence of positive integers with $a_n \geq 2$. One defines a so-called Cantor numeration system $Q = (q_n)_{n\geq 0}$ by $q_0 = 1$ and, for any non-negative integer n, $q_{n+1} = a_n q_n$, hence $q_n = a_{n-1} \cdots a_1 a_0$. With the standard convention that an empty product is equal to 1, that formula holds for $n = 0$, too. Then, any positive integer m possesses a unique expansion with respect to this numeration system by

$$m = \sum_{j\geq 0} \varepsilon_j(m) q_j, \quad \text{with} \quad 0 \leq \varepsilon_j(m) \leq a_j - 1. \tag{1.1}$$

The partial sum $\sum_{j<\ell} \varepsilon_j(m) q_j$ is the smallest non-negative representative of the residue class $m \pmod{q_\ell}$. In particular, if $a_n = q$ is constant, one retrieves the usual q-adic numeration. The Cantor numeration system is said to be *constant-like* if the sequence $(a_n)_{n\geq 0}$ is bounded.

An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ is called Q-additive, if $f(0) = 0$ and it satisfies the relation

$$f\left(\sum_{j=0}^{\infty} \varepsilon_j(n) q_j\right) = \sum_{j=0}^{\infty} f\left(\varepsilon_j(n) q_j\right). \tag{1.2}$$

In the q-adic case, such functions were introduced in [13]; some specific examples were studied earlier (cf. [6, 26]). In [9], H. Delange proved that a q-additive function f admits an asymptotic distribution function, if and only if the two series

$$\sum_{j=0}^{\infty}\left(\sum_{\varepsilon=0}^{q-1} f(\varepsilon q^j)\right) \quad \text{and} \quad \sum_{j=0}^{\infty}\sum_{\varepsilon=0}^{q-1} f(\varepsilon q^j)^2 \tag{1.3}$$

converge [9, théorème 3]. This result can be seen as the q-adic version of the Erdős-Wintner theorem on classical additive functions. Delange's investigations have found many generalisations in various directions: q-additive functions on subsequences of the integers have been

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studied by Kátai and others in [17, 18, 19, 22], additive functions with respect to more exotic number systems have been investigated in [3]. Furthermore, asymptotically normal distribution of q-additive functions such as the sum-of-digits function was studied in [5, 10, 11] and very precise fractal-type asymptotic estimates of their mean value have been given in [31].

The principal purpose of this paper is to give a new proof of Delange's theorem in terms of ergodic theory and probability theory and to extend it to constant-like Cantor numeration systems. The nature of the limiting distribution is also investigated. Furthermore, we extend this technique to more general numeration systems, namely Ostrowski numeration using denominators of convergents of irrational numbers as base sequence (cf. [4, 32]), and linear recurrent ones; in the latter case, we give a refinement of the results on the existence of asymptotic distribution functions of more general additive functions obtained in [3]. The method we present has the advantage to prove Delange's theorem much faster than by using the original Fourier analysis techniques (Delange's proof is based on Lévy's theorem on characteristic functions and thus needs quite long investigations on mean values of $q-$multiplicative functions). However, it does not allow to prove results along subsequences of zero density such as studied in [5] for polynomial subsequences and [21, 23] for the sequence of primes. Nevertheless, in Section 3 we present some results for special subsequences of positive density.

Similar to the investigations in the present paper, H. N. Shapiro [30] gave a purely probabilistic proof for the classical Erdős-Wintner theorem. In [27], E. Manstavičius constructed Kubilius-models for the study of q-additive functions.

2. Ergodic proof of Delange's theorem

Let us first fix some notation. As usual, $\mathbb{Z}_Q = \varprojlim \mathbb{Z}/q_n\mathbb{Z}$ denotes the profinite compact group of Q-adic integers and μ_Q its Haar measure. We will freely identify elements of $\mathbb{Z}_Q$ and sequences $x = (x_0, x_1, \dots)$ with $x_j \in \{0, 1, \dots, a_j - 1\}$ or use Hensel's representation $x = \sum_{j=0}^{\infty} x_j q_j$. As topological probability space, $\mathbb{Z}_Q$ is identified with the product $\prod_{n\geq 0}\{0, 1, \dots, a_n - 1\}$ endowed with the product topology and the product measure. A general reference for these compact groups is [16, p. 109].

For a given Q-additive function f and $n \in \mathbb{N}$ we define the random variables $f_n : \mathbb{Z}_Q \to \mathbb{R}$ by $f_n(x) = f(x_n q_n)$. Finally, recall that by definition f admits an asymptotic distribution function, if the sequence of empirical distribution functions

$$F_N(t) = \frac{1}{N} \sum_{n<N} \chi_t(f(n)) \tag{2.1}$$

converges to some distribution function F, where χ_t is the indicator function of the interval $(-\infty, t]$. We have the following theorem.

Theorem 1. *Let $f : \mathbb{N} \to \mathbb{R}$ be a Q-additive function with respect to a constant-like Cantor numeration system. Then the following statements are equivalent*

(1) *the function f admits an asymptotic distribution function F;*
(2) *the series*

$$\sum_{j=0}^{\infty} \frac{1}{a_j} \left(\sum_{\varepsilon=0}^{a_j-1} f(\varepsilon q^j) \right) \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{1}{a_j} \sum_{\varepsilon=0}^{a_j-1} f(\varepsilon q^j)^2$$

both converge

(3) *The series* $\widetilde{f}(x) = \sum_{n=0}^{\infty} f_n(x)$ *converges for almost all* $x \in \mathbb{Z}_q$.

Moreover, every point $y \in \mathbb{Z}_Q$, *for which* $\tilde{f}(y)$ *converges is generic for* $\chi_t \circ f$ *for every continuity point* t *of* F, *i. e.*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \chi_t(f(y+n)) = F(t).$$

Proof. Condition (2) is clearly equivalent to the convergence of the series in (1.3) if $(a_n)_n$ is constant equal to q. The idea of the proof is to circumvent the direct proof of the equivalence between (1) and (2) by introducing a further condition, and by showing that both of them are equivalent to this last condition.

We first prove that (2) implies (3), and therefore assume that (2) holds. Let us first notice that (2) can be interpreted as the convergence of both series $\sum \mathbb{E}(f_n)$ and $\sum \mathbb{E}(f_n^2)$. Since $\sigma^2(f_n) \le \mathbb{E}(f_n^2)$, the series $\sum \sigma^2(f_n)$ converges as well. Moreover, it follows from the convergence of $\sum \mathbb{E}(f_n^2)$ and from the boundedness of $(a_n)_n$ that $\|f_n\|_\infty$ tends to 0. Hence the convergence of any series of the type $\sum \mu_Q(|f_n| \ge a)$ for $a > 0$. Finally, the random variables f_n are independent. Hence the assumptions of Kolmogorov's three series theorem are satisfied, and the series $\sum f_n$ converges almost surely with respect to μ_Q.

Conversely, assume (3). Again by Kolmogorov's three series theorem, both series $\sum \mathbb{E}(f_n)$ and $\sum \sigma^2(f_n)$ converge. However, using $f(0) = 0$, Cauchy-Schwarz inequality yields

$$\mathbb{E}(f_n)^2 = \frac{1}{a_n^2} \left(\sum_{\varepsilon=1}^{a_n} f(\varepsilon q_n) \right)^2 \le \frac{a_n - 1}{a_n} \mathbb{E}(f_n)^2.$$

Then $\mathbb{E}(f_n^2) = \mathbb{E}(f_n)^2 + \sigma^2(f_n) \le \frac{a_n-1}{a_n}\mathbb{E}(f_n^2) + \sigma^2(f_n)$, hence $\mathbb{E}(f_n^2) \le a_n \sigma^2(f_n)$. Since $(a_n)_n$ is bounded, we get the convergence of $\sum \mathbb{E}(f_n^2)$ and (2) is proved.

We now prove that (1) implies (3). Assume (1). Then in particular,

$$F_{q_n}(t) = \frac{1}{q_n} \sum_{k<q_n} \chi_t(f(k))$$

tends to a distribution function $F(t)$ in any continuity point of F when n tends to infinity. Translated in terms of random variables, this means that the random series $\sum_n f_n$ converges in distribution. Since the random variables f_n are independent, this is equivalent to (3) (see [25, section 17.2 "convergence and stability"]).

It remains to prove that (3) implies (1), which is the crucial part of the argument. Roughly speaking, one has to show that it is enough to take the limit along the q_n's to get the limit in general. It is achieved by a technique from ergodic theory using Birkhoff's individual ergodic theorem to approximate an orbit by an other one that we can control. Notice that the argument below replaces Lemmas 1, 3, 4, as well as Propositions 1 to 6 and Theorem 1 of [9].

Definition and uniqueness of Haar measure ensure that the dynamical system $(\mathbb{Z}_Q, \tau)$ with $\tau : x \mapsto x + 1$ (the so-called *odometer*) is uniquely ergodic with invariant measure μ_Q. Since

f is measurable, one has $\chi_t \circ f \in L^1(\mathbb{Z}_Q)$ for any $t \in \mathbb{R}$. Then Birkhoff's ergodic theorem asserts that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \chi_t(f(n+x)) = \mu_Q\left(\{y \in \mathbb{Z}_Q;\ f(y) \le t\}\right) =: F(t)$$

for almost all $x \in \mathbb{Z}_Q$. Therefore, in other words it remains to prove that $x = 0$ is a generic point for every $\chi_t \circ f$ where t is a continuity point of F. Actually, we will prove directly that every point y, where the series defining $\tilde{f}$ in (3) converges, is generic.

Let $t \in \mathbb{R}$ and $\eta > 0$ such that $t \pm \eta$ are continuity points for F. Then the above application of Birkhoff's theorem ensures that there exists $x \in \mathbb{Z}_Q$ both generic for $\chi_{t-\eta} \circ f$ and $\chi_{t+\eta} \circ f$. For any positive integer N, let m be the non-negative integer such that $q_m \le N < q_{m+1}$; then we define

$$M = M(N) = q_{m+3} + \sum_{j=0}^{m} \varepsilon_j(y) q_j - \sum_{j=0}^{m+1} \varepsilon_j(x) q_j.$$

By construction, we have that $\varepsilon_j(x+M) = \varepsilon_j(y)$ for all $j \le m$. Since we already have proved that (3) implies (2), we know that $\|f_m\|_\infty$ tends to 0. Since the negative integers have measure 0, we can assume that x has infinitely many digits $\varepsilon_j(x) \ne a_j - 1$ (this means that x is not a negative integer). Then for $n < N$ we have

$$\begin{aligned} x + M &= \sum_{j=0}^{m} \varepsilon_j(y) q_j + \varepsilon_{m+2}(x) q_{m+2} + (\varepsilon_{m+s+1}(x) + 1) q_{m+s+1} + \sum_{j=m+s+2}^{\infty} \varepsilon_j(x) q_j \\ x + M + n &= \sum_{j=0}^{m+1} \varepsilon_j(y+n) q_j + \varepsilon_{m+2}(x) q_{m+2} + (\varepsilon_{m+s+1}(x) + 1) q_{m+s+1} + \sum_{j=m+s+2}^{\infty} \varepsilon_j(x) q_j \end{aligned}$$

for some $s \ge 2$ given by the carry in addition. Similarly, we have

$$y+n = \sum_{j=0}^{m} \varepsilon_j(y+n) q_j + \begin{cases} \sum_{j=m+1}^{\infty} \varepsilon_j(y) q_j & \text{if there is no carry beyond } j = m \\ (\varepsilon_{m+r}(y) + 1) q_{m+r} + \sum_{j=m+r+1}^{\infty} \varepsilon_j(y) q_j & \text{if there is a carry from } j = m \text{ to } j = m + r \\ 0 & \text{if there is a carry from } j = m \text{ to } \infty. \end{cases}$$

The indices r and s depend only on x, y, and N, but not on n. Then, for N suffiently large, we have

$$|f(x + M + n) - f(y + n)| < \eta$$

by the convergence of the series

$$\sum_{j=0}^{\infty} f(\varepsilon_j(\xi) q_j)$$

for $\xi = x$ and $\xi = y$ and the fact that $\lim_{j\to\infty} f(\varepsilon_j q_j) = 0$ by (2).

Thus, for N large enough, we have:

$$\frac{1}{N} \sum_{n<N} \chi_{t-\eta}(f(x+M+n)) \le \frac{1}{N} \sum_{n<N} \chi_t(f(y+n)) \le \frac{1}{N} \sum_{n<N} \chi_{t+\eta}(f(x+M+n)). \tag{2.2}$$

It just remains to prove that both left and right sides of (2.2) converge respectively to $F(t-\eta)$ and $F(t+\eta)$ and then use the continuity of F in η. This is done by a barycentric consideration:

$$\begin{aligned}\frac{1}{N+M}\sum_{n<N+M}\chi_{t\pm\eta}(f(x+n)) =& \frac{M}{N+M}\left(\frac{1}{M}\sum_{n<M}\chi_{t\pm\eta}(f(x+n))\right)\\ &+\frac{N}{N+M}\left(\frac{1}{N}\sum_{n<N}\chi_{t\pm\eta}(f(x+M+n))\right).\end{aligned}\tag{2.3}$$

By definition of M, we have $N < M < 2\big(\max\{a_n;\, n\in\mathbb{N}\}\big)^3 N$. Thus $M \asymp N$ and if two of the three sums present in the expression (2.3) converge to the same limit, so does the third one as well. Indeed, that is the case for the two first sums because of genericity of x, that converge to $F(t\pm\eta)$. Therefore, $N^{-1}\sum_{n<N}\chi_{t\pm\eta}(f(x+M+n))$ converges to $F(t\pm\eta)$ too and we have

$$F(t-\eta)\le \liminf \frac{1}{N}\sum_{n<N}\chi_t(f(y+n)) \le \limsup \frac{1}{N}\sum_{n<N}\chi_t(f(y+n)) \le F(t+\eta).$$

Since the set of continuity points of a distribution function is countable, hence dense, we can make η tend to 0, which shows the convergence of $N^{-1}\sum_{n<N}\chi_t(f(y+n))$ to $F(t)$, provided that F is continuous at t. Thus all the assertions of the theorem have been proved. □

It is natural to ask under which conditions the series $\sum_{n=0}^{\infty} f_n(x)$ converges everywhere. Besides, as noticed in [3, Proposition 5] an immediate application of Birkhoff's ergodic theorem proves the existence of an asymptotic distribution function in this case (the continuity of $\tilde{f}$ ensures that $\chi_t\circ f$ is Riemann-integrable and unique ergodicity of $(\mathbb{Z}_Q,\tau)$ yields uniform convergence of the ergodic means). An application of Cauchy's criterion gives the following equivalences.

Proposition 2. *Let Q be a constant-like Cantor numeration system and $f:\mathbb{N}\to\mathbb{R}$ be a Q-additive function. Then the following statements are equivalent:*

(1) $\sum f_n(x)$ *converges for all* $x\in\mathbb{Z}_Q$*;*
(2) $\sum \mathbb{E}(|f_n|)<\infty$*;*
(3) $\sum f_n$ *converges normally;*
(4) f *is continuous at* 0*;*
(5) f *can be extended to a continuous function on* $\mathbb{Z}_Q$*.*

Remark 1. It follows from Proposition 2 that the function f is either continuous in every point or nowhere continuous. Moreover, in the second case, we have

$$\forall x\in\mathbb{Z}_Q : \limsup_{y\to x}|f(y)| = +\infty.$$

In any case the second condition in (2) of Theorem 1 ensures that $f\in L^2(\mathbb{Z}_Q)$.

The nature of the limiting distribution can be easily solved. Let dF be the limiting distribution of the Q-additive function f. Then dF is the weak limit of $dF_0 * \cdots * dF_n$, where $dF_k = \frac{1}{a_k}\sum_{\varepsilon=0}^{a_k-1}\delta_{f(\varepsilon q_k)}$ and δ_a the Dirac measure concentrated in a.

Proposition 3. *Let Q be a Cantor numeration system and f a Q-additive function with limiting distribution dF. Then dF is of pure type (that is either purely atomic, purely singular continuous or purely absolutely continuous).*

Let k_n be the cardinality of the largest subset $A \subset \{0, \dots, a_n - 1\}$ such that the values $f(\varepsilon q_n)$ do not depend of $\varepsilon \in A$. Then dF is atomic if and only if the series $\sum(1 - k_n/a_n)$ converges. In particular, if Q is constant-like, then dF is purely atomic if and only if f_n is the zero function for n sufficiently large.

Proof. The random variables f_n are discrete and independent. Hence, by a theorem of Jessen and Wintner [20], the law of their infinite convolution is of pure type. Moreover, a result of Lévy [24] asserts that dF is purely atomic if and only if $\prod_{n=0}^{\infty} d_n$ converges, where d_n denotes the maximal jump of f_n, that is $d_n = \max_{x \in \mathbb{R}}(F_n(x) - F_n(x - 0))$ (both results of Jessen-Wintner and Lévy are [12, Lemma 1.22]). For the discrete random variable f_n, one has $d_n = k_n/a_n$, and the second part of the proposition follows. If $(a_n)_n$ is bounded, and f_n is not zero, then f_n is not constant (because $f_n(0) = 0$). Then $1 - k_n/a_n \geq 1/\max a_n$. Therefore, $(1 - k_n/a_n))_n$ tends to 0 if and only if it is ultimately 0, that is if f_n is ultimately 0. □

Remark 2. It seems to be hopeless to obtain a complete characterisation of additive functions admitting an absolutely continuous limiting distribution. This is supported by the results on the nature of Bernoulli convolutions, which correspond to the 2-additive functions $f(2^n) = \lambda^{-n}$. In this case the properties of the measure depend heavily on the arithmetical nature of λ (cf. [28]).

Non-constant like Cantor numeration systems. If $(a_n)_n$ is not bounded, then (1) and (2) (independently) still imply (3) in Theorem 1. The inequality of Cauchy-Schwarz that has been used does not ensure that (2) and (3) are still equivalent. A counterexample can be easily constructed as soon $(a_n)_n$ is not bounded. Furthermore, there is no logical dependence between (1) and (3):

Example 1. Let Q be a non constant-like Cantor numeration system. Since the sequence $(a_n)_n$ is unbounded, there exists a subsequence $(a_{\sigma(n)})_n$ such that the infinite product $\prod(1 - 1/a_{\sigma(n)})$ converges, with $\prod_{n \geq 0}(1 - /a_{\sigma(n)}) \geq 1/2$. Define $f(q_{\sigma(n)}) = 1$, and $f(\varepsilon q_m) = 0$ otherwise. Then $\mathbb{E}(f_{\sigma(n)}) = \mathbb{E}(f_{\sigma(n)}^2) = a_\sigma(n)^{-1}$ and $\mathbb{E}(f_m) = \mathbb{E}(f_m^2) = 0$ otherwise. Hence $\sum f_n$ converges almost surely (in that case, this is also an immediate application of the Borel-Cantelli lemma, since $\sum f_n(x)$ converges if and only if $\varepsilon_{\sigma(n)}(x) = 1$ only finitely often). Since $\chi_{1/2}(f(n)) = 1$ if and only if $f(n) = 0$, we have that

$$\frac{1}{q_{\sigma(n)}} \sum_{k < q_{\sigma(n)}} \chi_{1/2}(f(k)) = \prod_{j=0}^{n-1} \left(1 - \frac{1}{a_{\sigma(j)}}\right) > \prod_{j=0}^{\infty} \left(1 - \frac{1}{a_{\sigma(j)}}\right) \geq 1/2.$$

On the other hand, we have that

$$\frac{1}{2q_{\sigma(n)}} \sum_{k < 2q_{\sigma(n)}} \chi_{1/2}(f(k)) \leq \frac{1}{2},$$

so that the limit in (2.1) cannot exist for $t = 1/2$. Since f is integral-valued, its limiting distribution would be continuous at $t = 1/2$. Thus f does not admit a limiting distribution.

Example 2. We use the same construction as in Example 1 for the subsequence and consider $g((a_{\sigma(n)} - 1)q_{\sigma(n)}) = 1$, and $g(\varepsilon q_m) = 0$ otherwise. Clearly, the moments of f and g are equal. Therefore $\tilde{g}$ is defined almost surely on $\mathbb{Z}_Q$. We claim that g possesses a limit distribution.

Since g takes only integral values, it is sufficient to look at $\mathbb{1}_{\{k\}} \circ \tilde{g}$. Let $m_k(N) = \frac{1}{N}\#\{n < N;\, g(n) = k\}$. We have

$$\mu_Q(\tilde{g} = k) = \sum_{\substack{I \subset \mathbb{N} \\ |I|=k}} \prod_{m \in I} \frac{1}{a_{\sigma(m)}} \prod_{m \notin I} \left(1 - \frac{1}{a_{\sigma(m)}}\right)$$

$$m_k(q_N) = \sum_{\substack{I \subset [0, N-1] \\ |I|=k}} \prod_{m \in I} \frac{1}{a_{\sigma(m)}} \prod_{\substack{m \notin I \\ \sigma(m) < N}} \left(1 - \frac{1}{a_{\sigma(m)}}\right).$$

Furthermore, we have $m_k(\varepsilon q_N) = m_k(q_N)$ for any digit $\varepsilon < a_N$, and, more generally,

$$m_k(\varepsilon_n q_n + \varepsilon_{n-1} q_{n-1} + \cdots + \varepsilon_0) = \frac{\varepsilon_n q_n m_k(q_n) + \varepsilon_{n-1} q_{n-1} m_{k-s_1}(q_{n-1}) + \cdots + m_{k-s_j}(q_{n-j}) + \cdots}{\varepsilon_n q_n + \varepsilon_{n-1} q_{n-1} + \cdots + \varepsilon_0},$$

where s_j is the number of digits $\varepsilon_{n-\ell}$ with $\ell < j$ that are of the form $a_{\sigma(m)} - 1$, with the further convention that a negative index for m gives 0. Using that $m_k(q_N)$ tends to $\mu_Q(\tilde{g} = k)$ when N tends to infinity and $\lim a_{\sigma(m)} = \infty$, one obtains that $m_k(\varepsilon_n q_n + \varepsilon_{n-1} q_{n-1} + \cdots + \varepsilon_0) - m_k(q_n)$ tends to zero when n gets large, which means exactly that $m_k(N)$ tends to $\mu_Q(\tilde{g} = k)$. This is equivalent to 0 being a generic point.

3. Consequences

For this section we recall that two dynamical systems are called spectrally disjoint if the intersection of their respective spectra is $\{1\}$. Theorem 1 in [8, Chapter 10 § 1] states that the direct product of two dynamical systems is ergodic if and only if the two dynamical systems are ergodic and spectrally disjoint.

Proposition 4. *Let Q be a constant-like Cantor-numeration system. Let A be a compact abelian group, $\alpha \in A$, and $T_\alpha : g \mapsto g + \alpha$ be the rotation by α. Assume that T_α is ergodic and that*

$$\forall n : q_n \alpha \neq 0.$$

Let λ_A denote the Haar measure on A and let I be a subset of A of positive Haar measure and such that $\lambda_A(\partial I) = 0$. Finally, let f be a Q-additive function. Then the distribution function

$$\lim_{N \to \infty} \frac{1}{\#\{n < N \,;\, T_\alpha^n \xi \in I\}} \sum_{\substack{n < N \\ T_\alpha^n \xi \in I}} \chi_t(f(n)) = F(t)$$

exists, if and only if the conditions of Theorem 1 are satisfied.

Moreover, every point $(x, \xi) \in \mathbb{Z}_Q \times A$, for which $\tilde{f}(x)$ converges is generic for $\chi_t \circ f$ for every continuity point t of F, i. e.

$$\lim_{N \to \infty} \frac{1}{\#\{n < N \,;\, T_\alpha^n \xi \in I\}} \sum_{\substack{n < N \\ T_\alpha^n \xi \in I}} \chi_t(f(n + x)) = F(t).$$

Proof. We recall that the spectrum of the dynamical system $(\mathbb{Z}_Q, \tau)$ equals (cf. [29])

$$\{z \in \mathbb{C} \,;\, \exists n : z^{q_n} = 1\}.$$

On the other hand the condition on $\alpha \in A$ is equivalent to

$$\forall \chi \in \hat{A} \setminus \{\mathbf{1}\},\ \forall n : \chi(\alpha)^{q_n} \neq 1,$$

which shows that the two dynamical systems $(\mathbb{Z}_Q, \tau)$ and (A, T_α) are spectrally disjoint.

We consider the dynamical system $(\mathbb{Z}_Q \times A, \tau \times T_\alpha)$. Assume that (y, ξ) is a generic point for $\chi_t \circ f \otimes \mathbb{1}_I$, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \chi_t(f(n+y))\mathbb{1}_I(T_\alpha^n \xi) = F(t)\lambda_A(I). \tag{3.1}$$

This is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \chi_t(f(n+y))\chi(n\alpha + \xi) = 0$$

for all $\chi \in \hat{A} \setminus \{\mathbf{1}\}$. The equation $\chi(n\alpha + \xi) = \chi(n\alpha)\chi(\xi)$ immediately implies that this limit is uniform in $\xi \in A$ and therefore the limit in (3.1) is uniform in ξ.

Then following the same lines as the proof of "(3)⇒ (1)" in Theorem 1 yields that (x, η) is generic for any η and any x such that $\tilde{f}(x)$ is defined.

The proof of the necessity part follows the same lines as the proof of [19, Theorem 2]. □

Remark 3. Proposition 4 should be compared to [19, Theorem 2], where necessary and sufficient conditions for the existence of a distribution function for n ranging in $E \subset \mathbb{N}$ defined by irrational rotations on a torus are given. Proposition 4 gives a slight generalisation in terms of the group rotation and the underlying number system as well as a characterisation of the generic points. The necessity part still depends on [19, Theorem 2].

4. Extension to more general numeration systems

Let us recall that if $(G_n)_n$ is an increasing sequence of positive integers with $G_0 = 1$, we can expand every positive integer with respect to this sequence, *i.e.*

$$\forall n \in \mathbb{N},\ n = \sum_{k=0}^{\infty} \varepsilon_k G_k\ ,$$

this expansion being finite and unique, provided that

$$\forall K : \sum_{k=0}^{K-1} \varepsilon_k G_k < G_K. \tag{4.1}$$

The digits ε_k can be computed by the greedy algorithm. The sequence $(G_n)_n$ defines a so-called *numeration system.*

The notion of $Q-$additive function naturally extends to this of G-additive function, namely

$$f\left(\sum_{k=0}^{K} \varepsilon_k G_k\right) = \sum_{k=0}^{K} f(\varepsilon_k G_k) \tag{4.2}$$

for which one may investigate the existence of

$$F(t) := \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \chi_t\left(f(n)\right) =: \lim_{N\to\infty} F_N(t). \tag{4.3}$$

In Sections 5.1 and 5.2, we will partially extend the results of Section 2 to some families of numeration systems. Our approach uses ideas similar to those introduced in Section 2 to give sufficient conditions for the existence of an asymptotic distribution. The present section is devoted to the common framework of the study below.

For this purpose, we will use the G-adic compactification $\mathcal{K}_G$ of $\mathbb{N}$ as introduced in [14] and the embedding $\nu\colon \mathbb{N} \hookrightarrow \mathcal{K}_G$. Again, τ will denote the "addition of 1" map (extended to $\mathcal{K}_G$). For $x = (x_n)_n$ it can be defined as $\tau(x) = \lim_n(\nu(x_0G_0 + \cdots + x_nG_n + 1))$ (see [1]). Although there is no group structure on $\mathcal{K}_G$, it is possible to endow the dynamical system $(\mathcal{K}_G, \tau)$, the so-called *odometer*, with a τ-invariant probability measure μ_G which shall play the role of the Haar measure. For convenience, we will occasionally denote this measure by $\mathbb{P}$.

The first tool needed is a version of Kolmogorov's three-series theorem for dependent random variables (it is not surprinsing that only the sufficiency part is at disposal).

Lemma 1. [25, Section 29, Theorem D] *Let Y_n be a sequence of uniformly bounded random variables. Denote $\mathcal{F}_n$ the σ-algebra generated by $Y_0, \dots, Y_n$ and assume that*

$$\sum_{n=0}^{\infty} \mathbb{E}(Y_n \mid \mathcal{F}_{n-1}) \quad \textit{and} \quad \sum_{n=0}^{\infty} \mathbb{E}\left[(Y_n - \mathbb{E}(Y_n \mid \mathcal{F}_{n-1}))^2\right] \tag{4.4}$$

converge (almost surely for the first series). Then the series $\sum_{n=0}^{\infty} Y_n$ converges almost everywhere.

In terms of Theorem 1, the assumptions of Lemma 1 ensure the existence of $\tilde{f}$. Thus the lemma gives the implication (2) $\Rightarrow$ (3). Since the existence of the distribution function does not imply the existence of $\tilde{f}$ anymore (by the lack of independence), it remains to investigate the key part of the theorem, that is (3) implies (1). Hence we will get a sufficient condition for the distribution function to exist (we restrict ourselves to the genericity of 0). The proof works *mutatis mutandis*, provided that two conditions hold. The first one is that the sequence $(G_{n+1}/G_n)_n$ is bounded. With the notation of the proof, this ensures that $M(N) \ll N$. Actually, it is already the reason for which the implication (3) $\Rightarrow$ (1) can fail for non-constant like Cantor numeration systems. The second condition is more technical and ensures that if the expansion of $x + M$ begins with m zeroes, say, then the addition of a small number n does not change the digits beyond m, so that we have $f(x + M + n) = f(n) + f(x + M)$. That is obvious for Cantor numeration system, but not true in general, since the propagation of carries is considerably more complicated in the general case (see [1]). This condition is stated as the hypothesis below.

Hypothesis (H)**.** Let $G = (G_n)_n$ be a system of numeration and $\mathcal{K}_G$ the underlying odometer. For $x \in \mathcal{K}_G$ and $m \in \mathbb{N}$, let $x^{(>m)} = \sum_{j \geq m+1} \varepsilon_j(x) G_j$. Then we say that G satisfies Hypothesis (H) if:

$$\exists k \in \mathbb{N},\, \forall x \in \mathcal{K}_G,\, \forall m \geq k,\, \forall j > m : \varepsilon_j(\tau^{G_{m-k}}(x^{(>m)})) = \varepsilon_j(x).$$

Corollary 5. *Let $(G_n)_{n \in \mathbb{N}_0}$ be an increasing sequence of integers satisfying*

$$\limsup_{k \to \infty} \frac{G_{k+1}}{G_k} < \infty$$

and Hypothesis (H). Let $f : \mathbb{N} \to \mathbb{R}$ be a G-additive function. Assume that the series (4.4) *converge for $f_n(x) = f(x_nG_n)$. Then f admits an asymptotic distribution function.*

Proof. Hypothesis (H) implies that $\liminf(G_{n+1}/G_n) > 1$. Hence, by [2, Théorème 7] , the dynamical system (K_G, τ) is uniquely ergodic. Using the notation of the proof of Theorem 1, take $M = M(N) = G_{m+3} - \sum_{j\leq m+1} \varepsilon_j(x)G_j$, where x is a generic point belonging to $K_G \setminus \cup_{n\geq 1}\tau^{-n}(0)$. The proof is then an immediate adaptation of the Cantor case. □

In order to study the purity of the limiting distribution, we define a sequence $(X_n)_n$ of random variables on the probability space $(\mathcal{K}_G, \mu_G)$ by $X_n((x_0, x_1, x_2, \ldots)) = x_n$. The following proposition shows that the sequence of random variables (X_n) satisfies a Kolmogorov 0-1-law.

Proposition 6. *Let $G = (G_n)_{n\in\mathbb{N}}$ be a numeration system and $(\mathcal{K}_G, \tau)$ be the underlying odometer. Assume that $(\mathcal{K}_G, \tau)$ admits a τ-invariant ergodic measure μ_G. Then for any $A \in \mathcal{T} = \bigcap_{n\in\mathbb{N}} \sigma(X_n, X_{n+1}, \ldots)$, either $\mu_G(A) = 0$ or $\mu_G(A) = 1$ holds.*

Proof. Let $A \in \sigma(X_n, X_{n+1}, \ldots)$. Then we claim that $A \triangle \tau^{-1}(A) \subseteq \bigcup_{m\geq n}[\nu(G_m - 1)]$, where

$$[\nu(G_m - 1)] = \{x = (x_n)_n \in \mathcal{K}_G;\ x_0 = \varepsilon_0(G_m - 1), \ldots, x_{m-1} = \varepsilon_{m-1}(G_m - 1)\}.$$

To prove that we assume that $x \in A \triangle \tau^{-1}(A)$. Then $\tau(x) \in A$ and $x \notin A$, or $x \in A$ and $\tau(x) \notin A$. This means that for all n, the operation $x \mapsto x + 1$ has changed at least one of the digits $x_n, x_{n+1}, \ldots$. By definition of τ, this can only happen if the word $\varepsilon_0(G_m - 1) \cdots \varepsilon_{m-1}(G_m - 1)$ is a prefix of x for some $m \geq n$. Hence for $A \in \mathcal{T}$ we have

$$A \triangle \tau^{-1}(A) \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} [\nu(G_m - 1)] = \tau^{-1}(\{0\})).$$

Since μ_G is non-atomic (cf. [2, Theorem 8 (a)]), this implies $\mu_G(A \triangle \tau^{-1}(A)) = 0$. By ergodicity of μ_G this yields either $\mu_G(A) = 0$ or $\mu_G(A) = 1$. □

The following proposition is an analogue to the theorem of Jessen and Wintner [20].

Proposition 7. *Let $G = (G_n)_{n\in\mathbb{N}}$ be a numeration system and $(\mathcal{K}_G, \tau)$ be the underlying odometer. Assume that $(\mathcal{K}_G, \tau)$ admits a τ-invariant ergodic measure μ_G. Then the distribution function (if it exists) of a G-additive function is of pure type (either purely atomic, or singular continuous, or absolutely continuous).*

Proof. The proof is the same as the proof of [12, Lemma 1.22] using Proposition 6 instead of Kolmogorov's 0-1-law for independent random variables. □

The second part of Proposition 3 cannot be generalised offhand. Indeed, Lévy's proof is based on concentration inequalities, where the assumption of independence is hardly dispensable. Nevertheless, we will see a different approach in Proposition 9, that also works in that special weakly dependent setting.

5. Special systems of numeration

Quite contrary to the series (1.3), the series (4.4) are poorly explicit, and their convergence is not easily investigated: because the first series of (4.4) is a series of conditional expectations whose convergence has to hold almost surely, and because both use invariant measure, which is itself rather inexplicit. In this section, we examine special numeration systems allowing to give explicit conditions.

5.1. **Ostrowski expansions.** Let α be an irrational number in the interval $[0, 1/2]$ and write $\alpha = [0; a_1, a_2, \ldots]$ its continued fraction expansion. Define as usual $(p_n)_n$ and $(q_n)_n$ to be the sequences of numerators and denominators of the convergents of α. They are given by $q_{-1} = 0$, $q_0 = 1$ and $q_n = a_n q_{n-1} + q_{n-2}$. Then $(q_n)_{n\geq 0}$ is increasing and defines a so-called Ostrowski numeration system in the sense of Section 4. The corresponding compactification is denoted $\mathcal{K}_\alpha$ (for $\mathcal{K}_G$), whose elements $x = (x_n)_n$ are characterised by the inequalities

$$\begin{cases} x_0 \leq a_1 - 1, \\ \forall j \leq 1,\ x_j \leq a_{j+1}, \\ \forall j \leq 1,\ (x_j = a_{j+1} \Rightarrow x_{j-1} = 0) \end{cases} . \tag{5.1}$$

Recall that $\|x\|$ is the distance to the nearest integer. In the context of continued fraction, $\|q_n\alpha\| = |q_n\alpha - p_n|$.

Lemma 2 ([4, 32])**.** *Let $\mathbb{P}$ be the invariant probability measure on the odometer. Then, for any cylinder $C = [x_0 \cdots x_n]$ and for the n-th projection X_n on $\mathcal{K}_\alpha$, we have:*

$$\mathbb{P}([C]) = \begin{cases} \|q_n\alpha\| + \|q_{n+1}\alpha\| & \text{if } x_n = 0, \\ \|q_n\alpha\| & \text{if } x_n \neq 0; \end{cases} \tag{5.2}$$

$$\mathbb{P}(X_n = a) = \begin{cases} q_n(\|q_n\alpha\| + \|q_{n+1}\alpha\|) & \text{if } a = 0, \\ q_n\|q_n\alpha\| & \text{if } 1 \leq a \leq a_{n+1} - 1, \\ q_{n-1}\|q_n\alpha\| & \text{if } a = a_{n+1}. \end{cases} \tag{5.3}$$

From now on, we consider an Ostrowski numeration system associated with an irrational number α with bounded partial quotients (that is the analogue of the hypothesis to be constant-like for the Cantor numeration systems). As before, f denotes an additive function with respect to $(q_n)_n$ We begin with the second series of (4.4).

Lemma 3. *Let $\mathcal{F}_n$ be the σ-algebra generated by $X_0, \ldots, X_n$ and $f_n = f(X_n G_n)$. Then the following statements are equivalent:*

1. *The series* $\displaystyle\sum_{n=0}^{\infty} \mathbb{E}\left[(f_n - \mathbb{E}(f_n \mid \mathcal{F}_{n-1}))^2\right]$ *converges.*
2. *The series* $\displaystyle\sum_{n=0}^{\infty} \mathbb{E}(f_n^2)$ *converges.*
3. *The series* $\displaystyle\sum_{n=0}^{\infty} \sum_{\varepsilon=1}^{a_{n+1}} f(\varepsilon q_n)^2$ *converges.*

Proof. It is clear that (2) implies (1). Since $f_n \in L^2(\mathcal{K}_G)$, we have

$$\mathbb{E}\left[(f_n - \mathbb{E}(f_n \mid \mathcal{F}_{n-1}))^2\right] = \mathbb{E}(f_n^2) - \mathbb{E}\left(f_n \mathbb{E}(f_n \mid \mathcal{F}_{n-1})\right). \tag{5.4}$$

Using $f(0) = 0$ and Cauchy-Schwartz inequality, we get

$$\begin{aligned} \mathbb{E}\left(f_n \mathbb{E}(f_n \mid \mathcal{F}_{n-1})\right) &= \mathbb{P}(X_n > 0)\mathbb{E}\left(f_n \mathbb{E}(f_n \mid \mathcal{F}_{n-1}) \mid X_n > 0\right) \\ &\leq \mathbb{P}(X_n > 0)\left[\mathbb{E}(f_n^2 \mid X_n > 0)\right]^{1/2} \left[\mathbb{E}\left(f_n \mathbb{E}(f_n \mid \mathcal{F}_{n-1}) \mid X_n > 0\right)\right]^{1/2} \\ &\leq \mathbb{P}(X_n > 0)\mathbb{E}(f_n^2). \end{aligned}$$

Since $\mathbb{P}(X_n > 0)$ is bounded away from 1, this shows that (1) implies (2). The equivalence with (3) is then immediate, noticing that $\inf_{\varepsilon,n} \mathbb{P}(X_n = \varepsilon) > 0$ by the boundedness of the partial quotients a_n. $\square$

Less immediate is the first series, which is a series of random variables and not a numerical one. For $x = x_0 x_1 \cdots \in \mathcal{K}_\alpha$ and $n \in \mathbb{N}$, let $C_x^n = [x_0 \cdots x_n]$ and, for $k \in N$, $C_k^n = C_{\nu(k)}^n$. Using Lemma 2, we compute

$$\begin{aligned}
\mathbb{E}(f_n \mid \mathcal{F}_{n-1}) &= \sum_{k=0}^{q_n-1} \frac{1}{\mathbb{P}(C_k^n)} \int_{C_k^n} f_n(x)\, d\mathbb{P}(x) \mathbb{1}_{C_k^n} \\
&= \sum_{k=0}^{q_n-1} \frac{1}{\mathbb{P}(C_k^n)} \sum_{\varepsilon=0}^{a_{n+1}} f(\varepsilon q_n) \mathbb{P}([\varepsilon_0(k) \cdots \varepsilon_{n-1}(k)\varepsilon]) \mathbb{1}_{C_k^n} \\
&= \sum_{k=0}^{q_{n-1}-1} \frac{\|q_n\alpha\|}{\|q_{n-1}\alpha\| + \|q_n\alpha\|} \sum_{\varepsilon=1}^{a_{n+1}} f(\varepsilon q_n) \mathbb{1}_{C_k^n} + \sum_{k=q_{n-1}}^{q_n-1} \frac{\|q_n\alpha\|}{\|q_{n-1}\alpha\|} \sum_{\varepsilon=1}^{a_{n+1}-1} f(\varepsilon q_n) \mathbb{1}_{C_k^n} \\
&= \underbrace{\left(\frac{\|q_n\alpha\|}{\|q_{n-1}\alpha\| + \|q_n\alpha\|} \sum_{\varepsilon=1}^{a_{n+1}} f(\varepsilon q_n) \right)}_{\alpha_n} \mathbb{1}_{(X_{n-1}=0)} + \underbrace{\left(\frac{\|q_n\alpha\|}{\|q_{n-1}\alpha\|} \sum_{\varepsilon=1}^{a_{n+1}-1} f(\varepsilon q_n) \right)}_{\beta_n} \mathbb{1}_{(X_{n-1}\neq 0)},
\end{aligned} \tag{5.5}$$

where the equalities hold almost surely.

Lemma 4. *Let $a = (\alpha_n)_n$ and $b = (\beta_n)_n$ be two real (or complex) valued sequences. Let $a \oplus b$ be the set of the sequences $(u_n)_n$, such that, for all $n \in \mathbb{N}$, one has $u_n \in \{\alpha_n, \beta_n\}$. Then:*

$$\left(\forall u \in a \oplus b, \sum_{n\in\mathbb{N}} u_n \text{ converges} \right) \Longleftrightarrow \sum_{n\in\mathbb{N}} \alpha_n,\ \sum_{n\in\mathbb{N}} \beta_n,\ \text{and} \sum_{n\in\mathbb{N}} |\alpha_n - \beta_n| \text{ converge.}$$

Proof. We present the proof for real sequences; the extension to complex sequences is obvious. The sequences a, b, $\max(a,b)$, and $\min(a,b)$ are elements of $a \oplus b$. Hence so does the sequence $|a-b| = \max(a,b) - \min(a,b) \in a \oplus b$, which proves that the condition is necessary. Conversely, assume that the three series of the right-hand side converge. Let $u \in a \oplus b$; define $v = (v_n)_n$ by $v_n = \alpha_n - \beta_n$ if $u_n = \alpha_n$, and $v_n = \beta_n - \alpha_n$ if $u_n = \beta_n$. Then $|v| = |a - b|$, hence $\sum v_n$ is (absolutely) convergent. Morever, $u = \frac{1}{2}(a + v + b)$, hence the convergence of $\sum u_n$. $\square$

We get the following result:

Proposition 8. *Let $(q_n)_n$ an Ostrowski numeration system associated to an irrational number α, and assume that α has bounded partial quotients $(a_n)_n$. Let f be a $(q_n)_n$-additive function. Let α_n and β_n be as in 5.5. Assume that the four series below converge.*

$$\sum_{n=0}^{\infty} \sum_{\varepsilon=1}^{a_{n+1}} f(\varepsilon q_n)^2, \sum_{n=0}^{\infty} \alpha_n, \sum_{n=0}^{\infty} \beta_n, \sum_{n=0}^{\infty} |\alpha_n - \beta_n|.$$

Then f admits an asymptotic distribution function.

We now generalise the second part of Proposition 3.

Proposition 9. *Let $(q_n)_n$ be an Ostrowski numeration system associated to an irrational number α, and assume that α has bounded partial quotients. Let f be a $(q_n)_n$-additive function* *admitting an asymptotic distribution dF. Then dF is purely atomic if and only if f_n is ultimately 0.*

Proof. We now prove the second assertion of the theorem. Assume $f_n \equiv 0$ for $n \geq n_0$. Then f only takes finitely many values. Therefore dF is atomic.

Assume now that $f_{n_k} \not\equiv 0$ for an increasing sequence $(n_k)_k$. Without loss of generality, we may assume that $n_{k+1} - n_k \geq 3$ and that $\sup_{j\geq 1} \|f_{n_{k+j}}\|_\infty < \frac{1}{2}\min(A_k, B_k)$, where

$$A_k = \min\left\{|f(\varepsilon_{n_k} q_{n_k})|;\ 0 \leq \varepsilon_{n_k} \leq a_{n_k+1},\ f(\varepsilon_{n_k} q_{n_k}) \neq 0\right\}, \tag{5.6}$$

and

$$\begin{aligned} B_k = \min\big\{&|f(\varepsilon_{n_k-1} q_{n_k-1}) + f(\varepsilon_{n_k+1} q_{n_k+1})|; \\ &0 \leq \varepsilon_{n_k-1} \leq a_{n_k},\ 0 \leq \varepsilon_{n_k+1} \leq a_{n_k+2},\ f(\varepsilon_{n_k-1} q_{n_k-1}) + f(\varepsilon_{n_k+1} q_{n_k+1}) \neq 0\big\}. \end{aligned} \tag{5.7}$$

Let $t \in \mathbb{R}$ and set $A = \{x \in K\,;\, f(x) = t\}$. We define maps Φ_k on A by $\Phi_k((x_0, x_1, \ldots)) = (x'_0, x'_1, \ldots)$ as follows:

- if $f(x_{n_k} q_{n_k}) \neq 0$, then $x'_j = x_j$ for $j \neq n_k$ and $x'_{n_k} = 0$,
- if $f(x_{n_k} q_{n_k}) = 0$ and $f(x_{n_k-1} q_{n_k-1}) + f(x_{n_k+1} q_{n_k+1}) = 0$, then $x'_j = x_j$ for $|j - n_k| \geq 2$, $x'_{n_k-1} = x'_{n_k+1} = 0$, and x'_{n_k} is chosen so that $f(x'_{n_k} q_{n_k}) \neq 0$,
- if $f(x_{n_k} q_{n_k}) = 0$ and $f(x_{n_k-1} q_{n_k-1}) + f(x_{n_k+1} q_{n_k+1}) \neq 0$, then $x'_j = x_j$ for $|j - n_k| \neq 1$ and $x'_{n_k-1} = x'_{n_k+1} = 0$.

Then, (5.6) and (5.7) ensures that $\Phi_k(A) \cap \Phi_j(A) = \emptyset$ for $k \neq j$. Furthermore, we have by construction $\mu(A) \leq c\mu(\Phi_k(A))$, where the constant c is positive (it is possible to compute explicitely one value for c from (5.2): one gets $c = \min_n \frac{q_{n-1}\|q_{n+2}\alpha\|}{q_n(\|q_{n+2}\alpha\| + \|q_{n+3}\alpha\|)}$). It follows that $\mu(A) = 0$. Therefore, dF is not atomic. □

5.2. **Linear recurrent bases.** An other case of special interest is that where the numeration system is given by a recurrence sequence arrising from the β-numeration with decreasing coefficients. Namely, $a_0 \geq a_1 \geq \cdots \geq a_{d-1} \geq 1$, and

$$\begin{aligned} &G_{n+d} = a_0 G_{n+d-1} + \cdots + a_{d-1} G_n \quad \text{for } n \geq 0 \quad \text{with} \\ &G_0 = 1 \text{ and } G_k = a_0 G_{k-1} + \cdots + a_{k-1} G_0 + 1 \text{ for } k < d. \end{aligned} \tag{5.8}$$

The initial values are chosen as the "canonical initial values" from [15]. In this case, a finite sum $\sum_{k=0}^{\infty} \varepsilon_k G_k$ is the expansion of some integer if and only if

$$(\varepsilon_k, \ldots, \varepsilon_0, 0^\infty) < (a_0, \ldots, a_{d-1} - 1)^\infty \tag{5.9}$$

($<$ being understood as the lexicographical order) for every k. The strings $(\varepsilon_0, \varepsilon_1, \ldots)$ that verify this condition are called *admissible strings*. The dominating root of the characteristic equation $X^d - a_0 X^{d-1} - \cdots - a_{d-1} = 0$, say α, is a Pisot number; these equations have been studied by Brauer [7]. In particular, $\lim_{n\to\infty} \dfrac{G_n}{\alpha^n}$ exists and is non-zero.

It was proved in [14] that $(\mathcal{K}_G, \tau)$. Furthermore, the measure μ_G was computed explicitely there: let $\mathcal{C} = [x_0 \cdots x_k]$, be a cylinder in $\mathcal{K}_G$, and $F_m = \#\{k < G_m;\ \nu(k) \in \mathcal{C}\}$. Then

$$\mathbb{P}(\mathcal{C}) = \frac{F_{k+1}\alpha^{d-1} + (F_{k+2} - a_0 F_{k+1}) + \cdots + (F_{k+d} - a_0 F_{k+d-1} - \cdots - a_{d-2} F_{k+1})}{\alpha^{k+1}(\alpha^{d-1} + \cdots + \alpha + 1)}. \tag{5.10}$$

In [3, Theorem 4] it was proved that a G-additive function f admits an asymptotic distribution function, if the series

$$\sum_{n=0}^{\infty}\left|\sum_{\ell=0}^{a_s-1} f(a_0G_{n+d-1}+\cdots+a_{s-1}G_{n+d-s}+\ell G_{n+d-s-1})\right| \quad \text{for } s=0,\ldots,d-1 \tag{5.11}$$

$$\sum_{n=0}^{\infty}\sum_{\ell=0}^{a_0} f(\ell G_n)^2 \tag{5.12}$$

converge.

In the sequel, we look at the case $d=2$, and write $G_{n+2}=aG_{n+1}+bG_n$, where $a\geq b$. With the notation of (5.10), we have $F_{k+1}=1$, and $F_{k+2}=a+1$ if $x_k<b$, $F_{k+2}=a$ if $b\leq x_k\leq a$. Hence

$$\mathbb{P}([x_0x_1\cdots x_k])=\begin{cases}\alpha^{-k-1} & \text{if } x_k<b\\ \frac{1}{\alpha^k(\alpha+1)} & \text{if } b\leq x_k\leq a.\end{cases}$$

We shall now turn our attention to $\mathbb{E}(f_n\mid\mathcal{F}_{n-1})$ and use the notation of Section 5.1.

$$\begin{aligned}
\mathbb{E}(f_n\mid\mathcal{F}_{n-1}) &= \sum_{k=0}^{G_n-1}\frac{1}{\mathbb{P}(C_k^n)}\int_{C_k^n} f_n(x)\,d\mathbb{P}(x)\mathbb{1}_{C_k^n}\\
&=\sum_{k=0}^{G_n-1}\frac{1}{\mathbb{P}(C_k^n)}\sum_{\varepsilon=0}^{a} f(\varepsilon G_n)\mathbb{P}([\varepsilon_0(k)\cdots\varepsilon_{n-1}(k)\varepsilon])\mathbb{1}_{C_k^n}\\
&=\sum_{k=0}^{bG_{n-1}-1}\alpha^n\left(\sum_{\varepsilon=1}^{b-1} f(\varepsilon G_n)\alpha^{-n-1}+\sum_{\varepsilon=b}^{a} f(\varepsilon G_n)\frac{1}{\alpha^n(\alpha+1)}\right)\mathbb{1}_{C_k^n}\\
&\quad+\sum_{k=bG_{n-1}}^{G_n-1}\alpha^{n-1}(\alpha+1)\left(\sum_{\varepsilon=1}^{b-1} f(\varepsilon G_n)\alpha^{-n-1}+\sum_{\varepsilon=b}^{a-1} f(\varepsilon G_n)\frac{1}{\alpha^n(\alpha+1)}\right)\mathbb{1}_{C_k^n}\\
&=\left(\frac{1}{\alpha}\sum_{\varepsilon=1}^{b-1} f(\varepsilon G_n)+\frac{1}{\alpha+1}\sum_{\varepsilon=b}^{a} f(\varepsilon G_n)\right)\mathbb{1}_{(X_{n-1}<b)}\\
&\quad+\left(\frac{\alpha+1}{\alpha^2}\sum_{\varepsilon=1}^{b-1} f(\varepsilon G_n)+\frac{1}{\alpha}\sum_{\varepsilon=b}^{a-1} f(\varepsilon G_n)\right)\mathbb{1}_{(b\leq X_{n-1}\leq a)},
\end{aligned}$$

where the equalities hold almost surely. Lemma 3 can be immediately adaptated: the assertions (1) and (2) are still equivalent and are also equivalent to an appropriate form of (3), which now reads off

$$(3') \quad \textit{The series } \sum_{n=0}^{\infty}\sum_{\varepsilon=1}^{a} f(\varepsilon G_n) \textit{ converges.} \tag{5.13}$$

Hence we get the proposition below.

Proposition 10. *Let $G = (G_n)_n$ be a linear recurrent numeration system given by $G_{n+2} = aG_{n+1} + bG_n$. Let f be a G-additive function. Assume that the four series below converge.*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{\varepsilon=1}^{a} f(\varepsilon G_n)^2,\\
&\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}\sum_{\varepsilon=1}^{b-1} f(\varepsilon G_n) + \frac{1}{\alpha+1}\sum_{\varepsilon=b}^{a} f(\varepsilon G_n)\right),\\
&\sum_{n=0}^{\infty}\left(\frac{\alpha+1}{\alpha^2}\sum_{\varepsilon=1}^{b-1} f(\varepsilon G_n) + \frac{1}{\alpha}\sum_{\varepsilon=b}^{a-1} f(\varepsilon G_n)\right),\\
&\sum_{n=0}^{\infty}\left|\frac{1}{\alpha^2}\sum_{\varepsilon=1}^{b-1} f(\varepsilon G_n) + \frac{1}{\alpha(\alpha+1)}\sum_{\varepsilon=b}^{a-1} f(\varepsilon G_n) - \frac{1}{\alpha+1} f(aG_n)\right|.
\end{aligned}$$

Then f admits an asymptotic distribution function.

Similar to Proposition 9 one can prove the following proposition.

Proposition 11. *Let $G = (G_n)_n$ be a linear recurrent sequence. Let f be a G-additive function admitting an asymptotic distribution dF. Then dF is purely atomic if and only if f_n is ultimately 0.*

Example 3. Let $G_{n+2} = aG_{n+1} + aG_n$. Then the conditions of Proposition 10 can be slightly simplified. The function f admits an asymptotic distribution if the following series converge:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{\varepsilon=1}^{a} f(\varepsilon G_n),\ \sum_{n=0}^{\infty}\sum_{\varepsilon=1}^{a} f(\varepsilon G_n)^2,\\
&\sum_{n=0}^{\infty}\left|\sum_{\varepsilon=0}^{a-1} f(\varepsilon G_n) - af(aG_n)\right|,\ \sum_{n=0}^{\infty}\left(\alpha\sum_{\varepsilon=0}^{a-1} f(\varepsilon G_n) + af(aG_n)\right).
\end{aligned}$$

Example 4. The case $a = 1$ is interesting and can be easily generalised to obtain the so-called Multinacci sequence $G_{n+d} = G_{n+d-1} + G_{n+d-2} + \cdots + G_n$. For this numeration system, we get $\mathbb{P}([x_0 \cdots 01^s]) = (\alpha^d + \cdots + \alpha^{d-s})\alpha^{k+d}$ and the sufficient conditions are the convergence of the series

$$\sum_{n=0}^{\infty} |f(G_n)| \text{ and } \sum_{n=0}^{\infty} |f(G_n)|^2.$$

In particular, under this condition, the function f can be continuously extended to the odometer $\mathcal{K}_G$, for which case the existence of the limit distribution is obvious. The series in (5.11) yield the same trivial sufficient condition in this case.

The arguments used above for linear recurrent numeration systems of degree 2 are straightforward to extend to higher degrees. However, the explicit computation of the measures of the cylinders using (5.10) and the subsequent computation of the conditional expectations $\mathbb{E}(f \mid \mathcal{F}_{n-1})$ become more and more unpleasant, due to the rapidly increasing number of cases to distinguish.

References

[1] G. Barat, T. Downarowicz, A. Iwanik, and P. Liardet, *Propriétés topologiques et combinatoires des échelles de numération*, Colloq. Math. **84/85** (2000), no. , part 2, 285–306.

[2] G. Barat, T. Downarowicz, and P. Liardet, *Dynamiques associées à une échelle de numération*, Acta Arith. **103** (2002), 41–78.

[3] G. Barat and P. J. Grabner, *Distribution properties of G-additive functions*, J. Number Th. **60** (1996), 103–123.

[4] G. Barat and P. Liardet, *Dynamical systems originated in the Ostrowski alpha-expansion*, Ann. Univ. Sci. Budapest. Sect. Comput. **24** (2004), 133–184.

[5] N. L. Bassily and I. Kátai, *Distribution of the values of q-additive functions on polynomial sequences*, Acta Math. Hungar. **68** (1995), 353–361.

[6] R. Bellman and H. N. Shapiro, *On a problem in additive number theory*, Ann. of Math. (2) **49** (1948), 333–340.

[7] A. Brauer, *On algebraic equations with all but one root in the interior of the unit circle*, Math. Nachr. **4** (1951), 250–257.

[8] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinaĭ, *Ergodic theory*, Grundlehren der Mathematischen Wissenschaften, vol. 245, Springer-Verlag, New York, 1982, Translated from the Russian by A. B. Sosinskiĭ.

[9] H. Delange, *Sur les fonctions q-additive ou q-multiplicatives*, Acta Arith. **21** (1972), 285–298.

[10] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, J. Théor. Nombres Bordx. **10** (1998), 17–32.

[11] J.-M. Dumont and A. Thomas, *Gaussian asymptotic properties of the sum-of-digits function*, J. Number Th. **62** (1997), 19–38.

[12] P. D. T. A. Elliott, *Probabilistic number theory. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 239, Springer-Verlag, New York, 1979, Mean-value theorems.

[13] A. O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1968), 259–266.

[14] P. J. Grabner, P. Liardet, and R. F. Tichy, *Odometers and systems of numeration*, Acta Arith. **70** (1995), 103–123.

[15] P. J. Grabner and R. F. Tichy, *Contributions to digit expansions with respect to linear recurrences*, J. Number Th. **36** (1990), 160–169.

[16] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, 1979, Structure of topological groups, integration theory, group representations.

[17] K.-H. Indlekofer and I. Kátai, *Investigations in the theory of q-additive and q-multiplicative functions. I*, Acta Math. Hungar. **91** (2001), 53–78.

[18] ———, *Investigations in the theory of q-additive and q-multiplicative functions. II*, Acta Math. Hungar. **97** (2002), 97–108.

[19] K.-H. Indlekofer, I. Kátai, and Y.-W. Lee, *On q-multiplicative functions*, Publ. Math. Debrecen **61** (2002), 393–402.

[20] B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta function*, Trans. Amer. Math. Soc. **38** (1935), 48–88.

[21] I. Kátai, *Distribution of digits of primes in q-ary canonical form*, Acta Math. Hungar. **47** (1986), 341–359.

[22] ———, *Distribution of q-additive function*, Probability theory and applications, Math. Appl., vol. 80, Kluwer Acad. Publ., Dordrecht, 1992, pp. 309–318.

[23] ———, *Distribution of q-additive functions on the set of primes*, Liet. Mat. Rink. **47** (2007), 29–38.

[24] P. Lévy, *Sur les séries dont les termes sont des variables éventuelles indépendantes*, Studia Math. **3** (1931), 119–155.

[25] M. Loève, *Probability theory*, Third edition, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1963.

[26] K. Mahler, *On the translation properties of a simple class of arithmetical functions*, J. Math. and Phys. **6** (1927), 158–163.

[27] E. Manstavičius, *Probabilistic theory of additive functions related to systems of numeration*, New trends in probability and statistics, Vol. 4 (Palanga, 1996), VSP, Utrecht, 1997, pp. 413–429.

[28] Y. Peres, W. Schlag, and B. Solomyak, *Sixty years of Bernoulli convolutions*, Fractal Geometry and Stochastics II (C. Bandt, S. Graf, and M. Zähle, eds.), Progress in Probability, no. 46, Birkhäuser, 1999, pp. 39–65.
[29] M. Queffélec, *Substitution dynamical systems – spectral analysis*, Lecture Notes in Math., vol. 1294, Springer, 1987.
[30] H. N. Shapiro, *On the Erdős-Wintner theorem of probabilistic number theory*, Comm. Pure Appl. Math. **36** (1983), no. 5, 665–678.
[31] G. Tenenbaum, *Sur la non-dérivabilité de fonctions périodiques associées à certaines formules sommatoires*, The Mathematics of Paul Erdős (R. L. Graham and J. Nešetřil., eds.), Algorithms Comb., vol. 13, Springer, Berlin, 1997.
[32] A. M. Vershik and N. A. Sidorov, *Arithmetic expansions associated with the rotation of a circle and continued fractions*, St. Petersburg Math. J. **5** (1994), no. 6, 1121–1136.

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