ANALYTIC CONTINUATION OF A CLASS OF DIRICHLET SERIES

Peter J. Grabner and Jörg M. Thuswaldner

Institut für Mathematik A
Technische Universität Graz
Steyrergasse 30
8010 Graz, Austria

Abstract. We consider Dirichlet series of the type $\sum(\log k)^\eta(k(\log k)^\vartheta)^{-s}$. We prove the existence of an analytic continuation to the cut plane and give exact information about the singularity. We use this to generalize results, which occur in Ramanujan's second notebook.

1. Introduction and Statement of Results

The aim of this paper is to present generalizations of results due to Ramanujan [1, p. 308], Berndt and Evans [2] and Müller [5,6]. The first three authors study asymptotic expansions of sums of the form

$$\sum_{k=1}^{\infty}\exp\left(-xk^p\right)k^{m-1}\quad\text{and}\quad\sum_{k=1}^{\infty}\exp\left(-xk^p\right)k^{m-1}\log k \tag{1.1}$$

for $x\to 0+$. Berndt and Evans use a technique credited to Mellin, which involves certain Dirichlet series and their analytic continuation. Müller gives the analytic continuation of a large class of Dirichlet series, which are given by

$$\zeta_\Lambda(s)=\sum_{n=1}^{\infty}\lambda_n^{-s}, \tag{1.2}$$

where $\Lambda=(\lambda_n)$ has the asymptotic expansion

$$\sum_{k=-m}^{\infty}A_kn^{-k},\quad m\in\mathbb{N},\quad n\to\infty \tag{1.3}$$

with $A_{-m}>0$. He uses Ramanujan's result (cf. [1, p.308]) and an approximation technique to give a meromorphic continuation of these series to the whole complex plane. In [5] he computes the residues at the poles of these functions.

The authors are supported by the Austrian National Bank project Nr. 4995

Our approach will allow to study the asymptotic behaviour of sums of the form

$$\sum_{k=2}^{\infty} \exp\left(-k^{\alpha}(\log k)^{\beta} x\right) k^{\delta} (\log k)^{\varepsilon}, \quad \text{for } x \to 0+, \tag{1.4}$$

where $\alpha > 0$ and $\beta, \delta, \varepsilon$ are real numbers. Clearly this is related to the singular behaviour of the analytic continuation of the Dirichlet series

$$\sum_{k=2}^{\infty} \frac{k^{\delta}(\log k)^{\varepsilon}}{\left(k^{\alpha}(\log k)^{\beta}\right)^{s}}. \tag{1.5}$$

The correspondence is via Mellin transform and Mellin's inversion formula.

If we set $\beta = 0$ and let ε be a positive integer, this class reduces to a class similar to that in (1.1) and can be treated by the same methods (cf. [2, Theorems 3.1 and 3.2]). In the general case studied here there occur nonpolar singularities. In the analytic continuation of (1.5) we have a branch cut caused by the logarithm in the denominator and the non-integral power of the logarithm in the numerator. That is the matter why we are forced to use other methods to get the asymptotics of (1.4) resp. the analytic continuation of (1.5). These methods are similar to those used by A. Selberg to derive the asymptotics of the summatory function of the Dirichlet-coefficients of $(\zeta(s))^{\alpha}$ for $\alpha \in \mathbb{R}$ (cf. [7]).

Theorem 1. *Let η and ϑ be real numbers, then the Dirichlet series*

$$\zeta_{\eta,\vartheta}(s) = \sum_{k=2}^{\infty} \frac{(\log k)^{\eta}}{\left(k(\log k)^{\vartheta}\right)^{s}}$$

admits an analytic continuation to the whole complex plane except the line joining 1 and $-\infty$. This line gives a branch cut of the function, whose nature depends on the parameters. The singular expansion of the function around $s = 1$ starts with

$$\begin{aligned} &\Gamma(\eta-\vartheta+1)(s-1)^{\vartheta-\eta-1} && \text{for } \vartheta-\eta \notin \mathbb{N} \\ &\frac{(-1)^{m-1}}{(m-1)!}(s-1)^{m-1}\log\frac{1}{s-1} && \text{for } \vartheta-\eta = m \in \mathbb{N}. \end{aligned} \tag{1.6}$$

For the full expansion we refer to (2.2) *and* (2.3).

Remark 1. As mentioned above we get a meromorphic continuation of $\zeta_{\eta,\vartheta}$ in the special case $\vartheta = 0, \eta \in \mathbb{N}$. For a proof of this see [2, 8].

Remark 2. The term "singular expansion" shall describe the kind of singularity that the function has at the indicated point. Thus the function can be written as a sum of the singular part and a holomorphic function around this point. The terms in this expansion are ordered, such that later terms give smaller contributions to the asymptotic expansions of the sum (1.4).

The Dirichlet series (1.5) can be derived from the series studied in Theorem 1 by a linear change of the argument. From this observation and by a use of Mellin's inversion formula we get

Theorem 2. *Let $\alpha > 0$, $\beta \neq 0, \delta, \varepsilon$ be real numbers and $\eta = \varepsilon - \frac{\beta\delta}{\alpha}$ and $\vartheta = \frac{\beta}{\alpha}$. Then the function*

$$f_{\alpha,\beta,\delta,\varepsilon}(x) = \sum_{k=2}^{\infty} \exp\left(-k^{\alpha}(\log k)^{\beta}x\right) k^{\delta}(\log k)^{\varepsilon}$$

has the following asymptotics for $x \to 0+$, which has to be described by eight different cases:

(1) $\delta > -1$

$$\alpha^{\vartheta-\eta-1}\Gamma\left(\frac{\delta+1}{\alpha}\right) x^{-\frac{\delta+1}{\alpha}} \left(\log\frac{1}{x}\right)^{\eta-\vartheta} \left(1 + \frac{\beta(\vartheta-\eta)}{\log\frac{1}{x}}\log\log\frac{1}{x} + \cdots\right)$$

(2) $\delta = -1$ *and* $\varepsilon < -1$

$$\sum_{k=2}^{\infty}\frac{(\log k)^{\varepsilon}}{k} + \frac{\alpha^{-\varepsilon-1}}{1+\varepsilon}\left(\log\frac{1}{x}\right)^{\varepsilon+1}\left(1+\frac{\beta(\varepsilon+1)}{\log\frac{1}{x}}\log\log\frac{1}{x}+\cdots\right)$$

(3) $\delta = -1$ *and* $\varepsilon = -1$

$$\log\log\frac{1}{x} + \gamma + \frac{\beta}{\log\frac{1}{x}}\log\log\frac{1}{x} + \cdots$$

(4) $\delta = -1$ *and* $\varepsilon > -1$

$$\frac{\alpha^{-\varepsilon-1}}{1+\varepsilon}\left(\log\frac{1}{x}\right)^{\varepsilon+1}\left(1+\frac{\beta(\varepsilon+1)}{\log\frac{1}{x}}\log\log\frac{1}{x}+\cdots\right)$$

(5) $-1-\alpha < \delta < -1$

$$\sum_{k=2}^{\infty}(\log k)^{\varepsilon}k^{\delta} + \Gamma\left(\frac{\delta+1}{\alpha}\right)x^{-\frac{\delta+1}{\alpha}}\alpha^{\vartheta-\eta-1}\left(\log\frac{1}{x}\right)^{\eta-\vartheta} + \cdots$$

(6) $\vartheta - \eta \neq 1$ *and* $\delta = -1-\alpha$

$$\sum_{k=2}^{\infty}(\log k)^{\varepsilon}k^{\delta} + \frac{\alpha^{\eta-\vartheta+1}x}{\eta-\vartheta+1}\left(\log\frac{1}{x}\right)^{\eta-\vartheta+1} + \cdots$$

(7) $\vartheta - \eta = 1$ *and* $\delta = -1-\alpha$

$$\sum_{k=2}^{\infty}(\log k)^{\varepsilon}k^{\delta} - x\log\log\frac{1}{x} + \cdots$$

(8) $\delta < -1-\alpha$

$$\sum_{k=2}^{\infty}(\log k)^{\varepsilon}k^{\delta} - \alpha x\sum_{k=2}^{\infty}(\log k)^{\varepsilon+\beta}k^{\delta+\alpha} + \cdots.$$

In principle it is possible to give full asymptotic expansions by the methods used in the proof. We will supply all the tools needed for this, but do not state the corresponding formulæ because of their length.

Corollary. *If we set $\beta = 0$ we get simpler expressions for the main terms of the asymptotics in the cases*

(1) $\delta > -1$

$$\alpha^{-\varepsilon-1}\Gamma\left(\frac{\delta+1}{\alpha}\right)x^{-\frac{\delta+1}{\alpha}}\left(\log\frac{1}{x}\right)^{\eta-\vartheta}+\alpha^{-\varepsilon-1}x^{-\frac{\delta+1}{\alpha}}\sum_{k\geq 1}\left(\log\frac{1}{x}\right)^{\varepsilon-k}\frac{\Gamma^{(k)}\left(\frac{\delta+1}{\alpha}\right)}{k!}+\cdots$$

(2) $\delta = -1$ *and* $\varepsilon < -1$

$$\sum_{k=2}^{\infty}\frac{(\log k)^{\varepsilon}}{k}+\frac{\alpha^{-\varepsilon-1}}{1+\varepsilon}\left(\log\frac{1}{x}\right)^{\varepsilon+1}-\gamma\alpha^{-\varepsilon-1}\left(\log\frac{1}{x}\right)^{\varepsilon}+\cdots$$

(3) $\delta = -1$ *and* $\varepsilon = -1$

$$\log\log\frac{1}{x}+\gamma+\Gamma_0(0)\left(\log\frac{1}{x}\right)^{-2}+\cdots$$

(4) $\delta = -1$ *and* $\varepsilon > -1$

$$\frac{\alpha^{-\varepsilon-1}}{1+\varepsilon}\left(\log\frac{1}{x}\right)^{\varepsilon+1}-\gamma\alpha^{-\varepsilon-1}\left(\log\frac{1}{x}\right)^{\varepsilon}+\cdots.$$

Where $\Gamma_0(z) = \Gamma(z) - \frac{1}{z}$. *In the remaining cases the terms of Theorem 1 are not affected because the dominant singularity is polar.*

2. Proofs of the Theorems

Proof of Theorem 1. In order to give the analytic continuation of $\zeta_{\eta,\vartheta}(s)$ we use the Euler Maclaurin summation formula

$$\zeta_{\eta,\vartheta}(s)=\int_2^{\infty}\frac{(\log x)^{\eta}}{\left(x\,(\log x)^{\vartheta}\right)^{s}}\,dx+H(s), \tag{2.1}$$

to show that $H(s)$ is an entire function. From the m-th order Euler Maclaurin formula we get an analytic continuation of $H(s)$ to a holomorphic function in $\Re s > 1-m$.

$$H(s)=\frac{1}{2}f(2)-\sum_{k=1}^{m}\frac{B_{2k}}{2k!}f^{(2k-1)}(2)+R_m(s)$$

with

$$f_k(x)=\frac{(\log x)^{\eta}}{\left(x\,(\log x)^{\vartheta}\right)^{s}}.$$

Since $m$is arbitrary and $R_m(s)$ is holomorphic for $\Re s > 1-m$ we conclude that $H(s)$ is an entire function.

What remains is to investigate the analytic behaviour of the integral in (2.1). For this purpose we substitute $x = \exp(\frac{z}{s-1})$ for $\Re s > 1$ to get

$$\int_2^{\infty}\frac{(\log x)^{\eta}}{\left(x\,(\log x)^{\vartheta}\right)^{s}}dx=(s-1)^{\vartheta s-1-\eta}\int_{(s-1)\log 2}^{\infty}z^{\eta-\vartheta s}e^{-z}\,dz.$$

Because of the exponential decay of the integral we can do this also for non-real s in this half-plane. The integral which remains now is an incomplete Γ-function. We treat the behaviour of this function in the arc-region $|\arg(s-1)| \leq \psi < \pi$ in the following lemma.

Lemma 1. *The following equation holds for* $\vartheta-\eta\notin\mathbb{N}$

$$\int_{(s-1)\log 2}^{\infty} z^{\eta-\vartheta s}e^{-z}\,dz=\Gamma(\eta+1-\vartheta s)-\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{((s-1)\log 2)^{k+\eta+1-\vartheta s}}{k+\eta+1-\vartheta s}.$$

If $\vartheta-\eta=m\in\mathbb{N}$ *we have*

$$\int_{(s-1)\log 2}^{\infty} z^{\eta-\vartheta s}e^{-z}\,dz=\Gamma_{m-1}(\eta+1-\vartheta s)-\sum_{\substack{k=0\\k\neq m-1}}^{\infty}\frac{(-1)^k}{k!}\frac{((s-1)\log 2)^{k+\eta+1-\vartheta s}}{k+\eta+1-\vartheta s}$$
$$-\frac{(-1)^{m-1}}{(m-1)!}\frac{((s-1)\log 2)^{\vartheta(1-s)}-1}{\vartheta(1-s)},$$

where $\Gamma_m(z)=\Gamma(z)-\frac{(-1)^m}{m!(m+z)}$.

Proof. The proof of this lemma is done by splitting the usual Γ-integral into two parts and replacing the exponential function by its Taylor expansion for the integration from 0 to $(s-1)\log 2$. For $\eta-\vartheta<-1$ we use the following formula

$$\Gamma(z)=\int_0^\infty x^{z-1}\left(e^{-x}-\sum_{k=0}^{K-1}\frac{(-1)^k}{k!}x^k\right)dx\quad\text{for }-K<\Re z<-K+1.$$

For $\vartheta-\eta=m\in\mathbb{N}$ we subtract the residue of the Γ-function at $1-m$ and group it with the corresponding term in the sum. □

By the expressions we have obtained in the lemma and the singular expansion of $(s-1)^{\vartheta s-1-\eta}$ we get the following expansion around the singularity $s=1$ for $\zeta_{\eta,\vartheta}(s)$ for $\vartheta-\eta\notin\mathbb{N}$

$$(s-1)^{\vartheta-\eta-1}\sum_{n=0}^{\infty}\frac{(-\vartheta)^n}{n!}(s-1)^n\sum_{j=0}^{n}\binom{n}{j}\Gamma^{(j)}(\eta-\vartheta+1)\left(\log\frac{1}{s-1}\right)^{n-j}\tag{2.2}$$

and for $\vartheta-\eta=m\in\mathbb{N}$

$$\begin{aligned}&(s-1)^{m-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\vartheta^n(s-1)^n\sum_{k=0}^{n-1}\binom{n}{k}\Gamma_{m-1}^{(k)}(1-m)\left(\log\frac{1}{s-1}\right)^{n-k}+\\&\frac{(-1)^m}{(m-1)!}(s-1)^{m-1}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}\vartheta^n(s-1)^n\left(\log\frac{1}{s-1}\right)^{n+1}.\end{aligned}\tag{2.3}$$

The singular expansions start with the terms indicated in (1.6). This proves the theorem. □

Proof of Theorem 2. In order to compute the asymptotic expansion of $f_{\alpha,\beta,\delta,\varepsilon}(x)$ we compute its Mellin-transform

$$\Phi^*(s)=\int_0^\infty f_{\alpha,\beta,\delta,\varepsilon}(x)x^{s-1}\,dx=\Gamma(s)\sum_{k=2}^{\infty}\frac{k^\delta(\log k)^\varepsilon}{(k^\alpha(\log k)^\beta)^s}=\Gamma(s)\zeta_{\eta,\vartheta}(\alpha s-\delta)$$

for $\eta = \varepsilon - \frac{\beta\delta}{\alpha}$ and $\vartheta = \frac{\beta}{\alpha}$. We now use Mellin's inversion formula

$$f_{\alpha,\beta,\delta,\varepsilon}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta_{\eta,\vartheta}(\alpha s - \delta) x^{-s}\, ds$$

for some $c > \frac{\delta+1}{\alpha}$. We note here that by Lemma 1 the integrand has a singularity for $s = \frac{\delta+1}{\alpha}$.

The main ingredient of our proof is the correspondence between the type of the singularity of $f^*(s)$ and the asymptotic behaviour of $f(x)$, if the two functions are related by Mellin's inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} f^*(s) x^{-s}\, ds.$$

We summarize these well-known facts (cf.[3], [4]) in the following lemma.

Lemma 2. *Let $F^*(s)$ be analytic in $H_{\xi,\psi} = \{s \in \mathbb{C}\setminus\{0\} \mid \Re s > -\xi \wedge |\arg s| < \psi\}$ for some $\xi > 0$ and $\frac{\pi}{2} < \psi < \pi$. Then the following correspondences hold for real a and $b \in \mathbb{N}_0$ and $a \notin \mathbb{N}_0$*

$$f^*(s) = \mathcal{O}\left(|s|^a \left(\log\frac{1}{|s|}\right)^b\right) \Rightarrow f(x) = \mathcal{O}\left(\left(\log\frac{1}{x}\right)^{-1-a}\left(\log\log\frac{1}{x}\right)^b\right)$$

$$f^*(s) = o\left(|s|^a \left(\log\frac{1}{|s|}\right)^b\right) \Rightarrow f(x) = o\left(\left(\log\frac{1}{x}\right)^{-1-a}\left(\log\log\frac{1}{x}\right)^b\right)$$

$$f^*(s) \sim s^a\left(\log\frac{1}{s}\right)^b \Rightarrow$$

$$f(x) = \frac{1}{\left(\log\frac{1}{x}\right)^{a+1}} \sum_{\ell=0}^{b} \binom{b}{\ell} \left(\frac{1}{\Gamma}\right)^{(b-\ell)}(-a)\left(\log\log\frac{1}{x}\right)^{\ell} + o\left(\left(\log\frac{1}{x}\right)^{-a-1}\right).$$

If $a \in \mathbb{N}_0$ we have for $b \in \mathbb{N}$

$$f^*(s) = \mathcal{O}\left(|s|^a \left(\log\frac{1}{|s|}\right)^b\right) \Rightarrow f(x) = \mathcal{O}\left(\left(\log\frac{1}{x}\right)^{-1-a}\left(\log\log\frac{1}{x}\right)^{b-1}\right)$$

$$f^*(s) = o\left(|s|^a \left(\log\frac{1}{|s|}\right)^b\right) \Rightarrow f(x) = o\left(\left(\log\frac{1}{x}\right)^{-1-a}\left(\log\log\frac{1}{x}\right)^{b-1}\right)$$

$$f^*(s) \sim s^a\left(\log\frac{1}{s}\right)^b \Rightarrow$$

$$f(x) = \frac{1}{\left(\log\frac{1}{x}\right)^{a+1}} \sum_{\ell=0}^{b-1} \binom{b}{\ell} \left(\frac{1}{\Gamma}\right)^{(b-\ell)}(-a)\left(\log\log\frac{1}{x}\right)^{\ell} + o\left(\left(\log\frac{1}{x}\right)^{-a-1}\right).$$

With help of (2.2) and (2.3) we can find the singular expansion of $\Gamma(s)\zeta_{\eta,\theta}(\alpha s - \delta)$ in $s = \frac{\delta+1}{\alpha}$. Observing that for the cases (2) and (5–8) there comes an additional contribution from the first order pole of $\zeta_{\eta,\vartheta}(\alpha s - \delta)\Gamma(s)$ in $s = 0$ and applying Lemma 2 we derive the asymptotics stated in Theorem 2. □

References

1. B. Berndt, *Ramanujan's Notebooks, Part II*, Springer, Berlin, New York, 1989.
2. B. Berndt and R. Evans, *Extensions of asymptotic expansions from chapter 15 of Ramanujans second note book*, J. Reine Angew. Math. **361** (1985), 118–134.
3. G. Doetsch, *Handbuch der Laplace-Transformation*, Birkhäuser, Basel, 1958.
4. P. Flajolet and A.M. Odlyzko, *Singularity analysis of generating functions*, SIAM J. Disc. Math. **3** (1990), 216–240.
5. H. Müller, *On generalized Zeta-Functions at Negative Integers.*, Illinois J. Math. (2) **32** (1988), 222–229.
6. H. Müller, *Über die meromorphe Fortsetzung einer Klasse verallgemeinerter Zetafunktionen*, Arch. Math. **58** (1992), 265–275.
7. A. Selberg, *Note on a paper by L. G. Sathe*, J. Indian Math. Soc. B. **18** (1954), 83–87.
8. J.M. Thuswaldner, *Analytische Methoden zur asymptotischen Untersuchung von Funktionalgleichungen und zur probabilistischen Analyse kombinatorischer Algorithmen*, Thesis, Technische Universität Graz, 1995.