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SPECTRAL DISJOINTNESS OF DYNAMICAL SYSTEMS RELATED TO SOME ARITHMETIC FUNCTIONS

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Dedicated to the memory of Béla Brindza.

Abstract. We present a new and general approach to prove the spectral disjointness of dynamical systems related to digital expansions of natural numbers and Gaussian integers. The main tools are ideas from automata theory and rigid time in ergodic theory. This extends earlier work of T. Kamae and M. Queffélec.

1. Introduction

Let p be an integer, $p \geq 2$. An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ is called p-additive, if it satisfies the relation

$$f\left(\sum_{\ell=0}^{L} \varepsilon_\ell p^\ell\right) = \sum_{\ell=0}^{L} f(\varepsilon_\ell p^\ell), \tag{1.1}$$

where $\varepsilon_\ell \in \{0, \ldots, p-1\}$. In the sequel we will mainly be concerned with completely p-additive functions, i.e. p-additive functions which satisfy $f(\varepsilon p^\ell) = f(\varepsilon)$. A special instance of such a function is the p-ary sum-of-digits function $s_p(n)$ defined by

$$s_p\left(\sum_{\ell=0}^{L} \varepsilon_\ell p^\ell\right) = \sum_{\ell=0}^{L} \varepsilon_\ell.$$

We will also be concerned with p-multiplicative unimodular arithmetic functions $F : \mathbb{N} \to \mathbb{U}$ which satisfy the following multiplicative formula in place of (1.1)

$$F\left(\sum_{\ell=0}^{L} \varepsilon_\ell p^\ell\right) = \prod_{\ell=0}^{L} F(\varepsilon_\ell p^\ell).$$

As usual we set $e(t) = e^{2\pi i t}$, so that $e \circ f$ is p-multiplicative, if f is p-additive.

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In series of papers [10, 11, 12] T. Kamae has developed a method (involving Baker's theory of linear forms in logarithms of algebraic numbers) to prove the mutual singularity of the spectral measures related to the sum-of-digits function.

Let $\mathbb{Z}_p$ denote the totally disconnected compact group of p-adic integers (cf. [9]) equipped with its normalized Haar measure μ_p. For given p-additive function f we define a function $\psi_f : \mathbb{Z} \times \mathbb{Z}_p \to \mathbb{R}$ by

$$\psi_f(n,x) = \lim_{\substack{m \to x \\ m \in \mathbb{N}}} f(m+n) - f(m), \tag{1.2}$$

if the limit exists and $\psi(n,x) = 0$ otherwise. In general, the limit does not exist for $x \in \{-1, \dots, -n\}$, if $n > 0$ and for $x \in \{0, \dots, -n+1\}$, if $n < 0$. It is easy to see that ψ_f satisfies

$$\psi_f(m+n,x) = \psi_f(n,x) + \psi_f(m, n+x)$$

for μ_p-almost all x. This means that ψ_f is a cocycle which is, in addition, μ_p-continuous. As usual we set $e(t) = e^{2\pi i t}$. We define the group $G_f = \overline{\langle \{e(f(n)) \mid n \in \mathbb{N}\rangle}$ usually denoted simply by G and endowed with its normalized Haar measure h_G. Finally, we consider the transformation $T_{p,f}$ on $\mathbb{Z}_p \times G$ given by

$$T_{p,f}(x,u) = (x+1, ue(\psi_f(1,x))). \tag{1.3}$$

It is easy to see that $T_{p,f}$ is invertible and preserving the Haar measure on the compact group $\mathbb{Z}_p \times G$. Thus $(\mathbb{Z}_p \times G, T_{p,f}, \mu_p \otimes h_G)$ is a dynamical system, which is an extension of the odometer on $\mathbb{Z}_p$. Obviously, we have

$$T^n_{p,f}(x,u) = (x+n, ue(\psi_f(n,x))).$$

We shall study the spectral decomposition E of the unitary operator on $L^2(\mathbb{Z}_p \times G)$ associated to $T_{p,f}$. By general theory it is enough to study

$$\langle \gamma \circ T^n_{p,f}, \gamma \rangle = \int_{\mathbb{U}} x^n \langle E(dx)\gamma, \gamma \rangle$$

for all functions $\gamma \in L^2(\mathbb{Z}_p \times G)$. Thus, by the Bochner-Herglotz theorem, to every $\gamma \in L^2(\mathbb{Z}_p \times G)$ we associate a measure ρ_γ on $\mathbb{U}$, which is given by its Fourier coefficients

$$\hat{\rho}_\gamma(n) = \langle \gamma \circ T^n_{p,f}, \gamma \rangle$$

and our aim will be to determine the spectral type of $T_{p,f}$ *i.e.,* the Borel measure $\Xi_{p,f}$ on $\mathbb{U}$ (defined modulo equivalence of measures), such that $\rho_\gamma \ll \Xi_{p,f}$ for any $\gamma \in L^2$ and there exists a $\gamma_0 \in L^2$ such that $\rho_{\gamma_0} \sim \Xi_{p,f}$. The following decomposition

$$L^2(\mathbb{Z}_p \times G) = \bigoplus_{\xi \in \widehat{G}} L^2(\mathbb{Z}_p) \otimes \xi, \tag{1.4}$$

is invariant under the action of $T_{p,f}$, so that $\Xi_{p,f}$ can be written as ($*$ denotes the convolution on $\mathbb{U}$)

$$\Xi_{p,f} = \Delta_p + \Delta_p * \Lambda_{p,f}, \tag{1.5}$$

where Δ_p is discrete and comes from the action of addition by 1 on $L^2(\mathbb{Z}_p)$ and $\Lambda_{p,f}$ comes from the action of $T_{p,f}$ on the orthocomplement of $L^2(\mathbb{Z}_p)\otimes\mathbf{1}$. We shall see that Δ_p and $\Lambda_{p,f}$ are mutually singular (see Corollary 3). The action of $T_{p,f}$ on $L^2(\mathbb{Z}_p)\otimes\xi$, through the isometry $\varphi\mapsto\varphi\otimes\xi$, corresponds to the unitary representation $V_{f,\xi}$ of $\mathbb{Z}$ on $L^2(\mathbb{Z}_p)$ defined by

$$V^n_{f,\xi}(h)(x)=\xi(e(\psi_f(n,x)))h(x+n) \tag{1.6}$$

and the spectral measure $\rho_{\varphi\otimes\xi}$ is the one associated to φ with respect to $V_{f,\xi}$. Hence we first can restrict ourselves to the computation of $\hat{\rho}_\gamma(n)$ for $\gamma(x,u)=\chi(x)\xi(u)$ $(=\chi\otimes\xi(x,u))$ with $\chi\in\widehat{\mathbb{Z}}_p$ and $\xi\in\widehat{G}$, thus

$$\hat{\rho}_\gamma(n)=\chi(1)^n\int_{\mathbb{Z}_p}\xi(e(\psi_f(n,x)))\mu_p(dx)\quad(\gamma=\chi\otimes\xi). \tag{1.7}$$

Therefore, $\rho_\gamma=\delta_{\{\chi(1)\}}*\rho_{1\otimes\xi}$, where $\delta_{\{z\}}$ denotes the Dirac measure supported in z. Let us write ν_f for the spectral measure $\rho_{1\otimes\xi_1}$, setting $\xi_1:u\mapsto u$, so that

$$\hat{\nu}_f(n)=\int_{\mathbb{Z}_p}e(\psi_f(n,x))\,d\mu_p(x),\quad n\in\mathbb{Z}.$$

Notice that this equals the correlation coefficients

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}e(f(n+k)-f(k))$$

by unique ergodicity of the dynamical system $(\mathbb{Z}_p,\cdot+1,\mu_p)$ and μ_p-continuity of the functions $\psi_f(n,\cdot)$. Such correlation coefficients were studied in [1, 2] in the context of pseudo-randomness of sequences. If $\xi:u\mapsto u^s$ $(s\in\mathbb{Z})$, then $\rho_{1\otimes\xi}=\nu_{sf}$.

M. Queffélec [16] proved by means of Riesz-products that given ξ such that $\xi\circ\psi_f$ is not periodic, all spectral measures $\rho_{\chi\otimes\xi}$ are equivalent to ν_{sf} which is continuous and singular (with respect to the Lebesgue measure). The fact that ν_{sf} is singular continuous is also a consequence of the "principle of purity" (cf. [8]) implying that the spectral measure ν_f is either atomic, purely singular continuous, or absolutely continuous.

In our main result we will prove spectral disjointness of $\Lambda_{p,f}$ and $\Lambda_{q,g}$ provided that $p,q\geq 2$ are multiplicatively independent integers and f (or g) is not trivial, that is to say $\xi(e(f))\neq e(\frac{k}{p-1}s_p)$ for any non trivial character $\xi\in\hat{G}$ and any integer k. This is an extension of the above mentioned work of Kamae concerning the sum-of-digits function. Queffélec [17] applied general results about Riesz-products and Šreĭder characters (cf. [19]) to give a more conceptual approach to the results of Kamae. We develop a new method for the proof which avoids Baker's theory on linear forms in logarithms as well as Šreĭder characters. In a concluding section we extend our method to more complicated digital expansions, namely to canonical number systems for the Gaussian integers. A main ingredient of our approach is the application of addition automata and ζ-rigid time in ergodic theory.

2. Spectral disjointness

The following theorem is due to Kamae [10, 11, 12, 17] who proved it under the assumption that p and q are coprime. In this case even the measures $\Xi_{p,\alpha s_p}$ and $\Xi_{q,\beta s_q}$ are mutually singular.

Theorem 1. *Let $p, q \geq 2$ be two multiplicatively independent integers and let α and β be two irrational real numbers. Then the two measures $\Lambda_{p,\alpha s_p}$ and $\Lambda_{q,\beta s_q}$ are mutually singular.*

We will prove a more general theorem here.

Theorem 2. *Let $p, q \geq 2$ be two multiplicatively independent integers and let f and g be completely p-additive and q-additive functions. Assume that f is not trivial (i.e., $\xi(e(f)) \neq e(\frac{k}{p-1}s_p)$ for any non trivial character ξ on G_f any integers k), then the spectral measures $\Lambda_{p,f}$ and $\Lambda_{q,g}$ are mutually singular.*

We shall see that $\xi \circ e(f)$ is periodic if and only if $\xi \circ e(f) = e(\frac{k}{p-1}s_p)$ (see Proposition 2).

Addition of $n = \sum_{k=0}^{K} \varepsilon_k p^k$ to $x = \sum_{\ell=0}^{\infty} \delta_\ell p^\ell \in \mathbb{Z}_p$ can be described in terms of automata. For any digit $0 \leq \varepsilon \leq p-1$ we introduce the automaton $\mathcal{A}_\varepsilon$ defined by Figure 1. The

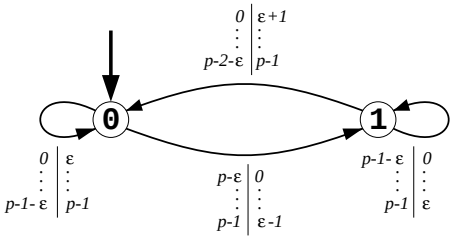

FIGURE 1. The automaton $\mathcal{A}_\varepsilon$

addition $n + x = \sum_{k=0}^{\infty} \eta_k p^k$ can be performed as follows: read the digit δ_0 and feed it into the automaton $\mathcal{A}_{\varepsilon_0}$ starting at the initial state. Denote the state reached after this operation by S_0. The digit written by the transducer is the digit η_0. Now read the digit δ_1 and feed it into the automaton $\mathcal{A}_{\varepsilon_1}$ this time starting in the state S_0. Denote the state reached after this operation by S_1. The digit written by the transducer is the digit η_1. Iterating this procedure gives all digits η_k.

To each automaton and each p-additive function, we attach the matrices

$$A_\varepsilon^{(m)} = \frac{1}{p} \begin{pmatrix} \sum_{\ell=0}^{p-\varepsilon-1} e(f_m(\varepsilon+\ell) - f_m(\ell)) & \sum_{\ell=p-\varepsilon}^{p-1} e(f_m(\varepsilon+\ell-p) - f_m(\ell)) \\ \sum_{\ell=0}^{p-\varepsilon-2} e(f_m(\varepsilon+\ell+1) - f_m(\ell)) & \sum_{\ell=p-\varepsilon-1}^{p-1} e(f_m(\varepsilon+\ell-p+1) - f_m(\ell)) \end{pmatrix}$$

where, for any non negative integer m, f_m denotes the p-additive function defined by

$$f_m(n) = f(p^m n).$$

Then we have

Proposition 1. *For any p-additive function f the following product formula*

$$\hat{\nu}_f\left(\sum_{k=0}^{K}\varepsilon_k p^k\right) = (1,0)A_{\varepsilon_0}^{(0)}\dots A_{\varepsilon_K}^{(K)}\begin{pmatrix}1\\ \hat{\nu}_{f_K}(1)\end{pmatrix} \tag{2.1}$$

holds.

Proof. Set for short $F_m(n,x) = e(\psi_{f_m}(n,x))$. Using the p-multiplicativity of $e(f)$ one gets for $n = \varepsilon_0 + pn'$:

$$\begin{aligned}\int_{\mathbb{Z}_p} F(n,x)\mu_p(dx) = &\sum_{a+\varepsilon_0<p} F(\varepsilon_0,a)\frac{1}{p}\int_{\mathbb{Z}_q} F_1(n',x)\mu_p(dx)\\ &+\sum_{a+\varepsilon_0\geq p} F(\varepsilon_0+a-p,a)\frac{1}{p}\int_{\mathbb{Z}_p} F_1(n'+1,x)\mu_q(dx)\end{aligned}$$

and an analogous equation for $\int_{\mathbb{Z}_p} F(n+1,x)\mu_p(dx)$, replacing e_0 by e_0+1 (with the usual convention that a sum over an empty set is equal to 0). Consequently

$$\begin{pmatrix}\hat{\nu}_f(n)\\ \hat{\nu}_f(n+1)\end{pmatrix} = A_{\varepsilon_0}^{(0)}\begin{pmatrix}\hat{\nu}_{f_1}(n')\\ \hat{\nu}_{f_1}(n'+1)\end{pmatrix} \tag{2.2}$$

and (2.1) follows by induction. □

Remark 1. By the known fact (cf. [3]) that p-additive functions which satisfy the hypotheses of Theorem 2 are pseudo-random, it follows that the corresponding spectral measure is not atomic. Furthermore, an application of (2.1) shows that for any $a \in \{0,\dots,p-1\}$

$$\begin{aligned}\hat{\nu}_f(a) &= \frac{1}{p}\sum_{\ell=0}^{p-a-1} e(f(a+\ell)-f(\ell)) + \frac{1}{p}\sum_{\ell=p-a}^{p-1} e(f(a+\ell-p)-f(\ell))\hat{\nu}_f(1)\\ \hat{\nu}_f(1) &= \frac{\sum_{\ell=0}^{p-2} e(f(\ell+1)-f(\ell))}{p-e(f(0)-f(p-1))}.\end{aligned} \tag{2.3}$$

If $\hat{\nu}_f(1) = 0$ (this cannot happen for $p=2$), then (2.3) shows that $\hat{\nu}_f(p-1) \neq 0$. Moreover, $\hat{\nu}_f(mp^n) = \hat{\nu}_f(m)$ for $m \in \mathbb{N}$; this implies that ν_f is not absolutely continuous.

Lemma 1. *Let $p \geq 2$ be an integer and f be a completely p-additive function, which does not satisfy $f(n) \equiv \frac{k}{p-1}s_p(n) \pmod 1$ for some integer k. Then there exists a constant θ, $0<\theta<1$, and a finite block of digits B (which contains at least two distinct digits) such that*

$$|\hat{\nu}_f(n)| \leq \theta^{\sigma_B(n)}, \tag{2.4}$$

where $\sigma_B(n)$ denotes the number of (non-overlapping) occurrences of the block B in the p-ary digital expansion of n.

Proof. We notice that the entries of the matrices $(a_{ij}^{(\varepsilon)}) = A_\varepsilon$ satisfy

$$|a_{ij}^{(\varepsilon)}| \le b_{ij}^{(\varepsilon)} \text{ for } B_\varepsilon = (b_{ij}^{(\varepsilon)}) = \frac{1}{p}\begin{pmatrix} p-\varepsilon & \varepsilon \\ p-\varepsilon-1 & \varepsilon+1 \end{pmatrix}. \tag{2.5}$$

If one of the above inequalities is strict for one ε (this cannot happen for $p = 2$), we see that all the entries of the matrix $A_\delta A_\varepsilon A_\eta$ are strictly bounded by the corresponding entries of the matrix $B_\delta B_\varepsilon B_\eta$. Thus we have $\|A_\delta A_\varepsilon A_\eta\| < 1$, where $\|\cdot\|$ denotes the matrix norm associated to the maximum norm on $\mathbb{C}^2$.

It remains to treat the case that there is equality in (2.5) for all i, j, ε. From $\varepsilon = 1$ and $i = j = 1$ we conclude that $f(\delta+1) - f(\delta) \equiv C \pmod 1$ for some constant C. Therefore $f(\delta) \equiv C\delta \pmod 1$, which implies that $f(n) \equiv Cs_p(n) \pmod 1$. In this case the matrix A_ε takes the form

$$A_\varepsilon = \frac{1}{p}\begin{pmatrix} (p-\varepsilon)\zeta^{\varepsilon} & \varepsilon\zeta^{\varepsilon-p} \\ (p-\varepsilon-1)\zeta^{\varepsilon+1} & (\varepsilon+1)\zeta^{\varepsilon-p+1} \end{pmatrix}$$

with $\zeta = e(C)$. Computing the entries of the product of two matrices shows that $\|A_\varepsilon A_\eta\| < 1$ for any pair $(\varepsilon, \eta) \notin \{(0,0), (p-1, p-1)\}$ except if $\zeta^{p-1} = 1$.

Taking any of the blocks considered in the two cases as the block B mentioned in the lemma and setting θ the norm of the corresponding matrix product, (2.1) gives the proof of the lemma. □

Remark 2. Observe that $||A_0^m|| = ||A_{p-1}^m|| = 1$ for any integers $m \ge 0$. For $p > 2$ and $f \not\equiv Cs_p \pmod 1$ the above proof shows that there is a digit $\varepsilon \ne 0$ such that $\sigma_B(n)$ can be replaced by the counting function of all occurrences of ε with at least 2 digits in between.

Proposition 2. *The measure ν_f is discrete, if and only if $f \equiv \frac{k}{p-1}s_p \pmod 1$ for $k \in \mathbb{Z}$. Otherwise, ν_f is singular continuous.*

Proof. Assume first that $f \not\equiv \frac{k}{p-1}s_p \pmod 1$ and apply Lemma 1 to bound the Fourier coefficients of ν_f in terms of the counting function of some block B of length L. Since $B \ne 0^L$, we can estimate $\sigma_B(n)$ in Lemma 1 from below by the number of occurrences of the "digit" B in the p^L-ary expansion of n. We apply Tschebysheff's inequality to obtain

$$\#\left\{n < N \mid \sigma_B(n) \le \frac{1}{2p^L}\log_p N\right\} \le \frac{4Lp^{2L}N}{\log_p N}.$$

Thus we have

$$\sum_{n<N}|\hat\nu_f(n)|^2 \le \sum_{\substack{n<N \\ \sigma(n) > \frac{1}{2p^L}\log_p N}} \theta^{\frac{1}{p^L}\log_p N} + \sum_{\substack{n<N \\ \sigma(n) \le \frac{1}{2p^L}\log_p N}} 1 \le N^{1+\frac{1}{p^L}\log_p\theta} + \frac{4Lp^{2L}N}{\log_p N}, \tag{2.6}$$

which implies

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n<N}|\hat\nu_f(n)|^2 = 0.$$

By the Wiener-Schoenberg theorem [18, 20] this means that ν_f has no point masses. By the principle of purity, ν_f is singular or absolutely continuous with respect to the Haar

measure on $\mathbb{U}$. The latter case never occurs since $\nu_f(p^k\ell) = \nu_f(\ell)$ for all integers $k \geq 0$ and any $\ell \in \mathbb{Z}$ while $\nu_f \neq 0$.

In the case $f \equiv \frac{k}{p-1} s_p \pmod 1$ we use the congruence $s_p(n) \equiv n \bmod (p-1)$ to see that $e(f(n))$ is periodic and therefore the measure is equal to the uniform distribution on the points $\langle e(\frac{k}{p-1})\rangle$. □

Remark 3. Notice that $e(\psi_f)$ is periodic if and only if $f(n) \equiv \frac{k}{p-1} s_p(n) \pmod 1$ for an integer k.

Remark 4. The support of the measure in the discrete case is the finite group $\langle e(\frac{k}{p-1})\rangle$. This shows that [16, Proposition 3] is not correct as stated there. Queffélec's proof can be corrected to show that the discrete measure is supported on an at most countable union of classes of the group $\langle \{e(p^{-k}) \mid k \in \mathbb{N}\}\rangle$ for general additive functions f.

In the following we will need the notation

$$B_q(n; \varepsilon_s \dots \varepsilon_0) = \#\{i\,;\ 0 \leq i \leq \log_q n,\ d_{i-s+j}(n) = \varepsilon_j, \text{and } 0 \leq j \leq s\},$$

where $d_j(n)$ denotes the j-th digit of n in base q representation and $\varepsilon_s \dots \varepsilon_0$ is an arbitrary block of digits in $\{0, \dots, p-1\}$.

Lemma 2. *Let $p, q \geq 2$ be multiplicatively independent integers and $r \in \mathbb{N}$, $r > 0$. Then, for any given block $\varepsilon_s \dots \varepsilon_0$, there exists an increasing sequence of integers n_k such that*

$$\lim_{k\to\infty} B_q(rp^{n_k}, \varepsilon_s \dots \varepsilon_0) = \infty.$$

Proof. We imitate the proof of [4, Theorem 2]. Let us consider the sum

$$\frac{1}{N} \sum_{n=1}^{N} B_q(rp^n, \varepsilon_s \dots \varepsilon_0). \tag{2.7}$$

We have to show that this sum tends to ∞ as $N \to \infty$. Let $K = [\frac{N}{\alpha}]$ and $m = \sum_{i=0}^{s} \varepsilon_i q^i$. For a positive integer $\ell \leq N$ we consider

$$A_\ell = \#\left\{(n,k)\,;\ 1 \leq n\,;\ s \leq k \leq K \text{ and } \ell + \frac{m}{q^{s+1}} \leq \frac{rp^n}{q^{k+1}} < \ell + \frac{m+1}{q^{s+1}}\right\}. \tag{2.8}$$

Notice that the inequalities in (2.8) involving ℓ are equivalent to the occurrence of the block $\varepsilon_s \dots \varepsilon_0$ in the q-adic digital expansion of rp^n at some position k. Taking logarithms in (2.8) and setting $\alpha = \frac{\log q}{\log p}$ we obtain

$$A_\ell = \#\Bigg\{k \leq K - s + 1\,;$$
$$\exists\, n,\ 1 \leq n \leq N, \frac{\log(\ell + \frac{m}{q^{s+1}})}{\log p} \leq n - (k+s)\alpha + \frac{\log r}{\log p} < \frac{\log(\ell + \frac{m+1}{q^{s+1}})}{\log p}\Bigg\} =$$
$$\#\{k \ \ 1 \leq k \leq K \text{ and } \{-(k+s)\alpha\} \in I_\ell\} + O(\log(\ell+1)),$$

where I_ℓ denotes the interval

$$\left[\frac{\log(\ell+\frac{m}{q^{s+1}})-\log r}{\log p}, \frac{\log(\ell+\frac{m+1}{q^{s+1}})-\log r}{\log p}\right)$$

modulo 1 of Lebesgue measure $|I_\ell|$. By the irrationality of α the sequence $(k\alpha)_{k\in\mathbb{N}}$ is uniformly distributed modulo 1 and thus we have

$$A_\ell = K|I_\ell| + o(N) + O(\log(\ell+1)),$$

where the $o(N)$-term is uniform in ℓ. Summing up we obtain for $2 \le \psi(N) < N$

$$\sum_{\ell=1}^{\psi(N)} A_\ell = K\sum_{\ell=1}^{\psi(N)} |I_\ell| + o(N\psi(N)) + O(\psi(N)\log\psi(N)).$$

Now choose the function $\psi(N)$ such that $\lim_{N\to\infty}\psi(N) = \infty$ and the $o(N\psi(N))$ term above is still $O(N)$. We now observe that the series

$$\sum_{\ell=1}^{\infty} |I_\ell|$$

is divergent and since

$$\frac{1}{N}\sum_{n=1}^{N} B_q(rp^n, \varepsilon_s \dots \varepsilon_0) \ge \frac{1}{N}\sum_{\ell=1}^{\psi(N)} A_\ell$$

we get $\lim_{N\to\infty} B_q(rp^n, \varepsilon_s \dots \varepsilon_0) = \infty$. Thus the proof of the Lemma is complete. □

3. Rigid times and proof of Theorem 2

For the proof of the main result we now introduce tools from ergodic theory. Let Γ be an infinite, countable discrete Abelian group, let $V : \gamma \to V^\gamma$ be a unitary representation of Γ on a Hilbert space H and let ζ be a complex number such that $|\zeta| \le 1$. An infinite subset S of Γ will be said a ζ-rigid time for V if the family $(V^s)_{s\in S}$ weakly converges to ζI (where I is the identity map) with respect to the filter of co-finite sets. In other words, for all h, h' in H and for all $\varepsilon > 0$, there exists a finite subset F of S such that $|\langle \tau^s h|h'\rangle - \zeta\langle h|h'\rangle| \le \varepsilon$ holds for any $s \in S \setminus F$. For short we write $\lim_{s\in S}\langle V^s h|h'\rangle = \zeta\langle h|h'\rangle$. By polarization, S is a ζ-rigid time for V if and only if for any $h \in H$

$$\lim_{s\in S}\langle V^s h|h\rangle = \zeta\|h\|^2. \tag{3.1}$$

Now we state the following general

Theorem 3. *Let V be a unitary representation of Γ on H and let ζ be a complex number of modulus ≤ 1. An infinite subset $S \subset \Gamma$ is a ζ-rigid time for V if and only if for any $h \in H$ and any $\varphi \in L^1(\hat\Gamma, \rho_h)$,*

$$\lim_{s\in S}\int_{\hat\Gamma}\varphi(u)u(s)\rho_h(du) = \zeta\int_{\hat\Gamma}\varphi(u)\rho_h(du), \tag{3.2}$$

where ρ_h denotes the spectral measure of h associated to V which is given by

$$\hat{\rho}_h(s) = \langle V^s h, h\rangle, \quad s \in \Gamma.$$

Proof. Assume that S is a ζ-rigid time for V. For any $\gamma \in \Gamma$, let C_γ be the character on $\hat{\Gamma}$ defined by $C_\gamma(u) = u(\gamma)$; one has by definition

$$\begin{aligned}\langle V^{s+\gamma} h|h\rangle &= \int_{\hat{\Gamma}} C_{s+\gamma}(u)\rho_h(du)\\ &= \int_{\hat{\Gamma}} u(s)u(\gamma)\rho_h(du).\end{aligned}$$

On the other hand,

$$\lim_{s\in S}\langle V^{s+\gamma}h|h\rangle = \zeta\langle V^\gamma h|h\rangle = \zeta\int_{\hat{\Gamma}} C_\gamma(u)\rho_h(du).$$

Therefore (3.2) holds for $\varphi = C_\gamma$. Clearly (3.2) also holds for any linear combination of characters and a straightforward density argument shows that (3.2) is also true for any $\varphi \in L^1(\hat{\Gamma}, \rho_h)$. Conversely, assuming that (3.2) is true for any h in H, φ in $L^1(\hat{\Gamma}, \rho_h)$ and taking $\varphi = \mathbf{1}$ we immediately get (3.1). □

The notion on ζ-rigid time furnishes the following simple test to derive the mutual singularity of two unitary representations:

Theorem 4. *Let V and V' be two unitary representations of Γ (on Hilbert spaces H and H', respectively) and assume that $S \subset \Gamma$ is a ζ-rigid time for V and a ζ'-rigid time for V' with $\zeta \neq \zeta'$. Then the spectral measures of V and V' are mutually singular.*

Proof. Let $h \in H$, $h' \in H'$ be of norm 1 and let ν_h, $\nu_{h'}$ be the corresponding spectral measures (which are both probability measures on $\hat{\Gamma}$). Let σ be a probability measure on $\hat{\Gamma}$ and assume that σ is absolutely continuous with respect to ρ_h and $\rho_{h'}$. By definition

$$\hat{\sigma}(\gamma) = \int_{\hat{\Gamma}} \frac{d\sigma}{d\rho_h}(u)u(\gamma)\rho_h(du) = \int_{\hat{\Gamma}} \frac{d\sigma}{d\rho_{h'}}(u)u(\gamma)\rho_{h'}(du),$$

and passing to the limit along S we obtain from Theorem 3

$$\lim_{s\in S}\hat{\sigma}(s) = \zeta = \zeta'.$$

This contradiction means that σ does not exist or, equivalently that ρ_h and $\rho_{h'}$ are mutually singular. □

Now we introduce particular weighted unitary representations related to multiplicative Γ-cocycles. Let K be a compact metrisable Abelian group with group law denoted additively and assume that Γ acts on K by means of translations. For any $\gamma \in \Gamma$, let τ^γ denote the translation which realizes the action of γ. We assume that this τ-action is ergodic and aperiodic on K endowed with its Haar probability measure μ. Aperiodicity of τ implies that $\gamma \mapsto \tau^\gamma(0_K)$ is one-one. For this reason we will identify γ by the group element $\tau^\gamma(0_K)$. We will also view τ as a unitary representation of Γ on the Hilbert space $L^2(K, \mu)$ and since Γ is infinite, we notice that there always exists a 1-rigid time S for τ.

As above, $\mathbb{U}$ denotes the group of complex numbers of modulus 1 equipped with its Haar measure λ and let $\varphi : \Gamma \times K \to \mathbb{U}$ be a τ-cocycle i.e. a measurable map such that

$$\varphi(\gamma+\gamma', x) = \varphi(\gamma, \tau^{\gamma'}x)\varphi(\gamma', x) \quad \mu\text{-a.e.}$$

We then define the skew product action τ_φ of Γ on the product space $(K \times \mathbb{U}, \mu \otimes \lambda)$ by

$$(\tau_\varphi)^\gamma(x, \zeta) = (x+\gamma, \zeta\varphi(\gamma, x)).$$

Finally, we define the unitary representation U_φ of Γ on $L^2(K)$ by

$$U_\varphi^\gamma(h)(x) = \varphi(\gamma, x)h(\tau^\gamma x). \tag{3.3}$$

Theorem 5. *Assume that S is a 1-rigid time for τ. Then for any τ-cocycle $\varphi : \Gamma \times K \to \mathbb{U}$, the set S is a ζ-rigid time for the representation U_φ if and only if for any $\chi \in \hat{K}$,*

$$\lim_{s\in S} \int_K \varphi(s, x)\chi(x)\mu(dx) = \begin{cases} \zeta & \text{if } \chi \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. For characters χ and χ' on K, we have

$$\langle U_\varphi^\gamma \chi | \chi' \rangle = \chi(\gamma) \int_K \varphi(\gamma, x)\chi(x)\overline{\chi'(x)}\mu(dx).$$

Assume that S is a ζ-rigid time for U_φ, then $\lim_{s\in S}\langle U_\varphi^s\chi|\chi'\rangle = \zeta\langle\chi|\chi'\rangle$. This proves (3.4). Conversely, assume (3.4) for any $\chi \in \hat{K}$, then $\lim_{s\in S}\overline{\chi(s)}\langle U_\varphi^s\chi|\chi'\rangle = \zeta\langle\chi|\chi'\rangle$ for any characters χ and χ' on K whereas $\lim_{s\in S}\chi(s) = 1$ due to the 1-rigidity of τ along S, hence

$$\lim_{s\in S}\langle U_\varphi^s\chi|\chi'\rangle = \zeta\langle\chi|\chi'\rangle.$$

By bilinearity, the same formula holds for any linear combinations of characters. We finally obtain by continuity, $\lim_{s\in S}\langle U_\varphi^s h|h'\rangle = \zeta\langle h|h'\rangle$ for all h, h' in $L^2(K, \mu)$, as expected. □

Readily, for any integer $\ell \neq 0$, the set $S = \{\ell p^n \ ; \ n \in \mathbb{N}\}$ is a 1-rigid time for the $\mathbb{Z}$-action $x \mapsto x + m$ ($m \in \mathbb{Z}$) on $\mathbb{Z}_p$. Going back to the cocycle ψ_f and the unitary representation $V_{f,\xi}$ we get the following consequence:

Theorem 6. *Assume that f is a completely p-additive arithmetic function. Then for any integer $\ell \neq 0$ the set $S := \{\ell p^n \ ; \ n \in \mathbb{N}\}$ is a ζ-rigid time for $V_{f,\xi}$, with $\xi : u \mapsto u^m$ ($u \in G$, $m \in \mathbb{Z}$) and*

$$\zeta = \hat{\nu}_{mf}(\ell).$$

Proof. In order to apply Theorem 5 we compute

$$I(n) = \int_{\mathbb{Z}_p} \xi(e(\psi_f(\ell p^n, x)))\chi(x)\mu(dx)$$

for any $\chi \in \widehat{\mathbb{Z}}_p$. Associated to χ, there exists an integer k such that $\chi(yp^k) = 1$ for any $y \in \mathbb{Z}_p$ and due to the p-additivity of f, one gets for any integer $n \geq k$

$$\begin{aligned} I(n) &= \frac{1}{p^n} \sum_{m=0}^{p^n-1} \int_{\mathbb{Z}_p} \xi(e(\psi_{f_n}(\ell, y)))\chi(m + p^n y)\mu(dy) \\ &= \left(\int_{\mathbb{Z}_p} \chi(x)\mu(dx) \right) \left(\int_{\mathbb{Z}_p} \xi(e(\psi_{f_n}(\ell, x)))\mu(dx) \right). \end{aligned}$$

The complete p-additivity of f implies

$$I(n) = \langle \chi | \mathbf{1} \rangle \hat{\nu}_{mf}(\ell)$$

so that conditions (3.4) are fulfilled. □

Corollary 1. *Let $m, n \in \mathbb{Z}$, then the two measures ν_{mf} and ν_{nf} are either equal or mutually singular.*

Proof. Assume that $\nu_{nf} \neq \nu_{mf}$. There exists an $\ell \in \mathbb{Z}$ such that $\hat{\nu}_{nf}(\ell) \neq \hat{\nu}_{mf}(\ell)$ which implies ν_{mf} and ν_{nf} mutually singular by using Theorems 6 and 4. □

Now we relate the measure ρ_γ given by (1.7) for $\gamma = \chi \otimes \xi$ to the measure ν_{mf}, where m is given by $\xi : u \mapsto u^m$. Recall that $\rho_\gamma = \delta_{\{\chi(1)\}} * \nu_{mf}$.

Proposition 3. *With the above notations, assume that $mf \neq \frac{k}{p-1} s_p \mod 1$. Then the measure ρ_γ is equivalent to ν_{mf}. Equivalently, all translations of ν_{mf} by p^k-th roots of unity are equivalent.*

Proof. Let μ be a measure on $\mathbb{U}$ and ζ be a primitive K-th root of unity. Define the measure

$$\kappa(dt) = \frac{1}{K^2} \left| \sum_{j=0}^{K-1} a(j)e(-jt) \right|^2 \sum_{k=0}^{K-1} \mu * \delta_{\{\zeta^k\}}(dt).$$

The two measures are then equivalent, if μ is continuous, since the trigonometric polynomial vanishes in at most finitely many points. A straightforward computation shows that the Fourier coefficients of κ and μ are related by the following formula, for $0 \leq \ell < K$,

$$\hat{\kappa}(\ell + Kn) = \frac{1}{K} \sum_{j=0}^{K-\ell-1} a(\ell + j)\overline{a(j)}\hat{\mu}(Kn) + \frac{1}{K} \sum_{j=K-\ell}^{K-1} a(\ell + j - K)\overline{a(j)}\hat{\mu}(K(n+1)). \tag{3.5}$$

Taking $K = p^k$, where k is the smallest exponent such that $\chi(1)^{p^k} = 1$, $\mu = \nu_{mf}$ (which is continuous by Proposition 2) and $a(j) = e(mf(j))\chi(1)^{js}$ in (3.5) ($0 \le s < p^k$) we obtain

$$\hat{\kappa}(\ell + p^k n) = \chi(1)^{\ell s}\left(\frac{1}{p^k}\sum_{j=0}^{p^k-\ell-1} e(m(f(\ell+j)-f(j)))\hat{\nu}_{mf}(n) + \right.$$
$$\left. \frac{1}{p^k}\sum_{j=p^k-\ell}^{p^k-1} e(m(f(\ell+j-p^k)-f(j)))\hat{\nu}_{mf}(n+1)\right) = \chi(1)^{\ell s}\hat{\nu}_{mf}(\ell+p^k n).$$

The last equation holds by an application of (2.2) to the function mf as a completely p^k-additive function. Thus $\kappa = \rho_{\chi^s\otimes\xi}$ and in particular ρ_γ is equivalent to $p^{-k}\sum_{j=0}^{p^k-1}\nu_{mf} * \delta_{\{\chi(1)^j\}}$, which by the same argument is equivalent to ν_{mf}. □

Corollary 2. *The spectral type of the unitary representation $V_{f,\xi}$ defined by (1.6), with $\xi : u \mapsto u^m$ such that $mf \not\equiv \frac{k}{p-1}s_p \pmod 1$, is equal to ν_{mf}, and for any $h \in L^2(\mathbb{Z}_p)$, any $\chi \in \widehat{\mathbb{Z}}_p$ the spectral measures $\rho_{h\otimes\xi}$ and $\rho_{(\chi\cdot h)\otimes\xi}$ $(= \rho_{h\otimes\xi} * \delta_{\{\chi(1)\}})$ are equivalent.*

Proof. It is enough to show that for any linear combination $h = \sum_{j=1}^n a_j\chi_i$ of characters χ_j on $\mathbb{Z}_p$ the measure $\rho_{h\otimes\xi}$ is absolutely continuous with respect to ν_{mf}, but as a classical result $\rho_{h\otimes\xi}$ is absolutely continuous with respect to $\sum_{j=1}^n \rho_{\chi_j\otimes\xi}$, and the result follows from the above proposition. □

In the next corollary, we sum up results which essentially derive from Corollary 2 and the decomposition (1.4):

Corollary 3. *For any completely p-additive arithmetic function f, the dynamical system $(\mathbb{Z}_p \times G, T_{p,f}, \mu_p \otimes h_G)$ is ergodic. Let Δ_p be the discrete part contribution of the translation $T : x \mapsto x+1$ on $\mathbb{Z}_p$, $\Lambda_{p,f}$ the part corresponding to the orthocomplement of $L^2(\mathbb{Z}_p, \mu_p) \otimes \mathbf{1}$ with $\Lambda_{p,f}^{(d)}$ (resp. $\Lambda_{p,f}^{(c)}$) its discrete (resp. continuous) part. The spectral type of $T_{p,f}$ (cf. (1.5)) has the form*

$$\Xi_{p,f} = \Delta_p * \Lambda_{p,f}^{(d)} + \Lambda_{p,f}^{(c)}.$$

Moreover, let J_f be the subgroup of integers m such that $mf = \frac{k}{p-1}s_p \mod 1$ for a suitable integer $k \in \mathbb{N}$, and let $m_0 \ge 0$ be defined by $J_f = m_0\mathbb{Z}$. Then

(i) if G_0 is the subgroup of the $(p-1)$-st roots of unity generated by the values of $m_0 f$, then the discrete part $\Lambda_{p,f}^{(d)}$ of $\Lambda_{p,f}$ is the spectral type of the translation $z \mapsto z\zeta_0$ where $\zeta_0 = e(m_0 f(1))$ is a generator of G_0 and the (ergodic) dynamical system $(\mathbb{Z}_p \times G_0, T_0, \mu_p \otimes h_{G_0})$ with $T_0(x,z) = (x+1, z\zeta_0)$ is a factor of $T_{p,f}$ under the factor map $(x,y) \mapsto (x, y^{m_0})$;

(ii) if $m_0 \ne 0$, G is finite and the continuous part $\Lambda_{p,f}^{(c)}$ of $\Lambda_{p,f}$ is given by the measure

$$\sum_{\substack{1\le m<\#G \\ m\not\equiv 0 \bmod m_0}} \nu_{mf}\,;$$

(iii) if $m_0 = 0$, $\Lambda_{p,f}$ $(= \Lambda^{(c)}_{p,f})$ *is equivalent to the measure*

$$\sum_{m=1}^{\infty} 2^{-m} \nu_{mf}.$$

In this case $\Xi_{p,f} = \Delta_p + \Lambda_{p,f}$.

Remark 5. In case (iii) of Corollary 3, if G is finite of cardinal r, then r and $p-1$ are relatively prime and the infinite sum can be replaced by the sum

$$\sum_{m=1}^{r-1} \nu_{mf}.$$

Proof of Theorem 2. From above, we may suppose that g is also not trivial. By complete p-additivity of f and Theorem 6, the set $S = \{\ell p^n \ ; \ n \in \mathbb{N}\}$ $(\ell \in \mathbb{N}, \ \ell \neq 0)$ is a ζ-rigid time with $\zeta = \hat{\nu}_f(\ell)$, and we can choose ℓ such that $\hat{\nu}_f(\ell) \neq 0$. Lemma 2 implies that there exists an increasing sequence of integers n_k such that the number of occurrences of a given non-zero block B in the q-ary expansion of ℓp^{n_k} tends to infinity. The estimate given in Lemma 1 then implies

$$\lim_k \hat{\nu}_g(\ell p^{n_k}) = 0. \tag{3.6}$$

Let H be the subspace of all $h \in L^2(\mathbb{Z}_q)$ such that $\rho_h \ll \nu_f$ where ρ_h denotes here the spectral measure h with respect to unitary representation $V^n_{g,e} : u \mapsto e(\psi_g(n,x))u(x+n)$ of $\mathbb{Z}$ on $L^2(\mathbb{Z}_q)$. The space H is invariant under $V_{g,e}$ and the unitary representation $\chi \mapsto M_\chi$ of $\widehat{\mathbb{Z}}_q$ given by $M_\chi(u)(x) = \chi(x)u(x)$, this latter fact following from Corollary 2. Assume that $H \neq \{0\}$ and choose $h \neq 0$, $h \in H$. For any function u in the orthocomplement of H we have $\langle V^n_{g,e}(h) \cdot u | \chi \rangle = 0$ for any $\chi \in \mathbb{Z}_q$ and $n \in \mathbb{Z}$. This implies $V^n_{g,e}(h) \cdot u = 0$ μ_q-a.e. or equivalently, $h(x+n)u(x) = 0$ for μ_q-almost all x and any $n \in \mathbb{Z}$, and consequently $u = 0$ μ_q-a.e. Therefore $H = L^2(\mathbb{Z}_q)$, in particular $\rho_{\mathbf{1}} = \nu_g \ll \nu_f$. But $\hat{\nu}_g(\ell p^{n_k}) = \int_{\mathbb{U}} \frac{d\nu_g}{d\nu_f}(t) e(\ell p^{n_k} t) \nu_f(dt)$, and

$$\lim_k \int_{\mathbb{U}} \frac{d\nu_g}{d\nu_f}(t) e(\ell p^{n_k} t) \nu_f(dt) = \zeta \neq 0$$

by Theorem 3, in contradiction with (3.6). Thus $H = \{0\}$ and we conclude that ν_g and ν_f are mutually singular. The same conclusion holds if we replace f and g by mf and $m'g$ with any integers m and m' such the characters $v \mapsto v^m$ and $v \mapsto v^{m'}$ are not trivial respectively on G_f and G_g. Now, the mutual singularity of $\Lambda_{p,f}$ and $\Lambda_{q,g}$ is an easy consequence of Corollary 3 part (iii). □

4. Gaussian integers

In this section we consider radix expansions for the Gaussian integers $\mathbb{Z}[i]$. It is well known that all bases b for canonical number systems in $\mathbb{Z}[i]$ are of the form $b = -a \pm i$,

$a \in \mathbb{N}$, $a \geq 1$ (cf.[13, 14]). Thus every Gaussian integer z can be written uniquely in the form

$$z = \sum_{\ell=0}^{L} \varepsilon_\ell b^\ell \quad \text{with } \varepsilon_\ell \in \mathcal{A} \quad (= \{0, \dots, a^2\})$$

that leads to the classical notion of b-additive arithmetic functions $f : \mathbb{Z}[i] \to \mathbb{R}$ which, by definition, verify $f(z) = \sum_{\ell=0}^{L} f(\varepsilon_\ell b^\ell)$. In the following we will mainly consider completely b-additive arithmetic functions defined by

$$f\left(\sum_{\ell=0}^{L} \varepsilon_\ell b^\ell\right) = \sum_{\ell=0}^{L} f(\varepsilon_\ell)$$

and still denote by G_f or simply G the closed subgroup of $\mathbb{U}$ generated by the values of $e \circ f$. The corresponding compact group $\mathcal{K}_b$ (replacing the q-adic integers for the classical radix expansions) with Haar measure μ_b as well as the related cocycle ψ_f (see *infra*) where f is the sum-of-digits function have been investigated in [7].

Let $(\mathcal{K}_b, T_b, \mu_b)$ be the group action of $\mathbb{Z}[i]$ by translation, namely $T_b^z(x) = x + z$ for any Gaussian integer z, and let $(\mathcal{K}_b \times G, T_{b,f}, \mu_b \otimes h_G)$ be the skew product above T_b defined by

$$T_{b,f}^z(x, u) = (x + z, ue(\psi_f(z, x))),$$

where ψ_f is the cocycle given by

$$\psi_f(z, x) = \lim_{\substack{\xi \to x \\ \xi \in \mathbb{Z}[i]}} f(z + \xi) - f(\xi),$$

if the limit exists and $\psi_f(z, x) = 0$ otherwise. For any $z \in \mathbb{Z}[i]$, the map $\psi_f(z, \cdot)$ is continuous almost everywhere. This fact was proved in [7] for the sum-of-digits function, but since the proof makes use only of the b-additive property, it is valid in full generality.

Theorem 7. *The dynamical system $(\mathcal{K}_b \times G, T_{b,f}, \mu_b \otimes h_G)$ is uniquely ergodic.*

Proof. The ergodicity will be a consequence of our next study. It can be also derived by mimicking the proof given in [7] for the sum-of-digits function. The uniqueness of the ergodic measure follows from the general result [see [7], Corollary 4]. □

Our first aim is the study of the spectral type of $T_{b,f}$, taking into account the decomposition analogous to (1.4), replacing $\mathbb{Z}_p$ by $\mathcal{K}_b$. The dual group of Γ is identified with the two-dimensional additive torus $\hat{\Gamma} = \mathbb{C}/\mathbb{Z}[i]$ so that $v \in \hat{\Gamma}$ corresponds to a unique character $\chi_v(z) = e(\Re(vz))$ of Γ and all characters of Γ are of this form. Moreover, by duality, all characters of $\hat{\Gamma}$ are of the form $\chi_z : v \mapsto \chi_v(z)$. Now, the dual group of $\mathcal{K}(b)$ is identified with the discrete subgroup

$$\hat{\Gamma}(b) := \{w \in \hat{\Gamma}\,;\ \exists k \in \mathbb{N},\, b^k w = 0\},$$

each character χ of $\mathcal{K}(b)$ being identified to $w(\chi) = \omega/b^k \in \hat{\Gamma}$, $\omega \in \mathbb{Z}[i]$, through the formula $\chi(x) = \chi_w(x) = e(\Re(w \sum_{j=0}^{k-1} x_j b^j))$. We use the same notations as before: ρ_h represent the spectral measure associated to any function h in $L^2(\mathcal{K}_b \times G)$ with respect to $T_{b,f}$. Recall

that ρ_h is a measure on $\hat{\Gamma}$ and in particular, for $h = \chi \otimes \xi$ with $\chi \in \widehat{\mathcal{K}_b}$ and $\xi : u \mapsto u^m$ $(\in \widehat{G})$, one has

$$\hat{\rho}_{\chi\otimes\xi}(z) = \chi(z) \int_{\mathcal{K}_b} e(\psi_{mf}(z,x))\mu_b(dx);$$

hence $\rho_{\chi\otimes\xi} = \delta_{\{w(\chi)\}} * \nu_{mf}$.

An important spectral property related to $T_{b,f}$ is the existence of rigid time. It is a simple observation that any set $S(\ell) := \{\ell b^n\,; n \in N\}$ (for any Gaussian integer $\ell \neq 0$) is a 1-rigid time for the $\mathbb{Z}[i]$-action T_b on $\mathcal{K}_b$ by translation. Now for any character ξ of G, let $V_{f,\xi}$ be the unitary representation of $\mathbb{Z}[i]$ on $L^2(\mathcal{K}_b)$ given by

$$V^z_{f,\xi}(h)(x) = \xi(e(\psi_f(z,x)))h(x+z).$$

The following theorem corresponds to Theorem 6 for p-additive function; the proof is almost identical and is left to the reader:

Theorem 8. *The set $S(\ell)$ is a ζ-rigid time for $V_{f,\xi}$, with $\xi : u \mapsto u^m$ and $\zeta = \hat{\nu}_{mf}(\ell)$.*

The corresponding Corollary 1 holds as well.

As in the case of p-adic integers discussed in Section 2, addition of $m + ni$ can be performed by a family of transducer automata. They are more complicated, due to the fact that there are 12 possible carries (including the carry 0) which form the set Σ of states. These automata are described in Figure 2. Since the situation is more complicated than in the p-adic case, we also give a brief verbal description. Let $d : \mathbb{Z}[i] \to \{0, 1, \ldots, a^2\}$ be the first digit function to base b *i.e.*, $d(z)$ is defined by the relation $z - d(z) \in b\mathbb{Z}[i]$ and notice that $(a^2+1)b^{-1} = -b-2a = b^2+(2a-1)b+(a-1)^2$. To add $m+ni = \varepsilon_0+\varepsilon_1 b+\cdots+\varepsilon_r b^r$ to any $x = \sum_{k=0}^{\infty} x_k b^k$ $(x_k \in \mathcal{A})$ in $\mathcal{K}_b$ such that $y = (m+ni)+x = \sum_{k=}^{\infty} y_k b^k$, we first compute $y_0 = \varepsilon_0 + x_0 - \eta(a^2+1)$ and report the carry $c_1 = \eta(-b-2a)$ where η is equal to 0 or 1 according to $\varepsilon_0 + x_0 \leq a^2$ or $a^2+1 \leq \varepsilon_0 + x_0$ $(< 2a^2+1)$. After computing the digits $y_0, \ldots y_{s-1}$, the next one, y_s, is obtained by computing $c_s + \varepsilon_s + x_s$ where c_s is the running carry to be added (and $\varepsilon_s = 0$ if $s > r$). This gives $y_s = d(c_s) + \varepsilon_s + x_s - \eta(a^2+1)$ and the next carry $c_{s+1} = (c_s - d(c_s)b^{-1} + \eta(-b-2a)$, where η is equal to 0, 1 or 2 in such a way that $0 \leq d(c_s) + \varepsilon_s + x_s - \eta(a^2+1) \leq a^2$. If we define the maps

$$S_\eta : z \mapsto (z - d(z))b^{-1} + \eta(-b-2a)$$

for $0 \leq \eta \leq 2$ then, Σ is the smallest subset of $\mathbb{Z}[i]$ containing 0 and satisfying both

$$(\forall\, c \in \Sigma)\ (d(c) = 0 \ \Rightarrow\ S_0(c) \in \Sigma\,\&\, S_1(c) \in \Sigma)$$

and

$$(\forall\, c \in \Sigma)\ (d(c) \neq 0 \ \Rightarrow\ \forall \eta \in \{0,1,2\}\,,\ S_\eta(c) \in \Sigma).$$

It is easy to see that there exists a positive integer M such that for any $z \in \mathbb{Z}[i]$, there is an integer $k \geq 0$ verifying for $i = 0, 1, 2$, $|S_i^k(z)| \leq M$ and moreover, $|S_i(z)| \leq M$ if $|z| \leq M$. Hence $|s| \leq M$ for any $s \in \Sigma$, proving that Σ is finite. By straightforward computation we obtain

$$\Sigma = \{0, 1, -A, C, -2, F, B, D, E, -B, A, -1\} \tag{4.1}$$

where

$$A = b + 2a, \quad B = b + 2a - 1, \quad C = -b - 2a - 2,$$
$$D = -b - 2a - 1, \quad E = b + 2a - 2, \quad F = -2b - 4a + 1.$$

The automaton corresponding to the addition of the digit ε is depicted Figure 2.

FIGURE 2. The automaton $\mathcal{A}_\varepsilon$ for base $-a + i$

In the labelled graph of Figure 2, the notation $[q]$ means that q has to be added to the digit that has just been read, if the result is still in $\mathcal{A}$ $(= \{0, \ldots, a^2\})$, so that if σ is the

current state, the next one is the extremity of the arrow issuing from σ and labeled by $[q]$. Notice that there is always exactly one possibility.

Finally, by ordering $\Sigma = \{\sigma_0, \dots, \sigma_{11}\}$ as it is given in(4.1), the corresponding matrices A_ε related to the computation of the Fourier coefficient $\hat{\nu}_f(\sum_{k=0}^{k} \varepsilon_k b^k)$ can be read off as (4.2)

$$\begin{pmatrix}
t(\varepsilon) & 0 & t(\varepsilon-a^2-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t(\varepsilon+1) & 0 & t(\varepsilon-a^2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t(\varepsilon+(a-1)^2) & t(\varepsilon-(a+1)^2) & 0 & 0 & 0 & t(\varepsilon-2a) \\
0 & 0 & 0 & t(\varepsilon-a^2-2a-3) & 0 & 0 & t(\varepsilon+a^2-2a-1) & 0 & 0 & 0 & 0 & t(\varepsilon-2a-2) \\
t(\varepsilon-2) & 0 & t(\varepsilon-a^2-3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t(\varepsilon+a^2-1) & 0 \\
0 & 0 & 0 & t(\varepsilon-a^2-4a) & t(\varepsilon-4a+1) & 0 & 0 & 0 & t(\varepsilon+a^2-4a+2) & 0 & 0 & 0 \\
0 & t(\varepsilon+2a-1) & 0 & 0 & 0 & t(\varepsilon-2a^2+2a-3) & 0 & 0 & 0 & t(\varepsilon-(a-1)^2-1) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t(\varepsilon+a^2-2a) & t(\varepsilon-a^2-2a-2) & 0 & 0 & 0 & t(\varepsilon-2a-1) \\
0 & t(\varepsilon+2a-2) & 0 & 0 & 0 & t(\varepsilon-2a^2+2a-4) & 0 & 0 & 0 & t(\varepsilon-(a-1)^2-2) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t(\varepsilon+(a-1)^2+1) & t(\varepsilon-a^2-2a) & 0 & 0 & 0 & t(\varepsilon-2a+1) \\
0 & t(\varepsilon+2a) & 0 & 0 & 0 & t(\varepsilon-2a^2+2a-2) & 0 & 0 & 0 & t(\varepsilon-(a-1)^2) & 0 & 0 \\
t(\varepsilon-1) & 0 & t(\varepsilon-a^2-2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t(\varepsilon+a^2) & 0
\end{pmatrix}$$

with

$$t(q) = \frac{1}{a^2+1} \sum_{k \in \mathcal{A} \cap -q+\mathcal{A}} e(f(k+q) - f(k)) \quad \mathcal{A} = \{0, \dots, a^2\},$$

$t(q)$ being null if $\mathcal{A} \cap -q + \mathcal{A}$ is empty (*i.e.*, $|q| > a^2$).

In fact, let f_m be the general notation for the b-additive function $z \mapsto f(b^m z)$, set $F_m(z,x) = e(\psi_{f_m}(z,x))$ for short and $x = x_0 + bx'$ for any $x \in \mathcal{K}_b$. For $z \in \mathbb{Z}[i]$ with partial b-expansion $z = \varepsilon + bz'$, and any $\sigma \in \Sigma$, using the b-multiplicativity of $e(f)$ and the fact that μ_b is a product measure, one has

$$\begin{aligned}
\hat{\nu}_f(z+\sigma) &= \int_{\mathcal{K}_b} \sum_{0 \le k \le a^2} \mathbf{1}_{\{x_0 = k\}} F(z+\sigma, x) \mu_b(dx) \\
&= \sum_{\eta=0}^{2} (A_\varepsilon)_{\sigma, S_\eta(\sigma)} \int_{\mathcal{K}_b} F_1(z' + S_\eta(\sigma), x') \mu_b(dx'),
\end{aligned}$$

the term in the above summation corresponding to $\eta = 2$ being omitted if $\varepsilon = 0$. This proves the following fundamental formula:

$$\begin{pmatrix} \hat{\nu}_f(z+\sigma_0) \\ \vdots \\ \hat{\nu}_f(z+\sigma_{11}) \end{pmatrix} = A_\varepsilon \begin{pmatrix} \hat{\nu}_{f_1}(z'+\sigma_0) \\ \vdots \\ \hat{\nu}_{f_1}(z'+\sigma_{11}) \end{pmatrix}. \tag{4.3}$$

Obviously, for any b-additive function f, $\|A_0\| = \|A_{a^2}\| = 1$ and if we denote by $B_\varepsilon = (b^{(\varepsilon)}_{\alpha,\beta})$ the matrix A_ε but with $f = 0$, each entry of $(a^2+1)B_\varepsilon$ are non negative integers, the sum of all terms on each line being equal to a^2+1, and the entries of the matrix $(a^{(\varepsilon)}_{\alpha,\beta}) = \mathcal{A}_\varepsilon$ satisfy $|a^{(\varepsilon)}_{\alpha,\beta}| \le b^{(\varepsilon)}_{\alpha,\beta}$. As in the above discussion, if there is no digit ε such that $||A_\varepsilon|| < 1$ then the first line of A_1 says that $f(\delta+1) - f(\delta) \equiv C \pmod 1$. Assuming in addition that f is completely additive, this implies that $f(z) \equiv C s_b(z) \pmod 1$. In that case $||A_\varepsilon|| = 1$ for any digit ε but there are many triples $(\varepsilon_1, \varepsilon_2, \varepsilon_2) \in \mathcal{A}$ such that one entries of the product $A_{\varepsilon_1} A_{\varepsilon_2} A_{\varepsilon_3}$ is strictly less than the corresponding entry of $B_{\varepsilon_1} B_{\varepsilon_2} B_{\varepsilon_3}$.

For our purpose, we only need to exhibit one such triple. In fact, choosing $((a-1)^2, 2a, 0)$ and working with the automata $\mathcal{A}_{(a-1)^2}$, $\mathcal{A}_{2a}$ and $\mathcal{A}_0$ we obtain

$$(A_{(a-1)^2}A_{2a}A_0)_{B,B} = (2a+1)2a\zeta^{a^2+1} + (a^2-2)(2a+1)\zeta^{-2a-1}.$$

where $\zeta = e(C)$. Therefore

$$|(A_{(a-1)^2}A_{2a}A_0)_{B,B}| < (B_{(a-1)^2}B_{2a}B_0)_{B,B} = (2a+1)2a + (a^2-2)(2a+1)$$

if $\zeta^{(a+1)^2+1} \neq 1$ (and $|(A_{(a-1)^2}A_{2a}A_0)_{B,B}| = (B_{(a-1)^2}B_{2a}B_0)_{B,B}$ otherwise). Consequently, for any digits ε and η, the inequality

$$||A_\varepsilon A_{(a-1)^2}A_{2a}A_0A_\eta|| < 1$$

holds if and only if ζ is not a $((a+1)^2+1)$-th root of unity. The case where $\zeta^{(a+1)^2+1} = 1$ is very particular as it was observed in [7] where it is shown that

$$\forall\,(z,z') \in \mathbb{Z}[i]^2\,,\ s_b(z+z') \equiv s_b(z) + s_b(z') \pmod{(a+1)^2+1}. \tag{4.4}$$

We have proved the analogue of Lemma 1, namely if f is completely b-additive and does not satisfy $f(z) \equiv \frac{k}{(a+1)^2+1}s_b(z) \pmod 1$ for some integer k then, there exists a constant θ, $0 < \theta < 1$, such that

$$|\hat{\nu}_f(z)| \leq \theta^{\sigma_B(z)} \tag{4.5}$$

for some non-overlapping block counting function $\sigma_B(z)$ where the digit block B can be chosen of length at most five. From (4.5) we infer the following:

Proposition 4. *For any completely b-additive function f, the measure ν_f is discrete if and only if $f \equiv \frac{k}{(a+1)^2+1}s_b \pmod 1$ for some $k \in \mathbb{Z}$. Otherwise, ν_f is singular continuous and for any $w \in \hat{\Gamma}(b)$, $\rho_{\chi_w \otimes \mathbf{1}}$ $(= \nu_f * \delta_{\{w\}})$ is equivalent to ν_f.*

Proof. Assume that $f \equiv \frac{k}{(a+1)^2+1}s_b \pmod 1$, then the map $\chi_f : z \mapsto e(f(z))$ is a character of $\mathbb{Z}[i]$ and ν_f is discrete. In fact, by direct computation, $\chi_f(z) = e(\Re\big(\frac{kz}{1+(a+1)i}\big))$ and $\hat{\nu}_f(z) = \chi_f(z)$ *i.e.*, $\nu_f = \delta_{\{\alpha\}}$, where $\alpha = \frac{k}{1+(a+1)i} + \mathbb{Z}[i]$.

In the case $f \not\equiv \frac{k}{(a+1)^2+1}s_b \pmod 1$, we introduce the set

$$\Delta_L := \left\{ \sum_{\ell=0}^{L-1} \varepsilon_\ell b^\ell\,;\ (\varepsilon_0, \ldots, \varepsilon_{L-1}) \in \mathcal{A}^L \right\}.$$

It follows from the geometric considerations in [6] that Δ_L satisfies the condition (4.8) below. Following the proof of Proposition 2 we get

$$\lim_{L\to\infty} \frac{1}{(a^2+1)^L} \sum_{z\in\Delta_L} |\hat{\nu}_f(z)|^2 = 0. \tag{4.6}$$

This, together with Theorem 9 *infra*, finishes the proof of continuity of the measure ν_f.

In order to prove $\nu_f * \delta_{\{w\}} \sim \nu_f$ for $w = \omega b^{-L}$ with $\omega \in \Delta_L$, $\omega \neq 0$, we introduce the probability measure

$$\kappa(du) = \frac{1}{(a^2+1)^L}\Big| \sum_{z\in\Delta_L} e(\Re(f(z)+wz)\chi_z(-u)\Big|^2 \frac{1}{(a^2+1)^L} \sum_{v\in\Delta_L} \mu * \delta_{\{vb^L\}}(du).$$

which precisely verifies for any $y \in \Delta_L$ and $y' \in \mathbb{Z}[i]$

$$\hat{\kappa}(y + b^L y') = \chi_w(y)\hat{\nu}_f(y + b^L y') \tag{4.7}$$

that is to say, $\kappa = \nu_f * \delta_{\{w\}}$. Equation (4.7) is obtained by iterating (4.3) with the b-additive function $F : z \mapsto \Re(f(z) + wz)$ which verifies $F_L \equiv \Re(f_L) \mod \mathbb{Z}[i]$. □

We could not find the following straight forward generalization of the Wiener-Schoenberg theorem in the literature. Thus we state it and give a short proof.

Theorem 9. *Let G be a compact metrizable abelian group and $(A_n)_n$ an increasing sequence of finite non empty subsets of $\hat{G}$ such that*

$$\lim_{n\to\infty} \frac{\#(\chi A_n \triangle A_n)}{\#A_n} = 0 \tag{4.8}$$

for all $\chi \in \hat{G}$. Let ν be a measure on G. Then the limit

$$\lim_{n\to\infty} \frac{1}{\#A_n} \sum_{\chi\in A_n} |\hat{\nu}(\chi)|^2 \tag{4.9}$$

exists and equals

$$\sum_{g\in G} \nu(\{g\})^2.$$

In particular, the measure has no point masses, if the limit (4.9) *is zero.*

Proof. The proof runs along the same lines as the proof of the classical Wiener-Schoenberg theorem [18, 20]. It depends on the fact that

$$\sum_{\chi\in A_n} |\hat{\nu}(\chi)|^2 = \iint\limits_{G\times G} \sum_{\chi\in A_n} \chi(u-v)\, d\nu(u)\, d\nu(v).$$

Define

$$f_n(x) := \frac{1}{\#A_n} \sum_{\chi\in A_n} \chi(x);$$

if we can show

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.10}$$

Lebesgue's theorem on dominated convergence yields existence of the limit (4.9) and its value.

In order to prove pointwise convergence in (4.10) we fix $x \neq 0$ (for $x = 0$ the convergence is trivial). Assume now that $|f_n(x)| \geq \varepsilon$ for some $\varepsilon > 0$ and infinitely many n. Then there

exists a convergent subsequence $f_{n_k}(x)$, whose limit is $\neq 0$. By our assumption on the sets A_n the sequence $\chi(x)f_{n_k}(x)$ is convergent to the same limit for any character χ. Taking χ such that $\chi(x) \neq 1$ gives a contradiction. □

From Proposition 4 and previous analysis we can transfer conclusions of Corollary 2 and Corollary 3 to b-additive arithmetic functions f, but replacing the index p by b, the function $\frac{k}{p-1}s_p \,(\mathrm{mod}\ 1)$ by $\frac{k}{(a+1)^2+1}s_b \,(\mathrm{mod}\ 1)$ and the group of $(p-1)$-th roots of unity by the $(1+(a+1)i)$-torsion subgroup of $\hat{\Gamma}$ *i.e.*, the group

$$\hat{\Gamma}_a := \{v \in \hat{\Gamma}\,;\ (1+i(a+1))v = 0\}.$$

To be complete, we need to know that $T_{b,f}$ is ergodic if $f \equiv \frac{k}{(a+1)^2+1}s_b \,(\mathrm{mod}\ 1)$, a result coming from the general theory [[7], Corollary 4], or simpler, due to the easy fact that $\hat{\Gamma}(b) \cap \hat{\Gamma}_a = \{0\}$ and (4.4) which says that $T_{b,f} = T_b \times R_a$ where, by definition, $R_a^z : y \mapsto ye(\Re\big(\frac{z}{1+(a+1)i}\big))$ on the group of $((a+1)^2+1)$-th roots of unity.

Let us mention the following simple consequences: first, as we have already noticed, $T_{b,f}$ is ergodic by the fact that the above spectral studies show that the eigenvalue 1 only comes from T_b and so occurs with multiplicity 1. Now, the following corresponds to the case where the eigenfunctions for $T_{b,f}$ are only those issuing from T_b:

Corollary 4. *Assume that f is completely b-additive and takes an irrational value. Then $G = \mathbb{U}$ and the spectral type $\Xi_{b,f}$ of $T_{b,f}$ has the form $\Xi_{b,f} = \Delta_b + \Lambda_{p,f}$ where Δ_b, the discrete part, is the spectral type of T_b and $\Lambda_{p,f}$, the singular continuous part, is given by $\sum_{m=1}^{\infty} 2^{-m}\nu_{mf}$.*

A b-additive arithmetical function f will be said *trivial* if there exists an integer k and a non trivial character $\chi_m : u \mapsto u^m$ of G_f such that

$$\chi_m(e(f)) = e\Big(\frac{k}{(a+1)^2+1}s_b\Big).$$

According to Proposition 4, this definition is equivalent to have ν_{mf} discrete for some m such that $0 < m < \#G$.

Now we want to prove an analogue of Theorem 2. Our proof will work only under the assumption of coprimality between the bases and we have to show that the corresponding Lemma 2 holds in this case. We recall that

$$\mathbb{Z}_m \cong \bigotimes_{p|m} \mathbb{Z}_p \quad (p \text{ prime })$$

as topological rings. Continuing with $b = -a + i$ $(a > 0)$, let p be an odd prime divisor of a^2+1. Since -1 is a square mod p, it is also a square in $\mathbb{Z}_p$. This means that $i \in \mathbb{Z}_p$ but we have to choose this square root of -1. We do this in connection with the factorization $p = \pi\bar{\pi}$ where π and $\bar{\pi}$ are conjugate and non equivalent Gaussian prime numbers. In fact, if $\pi = -\sigma + i\tau$, observe that the rational integers τ and σ are unities in $\mathbb{Z}_p$ and fix one square root i' of -1 in $\mathbb{Z}_p$ by assuming the relation

$$-\sigma + i'\tau = 0 \mod p\mathbb{Z}_p.$$

The specialization $i \to i'$ over $\mathbb{Z}$, allows us to identify $\mathbb{Z}[i]$ with $\mathbb{Z}[i']$ ($\subset \mathbb{Z}_p$) by the ring isomorphism $J : x+iy \mapsto x+i'y$. In addition, such a choice of i' implies that the p-valuation of $J(\pi)$ is 1 and for the conjugate prime, $J(\bar{\pi}) = -\sigma - i'\tau$ is a unity in $\mathbb{Z}_p$. Finally, let

$$i' = r_0 + r_1 p + r_2 p^2 + \dots$$

be the p-adic expansion of i', and set $i'[n] = \sum_{k=0}^{n-1} r_k p^k$ for any integers $n \geq 1$. Since $i'[n]$ is a square root of $-1 \bmod p^n$, the map $x + iy \mapsto x + i'[n]y + p^n\mathbb{Z}$ from $\mathbb{Z}[i]$ to $\mathbb{Z}/p^n\mathbb{Z}$ is a ring morphism with kernel $\pi^n\mathbb{Z}[i]$ which induces the ring isomorphism

$$\varphi_n : u + iv + \pi^n\mathbb{Z}[i] \mapsto u + i'[n]v + p^n\mathbb{Z}$$

from $\mathbb{Z}[i]/(\pi)^n$ onto $\mathbb{Z}/p^n\mathbb{Z}$. Thus, the family of morphisms φ_n defines an isomorphism φ between the projective limits $\mathbb{Z}[i]_\pi := \lim_{\leftarrow n} \mathbb{Z}[i]/\pi^n\mathbb{Z}[i]$ and $\mathbb{Z}_p = \lim_{\leftarrow n} \mathbb{Z}/p^n\mathbb{Z}$.

The case $b = -1 + i$ is particular because $-1 + i$ is the unique prime divisor (up to an equivalence) of 2 and -1 has no square root in $\mathbb{Z}_2$. As a consequence, $\mathcal{K}_{-1+i} = \mathbb{Z}_2[i]$.

Returning to $\mathcal{K}_b$ in full generality and let $\pi_1 \dots, \pi_r$ be the distinct prime divisors of b which are not above 2 (in case a is odd) and let p_j be the rational prime such that π_j is above p_j. Observe that the primes p_j are distinct, since $-a+i$ is not divisible by a rational prime. Select the square root i'_j of -1 in $\mathbb{Z}_{p_j}$ in accordance with π_j as above leading to the isomorphism $\varphi^{(j)}$ between $\mathbb{Z}[i]_{\pi_j}$ and $\mathbb{Z}_{p_j}$, which can be used to produce the isomorphism

$$\mathcal{K}_b \simeq \begin{cases} \bigotimes_{j=1}^r \mathbb{Z}_{p_j} \ (\simeq \mathbb{Z}_{a^2+1}) & \text{if } a \text{ is even;} \\ \mathbb{Z}_2[i] \otimes \left(\bigotimes_{j=1}^r \mathbb{Z}_{p_j} \right) & \text{otherwise.} \end{cases} \tag{4.11}$$

Theorem 10. *Let $b = -a+i$ and $c = -a'+i$ be coprime and f, g be completely b-additive, respectively c-additive. Assume that $e(f) \neq e(\frac{k}{(a+1)^2+1} s_b)$ for any integer k, then the spectral measures ν_f and ν_g are mutually singular.*

Proof. We assume that $e(g) \neq e(\frac{k}{(a+1)^2+1} s_b)$ for any $k \in \mathbb{Z}$ otherwise, ν_g is discrete while ν_f is continuous, and the theorem holds. Without loss of generality we may also assume that a' is even. Let $\pi_1 \dots, \pi_r$ be the distinct prime divisors of c, respectively above the (odd) primes $p_1, \dots, p_r$. Recall that the primes p_j are distinct and let $\varphi^{(j)} : \mathbb{Z}[i]_{\pi_j} \to \mathbb{Z}_{p_j}$ be the above isomorphism such that $\varphi^{(j)}(\pi_j)$ is equivalent to p_j. Define m to be the minimal positive integer such that $b^m \equiv 1 \mod c$ and define d_j as the maximal positive integer such that $\varphi^{(j)}(b^m) \in 1 + p_j^{d_j}\mathbb{Z}_{p_j} =: U_{d_j}(p_j)$. From [15, Theorem 5.7, Corollary] we infer that $\mathbb{Z}_{p_j} \simeq U_1(p_j)$ from which we conclude that $\mathbb{Z}_{p_j} \simeq U_{d_j}(p_j)$. Thus every element of $U_{d_j}(p_j)$ can be written as $z_j^{p^{d_j}\alpha}$, where z_j is a principal unity and $\alpha \in \mathbb{Z}_{p_j}$. By definition, we have

$$\varphi^{(j)}(b)^m = z_j^{p_j^{d_j}\alpha_j}$$

where α_j is a unity in $\mathbb{Z}_{p_j}$; it follows that $\overline{\langle \varphi^{(1)}(b)^m \rangle} = U_{d_j}$. Since the primes p_j are all distinct, $(\varphi^{(j)}(b)^m, \dots, \varphi^{(r)}(b)^m)$ generates $U_{d_1}(p_1) \times \cdots \times U_{d_r}(p_r)$. Thus from (4.11) we get $1 + c^d\mathcal{K}_c \subset \overline{\langle b^m \rangle}$ for any d greater than $\max\{d_1, \dots, d_r\}$. Therefore, for any block B of digits in $\{0, \dots, a'^2\}$, any Gaussian integer ℓ, and any $k \in \mathbb{N}$, choosing d large enough

with respect to ℓ, there exists a positive integer n_k such that $\sigma_B(\ell b^{n_k}) \geq k$. For a suitable choice of the digit block B, the bound (4.5) implies that

$$\lim_k \hat{\nu}_g(\ell b^{n_k}) = 0.$$

Following the proof of Theorem 2 with Proposition 4 in hand, we obtain first that ν_g is pure with respect to ν_f, *i.e.*, ν_g is absolutely continuous or singular with respect to ν_f. Since f is completely b-additive, there exists $\ell \in \mathbb{Z}[i]$ such that $\nu_f(\ell) \neq 0$. Therefore $\{\ell b^n\,;\, n \in \mathbb{N}\}$ is a ζ-rigid time for V_{f,ξ_1} with $\zeta \neq 0$, and therefore, by Theorem 3, ν_g has to be singular with respect to ν_f. □

Corollary 5. *With the assumptions of Theorem 10, if f and g are non trivial, then the spectral type of the dynamical systems T_{b_1,f_1} and T_{b_2,f_2} are mutually singular; in particular the direct product $T_{b_1,f_1} \times T_{b_2,f_2}$ is uniquely ergodic.*

We have deliberately paid attention to real valued p-additive or b-additive functions. This is not a real restriction since we work in fact with $e(f)$ and then use the corresponding p- or b-multiplicative notion. Indeed, our choice is motivated by the seminal paper of A. O. Gel'fond [5].

We end this study with an application to uniform distribution which is a consequence of Corollary 5 with more than two bases, taking into account the Tempel'man ergodic theorem:

Theorem 11. *Let $f^{(i)}$, $i = 1, \ldots, m$ be non trivial, completely b_i-additive functions such that the bases b_i are mutually coprime and set $G_i = G_{f^{(i)}}$. Then the sequence*

$$z \mapsto (e(f^{(1)}(z)), \ldots e(f^{(m)}(z)))$$

is uniformly distributed in $G_1 \times \cdots \times G_m$ in the sense that for any increasing sequence $(A_n)_n$ of finite non empty subsets of $\mathbb{Z}[i]$ satisfying (4.8) and for all continuous functions $\gamma : G_1 \times \cdots \times G_m \to \mathbb{R}$, one has

$$\lim_n \frac{1}{\#A_n} \sum_{z \in A_n} \gamma(e(f^{(1)}(z)), \ldots, e(f^{(m)}(z))) = \int_{G_1 \times \cdots \times G_m} \gamma \, dH \tag{4.12}$$

where $H = h_{G_1} \otimes \cdots \otimes h_{G_m}$.

Proof. From Theorem 10, the direct product $P := T_{b_1,f^{(1)}} \times \cdots \times T_{b_m,f^{(m)}}$ is ergodic and can be viewed as the skew product above the (unique) ergodic translation $T_{b_1} \times \cdots \times T_{b_m}$, built with the cocycle $(e(\psi_{f^{(1)}}), \cdots, e(\psi_{f^{(m)}}))$. Therefore P is uniquely ergodic and since P is $(\mu_{b_1} \otimes h_{G_1}) \otimes \cdots \otimes (\mu_{b_m} \otimes h_{G_m})$-continuous, all points in $(\mathcal{K}_{b_1} \times G_1) \times \cdots \times (\mathcal{K}_{b_m} \times G_m)$ are generic for P. Taking $\Omega := ((0, 1_{G_1}), \ldots, (0, 1_{G_m}))$ one has

$$P^z(\Omega) = ((z, e(f^{(1)}(z)), \ldots, (z, e(f^{(m)}(z)).$$

□

Clearly an analogous result holds for p_i-additive arithmetic functions in bases $p_i \geq 2$ which are assumed to be mutually coprime. In that case A_n is usually taken to be $\{0, 1, \ldots, n\}$.

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