DISTRIBUTION PROPERTIES OF G-ADDITIVE FUNCTIONS

GUY BARAT AND PETER J. GRABNER

ABSTRACT. We extend previous results of Delange [De] concerning the existence of distribution functions of certain q-adic digital functions to the case of digital expansions with respect to linear recurrences with decreasing coefficients. Furthermore we investigate a special case of digital functions and give a functional equation for the related distribution function. We prove uniqueness and continuity of the solution of this equation.

1. INTRODUCTION

Let us recall that if $\left(G_n\right)_n$ is an increasing sequence of positive integers with $G_0 = 1$, we can expand every positive integer with respect to this sequence, i.e.

$$\forall n \in \mathbb{N},\; n = \sum_{k=0}^{\infty} \varepsilon_k G_k\ , \tag{1.1}$$

this expansion being finite and unique, provided that for every K, $\sum_{k=0}^{K-1} \varepsilon_k G_k < G_K$. The digits ε_k can be computed by the greedy algorithm.

In this paper, we will only study the case that the base sequence is a recurrence sequence with decreasing integer coefficients $a_0 \geq a_1 \geq \cdots \geq a_{d-1} \geq 1$, namely :

$$\begin{aligned} &G_{n+d} = a_0 G_{n+d-1} + \cdots + a_{d-1} G_n \quad \text{for } n \geq 0 \quad \text{with} \\ &G_0 = 1 \text{ and } G_k = a_0 G_{k-1} + \cdots + a_{k-1} G_0 + 1 \text{ for } k < d. \end{aligned} \tag{1.2}$$

The initial values are chosen as the "canonical initial values" from [GT]. In this case, a finite sum $\sum_{k=0}^{\infty} \varepsilon_k G_k$ is the expansion of some integer iff

$$(\varepsilon_k, \ldots, \varepsilon_0, 0^\infty) < (a_0, \ldots, a_{d-1} - 1)^\infty \tag{1.3}$$

($<$ being understood as the lexicographical order) for every k. The $(\varepsilon_k, \cdots, \varepsilon_0)$ that verify this condition are called "admissible strings". The dominating root of the characteristic equation $X^d - a_0 X^{d-1} - \cdots - a_{d-1} = 0$, say α, is a Pisot number; these equations have been studied by Brauer [Br]. In particular, $\lim_{n\to\infty} \frac{G_n}{\alpha^n}$ exists and is non-zero. For detailed discussion of these numbers systems, we refer to [Fr,GT,GLT].

The first author is partially supported by URA-CNRS Nr. 225.
The second author is supported by the Schrödinger scholarship J00936-PHY

We want to study the notion of G-additive functions (which generalize A. O. Gelfond's [Ge] q-additive functions) and we will give a sufficient condition for the existence of a related distribution function. More precisely, we consider arithmetic functions which satisfy

$$f\left(\sum_{k=0}^{K}\varepsilon_k G_k\right)=\sum_{k=0}^{K}f(\varepsilon_k G_k) \tag{1.4}$$

for which we investigate existence and properties of

$$F(x):=\lim_{N\to\infty}\frac{1}{N}\sum_{n<N}\chi_x\left(f(n)\right)=:\lim_{N\to\infty}F_N(x) \tag{1.5}$$

where χ_x denotes the characteristic function of the interval $]-\infty,x[$. This is a continuation of investigations initiated by Delange [De], who gave a necessary and sufficient condition for the existence of a distribution function in the q-adic case. These investigations complement considerations concerning uniform distribution of G-additive (and also q-additive) functions modulo 1 (cf. [MF], [GT]).

We will give a sufficient condition for the existence of such a function in the Brauer case. For this purpose we will use a result of Kooman [Ko1,Ko2], who derived an extension of the theory of Poincaré and Perron concerning linear recurrences with non-constant coefficients. We will give some indications, why it seems to be difficult to give also necessary conditions.

In a last part we study in more details a special case of G-additive functions, which satisfy

$$f\left(0\varepsilon_0\ldots\varepsilon_K0^\infty\right)=\beta f\left(\varepsilon_0\ldots\varepsilon_K0^\infty\right) \tag{1.6}$$

where $\sum_{k=0}^{K}\varepsilon_kG_k$ is the G-expansion of any integer and $|\beta|$ being smaller than 1. We give a functional equation satisfied by the distribution function. The ideas which led to the functional equation also give a proof of the continuity of F for every non-trivial function f of this type.

We shall use the convention that $\sum_{k=0}^{-1}=0$. Double brackets will indicate intervals of $\mathbb{N}$.

2. Existence of the distribution function

According to Delange, we say that f admits a distribution function F if the limit (1.5) exists for every point of continuity of F. By basic Fourier-Stieltjes transform theory (see for example [Ch]), we know that a necessary and sufficient condition for that to hold is the pointwise convergence of the characteristic functions

$$\Phi_N(t)=\int_{-\infty}^{\infty}e^{itx}dF_N(x)=\frac{1}{N}\sum_{n<N}\int_{-\infty}^{\infty}e^{itx}\delta_{f(n)}(x)=\frac{1}{N}\sum_{n<N}e^{itf(n)}$$

to a function Φ continuous at 0. (In fact, an almost everywhere convergence is enough, provided that $\Phi(0)=1$, but we will not use this refinement.) Moreover, if that situation holds, then Φ is the characteristic function of F.

We first state three lemmata.

Lemma 1. *Let m be a positive integer, and $(x_k)_{0\le k<m}$ a finite sequence of real numbers. Then, the following inequality holds:*

$$\left|\sum_{k=0}^{m-1} e^{itx_k} - m\right| \le |t|\left|\sum_{k=0}^{m-1} x_k\right| + \frac{t^2}{2}\sum_{k=0}^{m-1} x_k^2. \tag{2.1}$$

Proof.

$$\begin{aligned}\left|\sum_{k=0}^{m-1} e^{itx_k} - m - it\sum_{k=0}^{m-1} x_k\right| &= \left|\sum_{k=0}^{m-1}\int_0^{tx_k}(e^{iu}-1)du\right| \\ &\le \sum_{k=0}^{m-1}\frac{t^2x_k^2}{2}\end{aligned}$$

because $|e^{iu}-1| \le |u|$ for every u. The lemma follows immediately.

The second lemma is a special case of a theorem due to Kooman [Ko1,Ko2].

Lemma 2. *Suppose we have a linear recurrence with non-constant coefficients*

$$Z_{n+d} = a_0^{(n)} Z_{n+d-1} + \cdots + a_{d-1}^{(n)} Z_n.$$

Suppose further that there exist a_k for $k = 0, \dots, d-1$, $a_{d-1} \ne 0$ such that

$$\sum_{n=0}^{\infty} |a_k^{(n)} - a_k| \tag{2.2}$$

converges for every $k = 0, \dots, d-1$ and that $P := X^d - a_0X^{d-1} - \cdots - a_{d-1}$ has distinct roots α_i. Then initial values $Z_0^{(i)}, \dots, Z_{d-1}^{(i)}$ (for $i = 1, \dots, d$) can be chosen such that the corresponding solutions $Z_n^{(1)}, \dots, Z_n^{(d)}$ satisfy

$$\lim_{n\to\infty}\frac{Z_n^{(i)}}{\alpha_i^n} = 1$$

for $i = 1, \dots, d$. In particular, if there is one dominating root α , then the limit

$$\lim_{n\to\infty}\frac{Z_n}{\alpha^n}$$

exists for every solution of the recurrence.

The third one is a result which is part of the folklore in the study of digital functions. Special cases are proved in [CRT] and [GLT].

Lemma 3. *Let g be a G-multiplicative function of modulus less or equal to 1, where G is a basis given by the recurrence* (1.2). *Assume further that*

$$\lim_{k\to\infty}\frac{1}{G_k}\sum_{n<G_k} g(n)$$

exists, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n<N} g(n)$$

exists also.

Theorem 4. *Let f be a G-additive function, such that the series*

$$\sum_{n=0}^{\infty}\left|\sum_{\ell=0}^{a_s-1} f(a_0G_{n+d-1}+\cdots+a_{s-1}G_{n+d-s}+\ell G_{n+d-s-1})\right| \tag{2.3}$$

converges for every $s=0,\dots,d-1$ and such that the series

$$\sum_{n=0}^{\infty}\sum_{\ell=0}^{a_0} f(\ell G_n)^2 \tag{2.4}$$

converges, then the distribution function exists and its Fourier-Stieltjes transform is equal to the limit for $N\to\infty$ of

$$\Phi_N(t)=\frac{1}{N}\sum_{n<N} e^{itf(n)}.$$

Proof.
Let Ξ_N be defined as :

$$\Xi_N(t) := G_N\Phi_{G_N}(t).$$

For an arbitrary integer n, let us consider the G-expansion of any integer m smaller than G_{n+d} : $m=\sum_{k=0}^{n+d-1}\varepsilon_kG_k$. Because of the admissibility of the string $(\varepsilon_{n+d-1},\cdots,\varepsilon_n)$, the G-expansion of m is of one of the following types :

$$\begin{cases} a_0G_{n+d-1}+a_1G_{n+d-2}+\cdots+a_{d-2}G_{n+1}+kG_n+r,\\ \qquad \text{with } k\le a_{d-1}-1 \text{ and } r<G_n\\ a_0G_{n+d-1}+a_1G_{n+d-2}+\cdots+a_{d-3}G_{n+2}+kG_{n+1}+r,\\ \qquad \text{with } k\le a_{d-2}-1 \text{ and } r<G_{n+1}\\ \cdots\cdots\cdots\cdots\\ a_0G_{n+d-1}+kG_{n+d-2}+r\\ \qquad \text{with } k\le a_1-1 \text{ and } r<G_{n+d-2}\\ kG_{n+d-1}+r\\ \qquad \text{with } k\le a_0-1 \text{ and } r<G_{n+d-1} \end{cases} \tag{2.5}$$

Conversely, all those sums represent a unique integer smaller than G_{n+d}. Thus, we have the following recurrence formula for Ξ :

$$\begin{aligned} \Xi_{n+d}(t) = &\left(\sum_{k=0}^{a_0-1} e^{itf(kG_{n+d-1})}\right)\Xi_{n+d-1}(t)+\\ &e^{itf(a_0G_{n+d-1})}\left(\sum_{k=0}^{a_1-1} e^{itf(kG_{n+d-2})}\right)\Xi_{n+d-2}(t)+\cdots+\\ &e^{itf(\sum_{j=0}^{d-2}a_jG_{n+d-1-j})}\left(\sum_{k=0}^{a_{d-1}-1} e^{itf(kG_n)}\right)\Xi_n(t) \end{aligned} \tag{2.6}$$

with

$$\Xi_s(t) = \sum_{n<G_s} e^{itf(n)}$$

for $s = 0, \ldots, d-1$.

First step : We prove that $\dfrac{\Xi_n(t)}{\alpha^n}$ has a limit for every t, where we recall that α is the dominating root. For that, we show that the relation (2.4) verifies the assumptions of Lemma 2.

Fixing t, we have $\Xi_{n+d}(t) = a_0^{(n)}(t)\Xi_{n+d-1}(t) + \cdots + a_{d-1}^{(n)}(t)\Xi_n(t)$, where, for any s, $0 \le s < d-1$,

$$a_s^{(n)}(t) = e^{itf(\sum_{j=0}^{s-1} a_j G_{n+d-1-j})} \left(\sum_{k=0}^{a_s-1} e^{itf(kG_{n+d-1-s})} \right).$$

Using Lemma 1, we get:

$$\begin{aligned} |a_s^{(n)}(t) - a_s| \le & |t| \left| \sum_{k=0}^{a_s-1} \sum_{j=0}^{s-1} (f(a_j G_{n+d-1-j}) + f(kG_{n+d-1-s})) \right| \\ & + \frac{t^2}{2} \sum_{k=0}^{a_s-1} \left(\sum_{j=0}^{s-1} (f(a_j G_{n+d-1-j}) + f(kG_{n+d-1-s})) \right)^2 \end{aligned}$$

The convergence of the series

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{a_s-1} \sum_{j=0}^{s-1} (f(a_j G_{n+d-1-j}) + f(kG_{n+d-1-s})) \right|$$

is exactly condition (2.3). Furthermore we have

$$\begin{aligned} & \left(\sum_{j=0}^{s-1} (f(a_j G_{n+d-1-j}) + f(kG_{n+d-1-s})) \right)^2 \le \\ & 2s \sum_{j=0}^{s-1} f(a_j G_{n+d-1-j})^2 + 2s^2 f(kG_{n+d-1-s})^2 \end{aligned}$$

by Schwarz inequality. Hence,

$$\begin{aligned} & \sum_{k=0}^{a_s-1} \left(\sum_{j=0}^{s-1} (f(a_j G_{n+d-1-j}) + f(kG_{n+d-1-s})) \right)^2 \le \\ & 2s a_s \sum_{j=0}^{s-1} f(a_j G_{n+d-1-j})^2 + 2s^2 \sum_{k=0}^{a_s-1} f(kG_{n+d-1-s})^2 \\ & \le 2(d-1)a_0 \sum_{j=0}^{d-2} f(a_j G_{n+d-1-j})^2 + 2(d-1)^2 \sum_{s=0}^{d-1} \sum_{k=0}^{a_0-1} f(kG_{n+d-1-s})^2. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{a_s-1}\left(\sum_{j=0}^{s-1}(f(a_jG_{n+d-1-j})+f(kG_{n+d-1-s}))\right)^2 \\ \leq 2(d-1)^2\,(a_0+d)\sum_{n=0}^{\infty}\sum_{\ell=0}^{a_0}f(lG_n)^2.$$

We are now allowed to apply Lemma 2, which tells us that $\dfrac{\Xi_n(t)}{\alpha^n}$ has a limit for every t.

Second step :

We use the result of the first step to prove that the limit of $\Theta_n(t) := \dfrac{\Xi_n(t)}{\alpha^n}$ is a continuous function of t. $\Theta_n(t)$ satisfies

$$\Theta_{n+d}(t) = \frac{a_0^{(n)}(t)}{\alpha}\Theta_{n+d-1}(t) + \frac{a_1^{(n)}(t)}{\alpha^2}\Theta_{n+d-2}(t)\cdots + \frac{a_{d-1}^{(n)}(t)}{\alpha^d}\Theta_n(t),$$

and the conditions of Lemma 2 remain satisfied for this recurrence.

Let K be a compact subset of $\mathbb{R}$. For every s and every n, the function $t \mapsto |a_s^{(n)}(t) - a_s|$ is continuous on K, and has a maximum on it that we note $m_s^{(n)}$. Let $(M_n)_n$ be the sequence defined by

$$M_{n+d} = \left(a_0 + m_0^{(n)}\right)M_{n+d-1} + \cdots + \left(a_{d-1} + m_{d-1}^{(n)}\right)M_n \text{ with} \\ M_k = \sup_{t\in K}|\Xi_k(t)| \text{ for } 0 \leq k \leq d-1.$$

By a trivial recurrence on n, we have $|\Xi_n(t)| \leq M_n$ for every $n \in \mathbb{N}$ and every $t \in K$. Moreover,

$$\begin{aligned}|\Xi_{n+d}(t) - \Xi_{n+d-1}(t)| &\leq \left|a_0^{(n)}(t) - 1\right||\Xi_{n+d-1}(t)| + \left|a_1^{(n)}(t)\right||\Xi_{n+d-2}(t)| + \cdots \\ &\qquad + \left|a_{d-1}^{(n)}(t)\right||\Xi_n(t)| \\ &\leq \left(a_0 + m_0^{(n)} - 1\right)M_{n+d-1} + \left(a_1 + m_1^{(n)}\right)M_{n+d-1} + \cdots \\ &\qquad + \left(a_{d-1} + m_{d-1}^{(n)}\right)M_n \\ &\leq |M_{n+d} - M_{n+d-1}|.\end{aligned}$$

We can apply Lemma 2 to the sequence $(M_n)_n$, every series $\sum_{n=0}^{\infty} m_s^{(n)}$ being equal to a convergent series $\sum_{n=0}^{\infty}|a_s^{(n)}(t_s) - a_s|$. Thus, $(M_n)_n$ is convergent, and the series $\displaystyle\sum_{n=0}^{\infty}(\Xi_{n+1}(t) - \Xi_n(t))$ is normally convergent. Hence, its limit is continuous.

Third step : Up to now, we have proved that $\dfrac{G_n\Phi_{G_n}(t)}{\alpha^n}$ admits a continuous limit. But $\displaystyle\lim_{n\to\infty}\frac{G_n}{\alpha^n}$ exists and is non-zero, so $\left(\Phi_{G_n}(t)\right)_n$ admits a continuous limit.

Now, f being G-additive, e^{itf} is G-multiplicative, and we can apply Lemma 3 to finish the proof of Theorem 4. $\square$

Remark : It seems to be difficult to obtain a necessary condition for the distribution function to exist. The limitation comes from the perturbation part of the proof, namely Kooman's Lemma 2. His result does not hold if absolute convergence of the coefficients is not assumed, but the $\frac{Z_n}{G_n}$ could converge, even if the series $\sum_n |a_k^{(n)} - a_k|$ diverge, as the following counterexample shows :

$$\begin{aligned} Z_{n+2} &= \left(a_1 - \frac{1}{n}\right) Z_{n+1} + \left(a_2 + \frac{G_{n+1}}{nG_n}\right) Z_n \\ &= a_1 Z_{n+1} + a_2 Z_n \end{aligned}$$

With $Z_0 = G_0$ and $Z_1 = G_1$.

In the following we will use ideas from [GLT] and [Ma] to give an alternative approach to the existence of a distribution function in a more specific case: According to [GLT], the G-compactification of $\mathbb{N}$ is defined as

$$\mathcal{K}_G := \{(\varepsilon_n)_{n\in\mathbb{N}}\,;\, \forall k \in \mathbb{N},\, (\varepsilon_k, \dots, \varepsilon_0)\,\text{is admissible}\}\,,$$

where $\mathcal{K}_G$ is endowed with the topology induced by that of the infinite product of discrete spaces $\prod_{n=0}^{+\infty} [\![0; a_0]\!]$. The addition of 1 can be extended on $\mathcal{K}_G$, where it is a continuous map. With this operator, it has been shown in the same paper that the so-called adding machine is uniquely ergodic. We will note $\mathbb{P}$ the corresponding probability measure, and consider $\mathcal{K}_G$ as a Borel probability space. We identify $\mathbb{N}$ and its image in $\mathcal{K}_G$. If C is the cylinder $[\varepsilon_0, \dots, \varepsilon_{K-1}]$, then C has probability

$$\frac{F_K\alpha^{d-1} + (F_{K+1} - a_0F_K)\alpha^{d-2} + \cdots + (F_{K+d-1} - a_0F_{K+d-2} - \cdots - a_{d-2}F_K)}{\alpha^K(\alpha^{d-1} + \alpha^{d-2} + \cdots + 1)}, \tag{2.7}$$

where $F_K := \#\big\{n < G_K\,;\, n \in C\big\}$. The difference between this formula and that of [GLT] is due to a modification of the notations and to a printing error in the previous paper.

Proposition 5. *Let f be a G-additive function. Then f can be extended to a continuous function on $\mathcal{K}_G$ if and only if*

$$\sum_{n=0}^{\infty}\sum_{\ell=0}^{a_0} |f(\ell G_n)| < \infty. \tag{2.8}$$

Then the distribution function of the extension is exactly F (except, maybe, at the points of discontinuity of F).

Proof. We first prove the equivalence of the existence of a continuation to $\mathcal{K}_G$ with (2.8): Suppose first that (2.8) holds. Then the series $\sum_{n=0}^{\infty} f(\varepsilon_n G_n)$ is absolutely convergent for any infinite string $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathcal{K}_G$. Thus, f can be extended on the compactum to $\tilde{f}$, whose continuity is a trivial consequence of (2.8) and of the product topology structure.

Conversely, suppose that

$$\sum_{n=0}^{\infty}\sum_{\ell=0}^{a_0}|f(\ell G_n)|=\infty.$$

Then there is an $\ell_0\in[\![0,a_0]\!]$ for which

$$\sum_{n=0}^{\infty}|f(\ell_0 G_n)|=\infty.$$

Then there is a $k\in[\![0,d-1]\!]$ such that the series

$$\sum_{n=0}^{\infty}|f(\ell_0 G_{k+dn})| \tag{2.9}$$

diverges. Define now

$$b_n=\begin{cases}\ell_0 & \text{if } f(\ell_0 G_{k+nd})>0\\ 0 & \text{else}\end{cases}\qquad c_n=\begin{cases}\ell_0 & \text{if } f(\ell_0 G_{k+nd})<0\\ 0 & \text{else.}\end{cases}$$

The strings $(0^{(k)}b_0 0^{(d-1)}b_1 0^{(d-1)}b_2\ldots)$ and $(0^{(k)}c_0 0^{(d-1)}c_1 0^{(d-1)}c_2\ldots)$ are admissible, since $(0^{(k)}(\ell_0 0^{(d-1)})^{(\infty)})$ is admissible. By the divergence of (2.9) at least one of those strings corresponds to an infinite value of the extension.

Since the adding machine is uniquely ergodic, every point, in particular 0, is generic. Thus, for any Riemann-integrable complex-valued function g on $\mathcal{K}_G$, we have

$$\frac{1}{N}\sum_{n=0}^{N-1}g(n)\underset{N\to\infty}{\longrightarrow}\int_{\mathcal{K}_G}g\,d\mathbb{P}. \tag{2.10}$$

In particular, if we take g to be $\chi_t\circ\tilde{f}$, and if we note $\tilde{F}$ the distribution function of the integrable function $\tilde{f}$, (2.10) yields $F(t)=\tilde{F}(t)$. Of course, the previous argument is valid if $\chi_t\circ\tilde{f}$ is Riemann-integrable, *i.e.* iff $\tilde{F}$ is continuous at the point t. $\square$

Remarks : Proposition 5 gives a new and extremely short proof for the existence of the distribution function, provided that (2.8) holds. Indeed, (2.8) is stronger than (2.3), and this probabilistic approach does not help in the most general case. Consider, e.g., $G_{n+2}=3G_{n+1}+3G_n$ and the G-additive function defined by $f(G_n)=\frac{1}{n+1}$, $f(2G_n)=-\frac{1}{n+1}$ and $f(3G_n)=0$. Then the series (2.3) and (2.4) converge, whereas (2.8) diverges. This is also a strong difference between Mauclaire's study of arithmetic additive functions [Ma] and the G-additive functions we deal with.

3. Application to a particular case

Recall that the shift adjoint operator associated to the basis $(G_n)_n$ is defined as follows:

$$S:\mathbb{N}\longrightarrow\mathbb{N}$$
$$\sum_k \varepsilon_k G_k\longmapsto\sum_k\varepsilon_k G_{k+1}.$$

In this part, we study a special kind of G-additive functions, which satisfy

$$\forall n \in \mathbb{N} \,:\, f(S(n)) = \beta f(n) \tag{3.1}$$

where $|\beta| < 1$ (this is clearly a rewriting of (1.6)). A typical example of such a function is $f(\sum \varepsilon_k G_k) := \sum \varepsilon_k \beta^k$. Anyway, f is given by its values on $[\![1; a_0]\!]$ as it yields from $f(\sum \varepsilon_k G_k) = \sum \beta^k f(\varepsilon_k)$.

Proposition 6. *If $|\alpha\beta| < 1$, then F has zero derivative almost everywhere, in the sense of Lebesgue measure μ. The Hausdorff dimension of the set where the derivative does not vanish is at most $\frac{\log \alpha}{\log \frac{1}{|\beta|}}$.*

Proof. Clearly,

$$\overline{f(\mathbb{N})} \subset \bigcap_{K=0}^{\infty} \bigcup_{\substack{(\varepsilon_K, \dots, \varepsilon_0) \\ \text{admissible}}} \left[f\left(\sum_{k=0}^{K} \varepsilon_k G_k\right) - \left|\beta^K f(\varepsilon_K)\right| \,;\, f\left(\sum_{k=0}^{K} \varepsilon_k G_k\right) + \left|\beta^K f(\varepsilon_K)\right| \right].$$

Moreover, the set of admissible $(\varepsilon_K, \dots, \varepsilon_0)$ has cardinal G_{K+1}, and if we define λ to be $\max\limits_{0 \le \varepsilon \le a_0} |f(\varepsilon)|$, then

$$\mu\left(\bigcup_{\substack{(\varepsilon_K, \dots, \varepsilon_0) \\ \text{admissible}}} \left[f\left(\sum_{k=0}^{K} \varepsilon_k G_k\right) - \left|\beta^K f(\varepsilon_K)\right| \,;\, f\left(\sum_{k=0}^{K} \varepsilon_k G_k\right) + \left|\beta^K f(\varepsilon_K)\right| \right] \right)$$

is smaller than or equal to $2\lambda|\beta|^K G_{K+1}$ and hence belongs to $\mathcal{O}\left(|\alpha\beta|^K\right)$. Suppose now $|\alpha\beta| < 1$. $\overline{f(\mathbb{N})}$ is an enumerable intersection of sets whose measure tends to zero. Hence,

$$\mu\left(\overline{f(\mathbb{N})}\right) = 0.$$

This shows that the derivative of F is 0 on the complement of a set of measure 0. The statement concerning Hausdorff dimension is an immediate consequence of the fact that $\overline{f(\mathbb{N})}$ can be covered by G_k intervals of length $\lambda\beta^k$ for all positive integers k. For suitable choice of f at the digits there is equality, but it is also possible that the dimension breaks down. □

We now try to obtain a functional equation for F. For this, we exhibit a tiling of $[\![0; G_K[\![$ involving the shift operator, to allow us to use (3.1).

Let K be an integer, and $n \in [\![0; G_{K+1}[\![$. Writing $n = \sum_{k=0}^{K} \varepsilon_k G_k$, we define

$$\ell(n) := \min\left\{k \in [\![0; K]\!];\, \varepsilon_k < a_{d-1}\right\} \,;$$

If the previous set is empty, $\ell(n)$ is defined as -1. Thus, n can be uniquely written as follows :

$$\begin{aligned} &n = S^{\ell+1}(m) + r \text{ with } 0 \le \ell \le K,\ m < G_{K-\ell} \text{ and } r \in \mathcal{P}_\ell \\ &\text{or } n \in \mathcal{Q}_K := \left\{ \sum_{k=0}^{K} \varepsilon_k G_k \,;\, \varepsilon_k \ge a_{d-1} \text{ for each } k \right\} \end{aligned} \tag{3.2}$$

where

$$\mathcal{P}_\ell = \left\{ m \in \mathbb{N}\,;\, m = \sum_{k=0}^{\ell} \varepsilon_k G_k,\ \forall k < \ell :\ \varepsilon_k \geq a_{d-1},\ 0 \leq \varepsilon_\ell < a_{d-1} \right\}.$$

We are now able to write F_{G_K} in a different way.

$$\begin{aligned} G_{K+1} F_{G_{K+1}}(t) &= \sum_{n < G_{K+1}} \chi_t\left(f(n)\right) \\ &= \sum_{\ell=0}^{K} \sum_{\substack{x \in \mathcal{P}_\ell \\ m < G_{K-\ell}}} \chi_t\left(f\left(S^{\ell+1}(m) + x\right)\right) + \sum_{y \in \mathcal{Q}_K} \chi_t\left(f(y)\right). \end{aligned}$$

If $\beta > 0$, we get

$$G_{K+1} F_{G_{K+1}}(t) = \sum_{\ell=0}^{K} \sum_{x \in \mathcal{P}_\ell} \sum_{m=0}^{G_{K-\ell}-1} \chi_{\frac{t-f(x)}{\beta^{\ell+1}}}\left(f(m)\right) + \sum_{y \in \mathcal{Q}_K} \chi_t\left(f(y)\right). \tag{3.3}$$

If $\beta < 0$, we get

$$\begin{aligned} G_{K+1} F_{G_{K+1}}(t) &= \sum_{\substack{\ell=0 \\ \ell \equiv 1[2]}}^{K} \sum_{x \in \mathcal{P}_\ell} \sum_{m=0}^{G_{K-\ell}-1} \chi_{\frac{t-f(x)}{\beta^{\ell+1}}}\left(f(m)\right) + \sum_{y \in \mathcal{Q}_K} \chi_t\left(f(y)\right) \\ &\quad + \sum_{\substack{\ell=0 \\ \ell \equiv 0[2]}}^{K} \sum_{x \in \mathcal{P}_\ell} \sum_{m=0}^{G_{K-\ell}-1} \left(1 - \chi_{\frac{t-f(x)}{\beta^{\ell+1}}}\left(f(m)\right)\right). \end{aligned} \tag{3.4}$$

For convenience, we will henceforth consider positive β. We will mention the results obtained for negative ones, the computations being quite identical. We state now some intermediate results.

Lemma 7. *Let us define $p_\ell := \#\mathcal{P}_\ell$. Then the generating function of the p_ℓ is given by*

$$P(z) = \sum_{\ell=0}^{\infty} p_\ell z^\ell = a_{d-1} \frac{1 + z + \cdots + z^{d-1}}{1 - (a_0 - a_{d-1})z - \cdots - (a_{d-2} - a_{d-1})z^{d-1}}.$$

In particular $\displaystyle\sum_{\ell=0}^{\infty} \frac{p_\ell}{\alpha^\ell}$ is convergent to α.

Proof. By (3.2), we have the following recurrence formula :

$$G_{K+1} = \sum_{\ell=0}^{K} p_\ell G_{K-\ell} + \frac{p_{K+1}}{a_{d-1}}.$$

Translating this recurrence into a relation of the generating functions $P(z)$ and $G(z)$ of p_n and G_n resp. yields

$$G(z) = zG(z)P(z) + \frac{1}{a_{d-1}} P(z) \quad \text{or } P(z) = \frac{G(z)}{zG(z) + \frac{1}{a_{d-1}}}.$$

$G(z)$ is given by (cf. [GT])

$$G(z) = \frac{1 + z + \cdots + z^{d-1}}{1 - a_0 z - \cdots - a_{d-1} z^{d-1}}.$$

Thus we have the desired result for $P(z)$ and

$$\alpha = P\left(\frac{1}{\alpha}\right) = \sum_{n=0}^{\infty} \frac{p_n}{\alpha^n}.$$

□

Remark . It is an immediate consequence of this lemma and the definition of Hausdorff dimension that the dimension of $\overline{f(\bigcup_\ell \mathcal{P}_\ell)}$ is at most $\frac{\log \gamma}{\log \frac{1}{|\beta|}}$ for $|\beta| \le \frac{1}{\alpha}$, where γ is the dominating root of the polynomial

$$z^{d-1} - (a_0 - a_{d-1}) z^{d-2} - (a_1 - a_{d-1}) z^{d-3} - \cdots - (a_{d-2} - a_{d-1}).$$

Remark . Notice that in the case $a_0 = a_1 = \ldots = a_{d-1} = a$ the $\mathcal{P}_\ell$ are only non-empty for $\ell = 0, \ldots, d-1$. Furthermore, in the case $a_1 = a_2 = \ldots = a_{d-1} = a$, $a_0 = a + 1$ the number of elements of the $\mathcal{P}_\ell$ is ad for $\ell \ge d-1$. These are the only cases where p_ℓ remains bounded.

Corollary 8. *The following series converges normally :*

$$\sum_{\ell=0}^{\infty} \frac{1}{\alpha^{\ell+1}} \sum_{x \in \mathcal{P}_\ell} F\left(\frac{t - f(x)}{\beta^{\ell+1}}\right).$$

Proof.

$$\left| \frac{1}{\alpha^{\ell+1}} \sum_{x \in \mathcal{P}_\ell} F\left(\frac{t - f(x)}{\beta^{\ell+1}}\right) \right| \le \frac{p_\ell}{\alpha^{\ell+1}},$$

whose sum converges by the previous Lemma 7. We denote H the function defined by Corollary 8. □

Dividing (3.3) by G_{K+1} and subtracting $H(t)$ we get

$$\begin{aligned} F_{G_{K+1}}(t) - H(t) &= \sum_{\ell=0}^{K} \frac{G_{K-\ell}}{G_{K+1}} \sum_{x \in \mathcal{P}_\ell} \frac{1}{G_{K-\ell}} \sum_{m=0}^{G_{K-\ell}-1} \chi_{\frac{t-f(x)}{\beta^{\ell+1}}} (f(m)) \\ &- \sum_{\ell=0}^{K} \frac{1}{\alpha^{\ell+1}} \sum_{x \in \mathcal{P}_\ell} F\left(\frac{t - f(x)}{\beta^{\ell+1}}\right) - \sum_{\ell=K+1}^{\infty} \frac{1}{\alpha^{\ell+1}} \sum_{x \in \mathcal{P}_\ell} F\left(\frac{t - f(x)}{\beta^{\ell+1}}\right) \\ &+ \frac{1}{G_{K+1}} \sum_{y \in \mathcal{Q}_K} \chi_t\left(f(y)\right). \end{aligned}$$

Thus, for any integer S smaller than K we have

$$
\begin{aligned}
|F_{G_{K+1}}(t)-H(t)| \le& \sum_{\ell=0}^{S}\left|\frac{G_{K-\ell}}{G_{K+1}}-\frac{1}{\alpha^{\ell+1}}\right|\sum_{x\in\mathcal{P}_\ell}\frac{1}{G_{K-\ell}}\sum_{m=0}^{G_{K-\ell}-1}\chi_{\frac{t-f(x)}{\beta^{\ell+1}}}(f(m))\\
&+\sum_{\ell=S+1}^{K}\left|\frac{G_{K-\ell}}{G_{K+1}}-\frac{1}{\alpha^{\ell+1}}\right|\sum_{x\in\mathcal{P}_\ell}\frac{1}{G_{K-\ell}}\sum_{m=0}^{G_{K-\ell}-1}\chi_{\frac{t-f(x)}{\beta^{\ell+1}}}(f(m))\\
&+\sum_{\ell=0}^{S}\frac{1}{\alpha^{\ell+1}}\sum_{x\in\mathcal{P}_\ell}\left(\frac{1}{G_{K-\ell}}\sum_{m=0}^{G_{K-\ell}-1}\chi_{\frac{t-f(x)}{\beta^{\ell+1}}}(f(m))-F\left(\frac{t-f(x)}{\beta^{\ell+1}}\right)\right)\\
&+\sum_{\ell=S+1}^{K}\frac{1}{\alpha^{\ell+1}}\sum_{x\in\mathcal{P}_\ell}\left(\frac{1}{G_{K-\ell}}\sum_{m=0}^{G_{K-\ell}-1}\chi_{\frac{t-f(x)}{\beta^{\ell+1}}}(f(m))-F\left(\frac{t-f(x)}{\beta^{\ell+1}}\right)\right)\\
&+\sum_{\ell=K+1}^{\infty}\frac{1}{\alpha^{\ell+1}}\sum_{x\in\mathcal{P}_\ell}F\left(\frac{t-f(x)}{\beta^{\ell+1}}\right)+\frac{1}{G_{K+1}}\sum_{y\in\mathcal{Q}_K}\chi_t(f(y))\\
\le& \sum_{\ell=0}^{S}\left|\frac{G_{K-\ell}}{G_{K+1}}-\frac{1}{\alpha^{\ell+1}}\right|p_\ell+C\sum_{\ell=S+1}^{K}\frac{p_\ell}{\alpha^{\ell+1}}\\
&+\sum_{\ell=0}^{S}\frac{1}{\alpha^{\ell+1}}\sum_{x\in\mathcal{P}_\ell}\left(\frac{1}{G_{K-\ell}}\sum_{m=0}^{G_{K-\ell}-1}\chi_{\frac{t-f(x)}{\beta^{\ell+1}}}(f(m))-F\left(\frac{t-f(x)}{\beta^{\ell+1}}\right)\right)\\
&+2\sum_{\ell=S+1}^{K}\frac{p_\ell}{\alpha^{\ell+1}}+\sum_{\ell=K+1}^{\infty}\frac{p_\ell}{\alpha^{\ell+1}}+\frac{p_{K+1}}{a_{d-1}G_{K+1}}
\end{aligned}
$$

where C is a constant such that $\frac{G_{K-\ell}}{G_{K+1}}\le(C-1)\alpha^{\ell+1}$ (The existence of C comes from this of two constants A and B verifying $A\alpha^n\le G_n\le B\alpha^n$ for every n).

Let $\varepsilon>0$. There exists S in $\mathbb{N}$ such that

$$\max(2,C)\sum_{\ell=S+1}^{\infty}\frac{p_\ell}{\alpha^{\ell+1}}<\varepsilon.$$

Now, there exists an integer K_0 greater than S such that for every $K>K_0$ the following majorations hold :

- $\displaystyle\sum_{\ell=0}^{S}\left|\frac{G_{K-\ell}}{G_{K+1}}-\frac{1}{\alpha^{\ell+1}}\right|p_\ell<\varepsilon$
- $\forall\ell<S,\ \forall x\in\mathcal{P}_\ell:\ \displaystyle\left|F_{G_{K-\ell}}\left(\frac{t-f(x)}{\beta^{\ell+1}}\right)-F\left(\frac{t-f(x)}{\beta^{\ell+1}}\right)\right|\le\frac{\varepsilon\alpha^{\ell+1}}{(S+1)p_\ell}$
- $\displaystyle\frac{p_{K+1}}{a_{d-1}G_{K+1}}<\varepsilon.$

Then, for any $K>K_0$, we have $\left\|F_{G_{K+1}}-H\right\|_\infty\le 7\varepsilon$. Thus we have proved the following :

Theorem 9. *i/ For positive β, F is a solution of the functional equation*

$$Y(t)=\sum_{\ell=0}^{\infty}\frac{1}{\alpha^{\ell+1}}\sum_{x\in\mathcal{P}_\ell}Y\left(\frac{t-f(x)}{\beta^{\ell+1}}\right). \tag{3.5}$$

ii/ For negative β, F is a solution of the functional equation

$$Y(t)=\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\alpha^{\ell+1}}\sum_{x\in\mathcal{P}_\ell}Y\left(\frac{t-f(x)}{\beta^{\ell+1}}\right)+\sum_{\ell=0}^{\infty}\frac{p_{2\ell}}{\alpha^{2\ell+1}} \tag{3.6}$$

In order to be able to use the above functional equations to determine the distribution functions we prove the following:

Theorem 10. *The solutions of the functional equations* (3.5) *and* (3.6) *are unique among distribution functions, where it is understood that two distribution functions yielding the same measure are identified.*

Proof. We first prove that a solution of the functional equation (3.5) has to have compact support in the sense that "$0<f<1$" is bounded. Since the function $f(x)$ is bounded we can choose t_0 such that $t<\frac{t-f(x)}{\beta^{\ell+1}}$ for all $t>t_0$ and for all ℓ and $x\in\mathcal{P}_\ell$. Then we have for such t and any solution F of (3.5)

$$F(t)=\sum_{\ell=0}^{\infty}\frac{1}{\alpha^{\ell+1}}\sum_{x\in\mathcal{P}_\ell}F\left(\frac{t-f(x)}{\beta^{\ell+1}}\right)\geq\sum_{\ell=0}^{\infty}\frac{1}{\alpha^{\ell+1}}p_\ell F(t)=F(t);$$

the last equality is a consequence of Lemma 7. Therefore we have

$$F(t)=F\left(\frac{t-f(x)}{\beta^{\ell+1}}\right) \tag{3.7}$$

for all ℓ. Take now $\ell=1$, x fixed and t_1 large enough; then F is constant on the interval $[t_1,\frac{t_1-f(x)}{\beta^2}]$. Take $t_2=\frac{t_1-f(x)}{\beta^2}>\frac{t}{\beta}$ and continue this geometrically increasing process, we derive that the function is constant on the interval $[t_1,\infty[$; this constant has to be 1. The argument also applies for large negative values of t.

For negative β one has to split summation in (3.6) into odd and even ℓ; the same arguments as above yield $F(t)+F(-t)=1$ for large enough $|t|$. Then again (3.7) has to hold, and thus F has compact support.

To prove uniqueness we compute the moment generating function of (3.5) and (3.6) i.e.

$$\hat{F}(z)=\int_{-\infty}^{\infty}e^{tz}dF(t)$$

(we have omitted the "i" in the usual Fourier-Stieltjes transform just for convenience). The transform of any solution is an entire function because of the property proved above. It satisfies

$$\hat{F}(z)=\sum_{\ell=1}^{\infty}\frac{1}{\alpha^\ell}\hat{F}(\beta^\ell z)G_\ell(z), \tag{3.8}$$

where

$$G_\ell(z)=\sum_{x\in\mathcal{P}_{\ell-1}} e^{zf(x)}=\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{x\in\mathcal{P}_{\ell-1}} f(x)^n.$$

Since $\hat{F}$ is an entire function we can write its power series expansion

$$\hat{F}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\varphi(n),\quad \varphi(0)=1$$

and insert into (3.6) which yields
(3.9)

$$\varphi(n)\left(1-\frac{\beta^n}{\alpha}P\left(\frac{\beta^n}{\alpha}\right)\right)=\sum_{k=0}^{n-1}\binom{n}{k}\varphi(k)\sum_{\ell=1}^{\infty}\left(\frac{\beta^k}{\alpha}\right)^{\ell}\sum_{x\in\mathcal{P}_{\ell-1}}(f(x))^{n-k},\quad \text{for } n\geq 1$$

where $P(z)$ is the generating function of the number of elements of the $\mathcal{P}$'s whose properties have been studied in Lemma 7. (3.9) is a recurrence relation for the $\varphi(n)$, thus all of them are uniquely determined by this equation and therefore the distribution function is uniquely determined in any point of continuity. $\square$

Proposition 5 gives the existence of a continuous extension of f to $\mathcal{K}_G$. We will use the properties of this extension in order to prove the continuity of F.

Lemma 11. *Let $s\in\mathbb{N}^*$. Then the following event is almost sure w.r.t $\mathbb{P}$:*

$$A_s:=\Big\{\omega=(\varepsilon_0,\dots,\varepsilon_k,\dots)\in\mathcal{K}_G\,;$$
$$\exists(\ell_1,\dots,\ell_s)\in\mathbb{N}^s,\,\forall j\in[\![0;s-1]\!],(\varepsilon_{\ell_1+\ldots+\ell_j+j},\dots,\varepsilon_{\ell_1+\ldots+\ell_{j+1}+j})\in\mathcal{P}_{\ell_{j+1}}\Big\}.$$

Proof.

$$A_s^C=\big\{\omega\in\mathcal{K}_G\,;\,\#\{j\in\mathbb{N};\varepsilon_j\geq a_{d-1}\}<s\big\}\quad\text{yields to}$$
$$\mathbb{P}\left(A_s^C\right)=\sum_{m=0}^{s-1}\sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_m=0}^{\infty}\sum_{x_1\in\mathcal{P}_{\ell_1}}\cdots\sum_{x_m\in\mathcal{P}_{\ell_m}}\mathbb{P}\big([x_1\dots x_m]\big)\mathbb{P}(A_0^C)$$

It is hence sufficient to prove that $\mathbb{P}(A_0)=1$. But $\mathbb{P}(A_0)=\sum_{\ell=0}^{\infty}\sum_{x\in\mathcal{P}_\ell}\mathbb{P}\big([x]\big)$ and, because $F_{\ell+1+j}=G_j$ for any $x\in\mathcal{P}_\ell$, (2.7) gives

$$\mathbb{P}\big([x]\big)=\frac{G_0\alpha^{d-1}+(G_1-a_0G_0)\alpha^{d-2}+\cdots+(G_{d-1}-a_0G_{d-2}-\cdots-a_{d-2}G_0)}{\alpha^{\ell+1}(1+\alpha+\cdots+\alpha^{d-1})}.$$

Thus, by Lemma 7, $\mathbb{P}(A_0)=\sum_{\ell=0}^{\infty}\frac{p_\ell}{\alpha^{\ell+1}}=1.$

Theorem 12. *Let f be of the* (3.1) *type, and F its distribution function. Then F is continuous iff f is not identically 0.*

Proof. If f is 0, then F is the Heaviside function $\chi_{[0,\infty[}$, which is not continuous at $t = 0$.

In general, if f satisfies (3.1), then f verifies the assumptions of Proposition 5 too. In terms of $\mathbb{P}$, the continuity of F means that $\mathbb{P}(f = t) = 0$ for every t (we henceforth omit the tilde). Let us note π_ℓ to be the ℓ-th projection on $\mathcal{K}_G$. Then f is given by $f(\omega) = \sum_{\ell=0}^{\infty} f \circ \pi_\ell(\omega)\beta^\ell$. Let $t \in \mathbb{R}$. Then, considering $\omega \in \mathcal{K}_G$, taking an s in $\mathbb{N}^*$ and conditioning w.r.t. the almost sure event A_s, we have

$$
\begin{aligned}
&\mathbb{P}\big(f(\omega) = t\big) = \mathbb{P}\big(f(\omega) = t \mid A_s\big)\\
&= \sum_{\ell_1=0}^{\infty} \cdots \sum_{\ell_s=0}^{\infty} \mathbb{P}\Big(f(\omega) = t \mid \forall 0 \le j < s, (\varepsilon_{\ell_1+\cdots+\ell_j+j}, \dots, \varepsilon_{\ell_1+\cdots+\ell_{j+1}+j}) \in \mathcal{P}_{\ell_{j+1}}\Big)\\
&= \sum_{\ell_1=0}^{\infty} \cdots \sum_{\ell_s=0}^{\infty} \sum_{x_1 \in \mathcal{P}_{\ell_1}} \cdots \sum_{x_s \in \mathcal{P}_{\ell_s}} \mathbb{P}\left((\pi_0(\omega), \dots, \pi_{\ell_1+\ldots+\ell_s+s-1}(\omega)) = (x_1, \dots, x_s)\right)\\
&\qquad\qquad \times \mathbb{P}\Big(f(\omega) = t \mid (\pi_0(\omega), \dots, \pi_{\ell_1+\ldots+\ell_s+s-1}(\omega)) = (x_1, \dots, x_s)\Big)\\
&= \sum_{\ell_1=0}^{\infty} \cdots \sum_{\ell_s=0}^{\infty} \sum_{x_1 \in \mathcal{P}_{\ell_1}} \cdots \sum_{x_s \in \mathcal{P}_{\ell_s}} \mathbb{P}\Big((\pi_0(\omega), \dots, \pi_{\ell_1+\cdots+\ell_s+s-1}(\omega)) = (x_1, \dots, x_s)\Big)\\
&\qquad\qquad \times \mathbb{P}\left(f(\omega) = \frac{t - f(x_1 \ldots x_s)}{\beta^{\ell_1+\cdots+\ell_s+s}}\right).
\end{aligned}
$$

We apply now the previous equality to some t for which $\mathbb{P}\big(f(\omega) = t\big)$ attains its greatest value. Because of

$$
\sum_{\ell_1=0}^{\infty} \cdots \sum_{\ell_s=0}^{\infty} \sum_{x_1 \in \mathcal{P}_{\ell_1}} \cdots \sum_{x_s \in \mathcal{P}_{\ell_s}} \mathbb{P}\Big((\pi_0(\omega), \dots, \pi_{\ell_1+\cdots+\ell_s}(\omega)) = (x_1, \dots, x_s)\Big) = 1,
$$

we have $\mathbb{P}\left(f(\omega) = \dfrac{t - f(x_1 \ldots x_s)}{\beta^{\ell_1+\cdots+\ell_s+s}}\right) = \mathbb{P}\big(f(\omega) = t\big)$ for any $x = x_1 \cdots x_s$ where x_j belongs to $\mathcal{P}_{\ell_j}$. It is sufficient to remark that $(t - f(x))/\beta^{\ell_1+\ldots+\ell_s+s}$ takes infinitely many values (provided that f is not identically 0) to get a contradiction if $\mathbb{P}\big(f(\omega) = t\big) > 0$.

Indeed, suppose there exists some $\varepsilon < a_{d-1}$ such that $f(\varepsilon) \neq 0$. Considering $x = 0^{(\ell)}\varepsilon$, $t/\beta^{\ell+1} - f(\varepsilon)/\beta$ takes infinitely many values if $t \neq 0$. But for $t = 0$, we get a non zero value, namely $f(\varepsilon)/\beta$, at which f reaches its maximum, and to which we are allowed to apply the argument above. On the other side, if such an ε does not exist, there is an $\eta \ge a_{d-1}$ such that $f(\eta) \neq 0$. Considering $x = \eta 0^{(\ell)}$, we apply the same argument *mutatis mutandis*, which ends the proof. □

4. Concluding Remarks

Finally we want to present two special examples which show the applicability of the functional equations (3.5) and (3.6). We will show the uniform distribution of two van der Corput type digital sequences (for the definition of van der Corput sequence we refer to [KN]).

Proposition 13. *Let a and d be positive integers and define G_n by the recurrence $G_{n+d} = a(G_{n+d-1} + \ldots + G_n)$ and the corresponding canonical initial values. Then the G-adic van der Corput sequence*

$$f\left(\sum_{k=0}^{K} \varepsilon_k G_k\right) = \sum_{k=0}^{K} \frac{\varepsilon_k}{\alpha^{k+1}}$$

is uniformly distributed in $[0,1]$. The discrepancy of $f(n)$ i.e.

$$D_N = \sup_{x\in[0,1]} \left| \frac{1}{N} \sum_{n<N} \chi_x\left(f(n)\right) - x \right|$$

is $\mathcal{O}(\frac{\log N}{N})$.

Proof. The proof is done just by inserting the distribution function of uniform distribution into (3.5). Since the solution is unique by Theorem 10, we are done. The proof of the estimate for the discrepancy is a rephrasing of the proof for the q-adic van der Corput sequence (cf. [KN]). $\square$

Proposition 14. *Let a be a positive integer and define G_n by the recurrence $G_{n+2} = (a+1)G_{n+1} + aG_n$ and the initial values $G_0 = 1$ and $G_1 = a+2$. Define a G-additive function by $f(\varepsilon) = \frac{\varepsilon}{\alpha}$ for $0 \le \varepsilon \le a$ and $f(a+1) = \frac{a}{\alpha - 1}$ and*

$$f\left(\sum_{k=0}^{K} \varepsilon_k G_k\right) = \sum_{k=0}^{K} \frac{f(\varepsilon_k)}{\alpha^{k}}.$$

Then $f(n)$ is uniformly distributed in $[0,1]$.

Proof. The proof is again done by inserting the distribution function of uniform distribution into (3.5) and observing that

$$\bigcup_{\ell=0}^{\infty} \bigcup_{x\in\mathcal{P}_\ell} \left[f(x), f(x) + \frac{1}{\alpha^\ell}\right) = [0,1)$$

is a disjoint union. $\square$

The two examples we presented here, are the only uniformly distributed G-additive functions we could find. In the general case the functional equation expresses some self-similarity structure of the graph of the distribution function which shall be illustrated by the following picture. We have computed the distribution function for $G_{n+3} = 3G_{n+2} + G_{n+1} + G_n$ with initial values $G_0 = 1$, $G_1 = 4$ and $G_2 = 14$ and

$$f\left(\sum_{k=0}^{K} \varepsilon_k G_k\right) = \sum_{k=0}^{K} \frac{\varepsilon_k}{\alpha^{k+1}}.$$

The deviation from uniform distribution comes from the fact that the reversion of an admissible string in G-adic expansion need not be an admissible string in α-expansion (cf. [Pa]).

Acknowledgements. This work was done during the second authors visit at the Dynamics, Statistics and Algorithms group of the "Université de Provence". The authors want to thank Pierre Liardet and Robert Tichy for many helpful discussions. The first author is indebted to Enrique Andjel for the idea which led to the proof of Theorem 12.

References

[Br] A. Brauer, *On algebraic equations with all but one root in the interior of the unit circle*, Math. Nachr. **4** (1951), 250–257.

[Ch] K. Chandrasekharan, *Classical Fourier transforms*, 1980, Universitex, Springer Verlag.

[CRT] J. Coquet, G. Rhin et Ph. Toffin, *Représentation des entiers naturels et indépendence statistique 2*, Ann. Inst. Fourier **31** (1981), 1–15.

[De] H. Delange, *Sur les fonctions q-additives et q-multiplicatives*, Acta Arith. **21** (1972), 258–298.

[Fr] A.S. Fraenkel, *Systems of numeration*, Amer. Math. Monthly **92** (1985), 105–114.

[Ge] A.O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1968), 259–266.

[GLT] P.J. Grabner, P. Liardet and R.F. Tichy, *Odometers and Systems of Numeration*, Acta Arith. **70** (1995), 103–123.

[GT] P.J. Grabner and R.F. Tichy, *Contributions to Digit Expansions with Respect to Linear Recurring Sequences*, J. Number Th. **36** (1990), 160–169.

[Ko1] R.J. Kooman, *Convergence Properties of Recurrence Sequences*, 1989, Thesis, University of Leiden.

[Ko2] R.J. Kooman, *Decomposition of matrix sequences*, Indag. Math. (new series) **5** (1994), 61–79.

[KN] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, J. Wiley and Sons, New York, 1974.

[Ma] J.-L. Mauclaire, *Intégration et théorie des nombres*, Hermann, Paris, 1986.

[MF] M. Mendès-France, *Les suites à spectre vide*, J. Number Th. **5** (1973), 1–15.

[Pa] W. Parry, *On the β-Expansion of Real Numbers*, Acta Math. Acad. Sci. Hung. **12** (1961), 401–416.

G. Barat
DSA
Université de Provence
CMI, Château Gombert
39, rue Joliot-Curie
13453 Marseille, Cedex 13
France

P. Grabner
Institut für Mathematik A
Technische Universität Graz
Steyrergasse 30
8010 Graz
Austria

E-mail address: barat@arioch.univ-mrs.fr grabner@weyl.math.tu-graz.ac.at